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CONTENTS

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ON (CO)HOMOLOGICAL PROPERTIES OF REMAINDERS OF STONE-ČECH COMPACTIFICATIONS

V. BALADZE

Abstract. In the paper are defined the Čech border homology and cohomology groups of closed pairs of normal spaces and showed that they give intrinsic characterizations of Čech (co)homology groups based on finite open coverings, cohomological coefficients of cyclicity, small and large cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces.

Introduction

The investigation and discussion presented in this paper are centered around the following problem:

Find necessary and sufficient conditions under which a space of given class has a compactification whose remainder has the given topological property (cf. [35], Problem I, p. 332, and Problem II, p. 334).

This problem for different topological invariants and properties was studied by several authors (see [1–3, 5–8, 11–14, 19–25, 27, 30–36]).

The present paper is motivated by this general problem. Specifically, we study this problem for the properties: Čech (co)homology groups based on finite open covers, cohomological coefficients of cyclicity and cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces are given groups and given numbers, respectively.

In the paper we define the Čech type covariant and contravariant functors which coefficients in an abelian group G,

$$\check{\mathrm{H}}^{\infty}_n(-,-;G) : \mathscr{N}_p^{\mathbf{2}} \to \mathscr{A}b$$

and

$$\hat{\mathrm{H}}^n_{\infty}(-,-;G) : \mathscr{N}_p^{\mathbf{2}} \to \mathscr{A}b,$$

from the category $\mathscr{N}_p^2$ of closed pairs of normal spaces and proper maps to the category $\mathscr{A}b$ of abelian groups and homomorphisms. The construction of these functors is based on all border open covers of pairs $(X, A) \in ob(\mathscr{N}_p^{\mathbf{2}})$ (see Definition 1.1 and Definition 1.2).

One of our main results of the paper is the following theorem (see Theorem 2.1). Let $\mathscr{M}_p^2$ be the category of closed pairs of metrizable spaces and proper maps. For each closed pair $(X, A) \in ob(\mathscr{M}_p^{\mathbf{2}})$, one has

$$\check{H}^f_n(\beta X \setminus X, \beta A \setminus A; G) = \check{H}^{\infty}_n(X, A; G)$$

and

$$\hat{H}^n_f(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}^n_{\infty}(X, A; G),$$

where $\check{H}^f_n(\beta X \setminus X, \beta A \setminus A; G)$ and $\hat{H}^n_f(\beta X \setminus X, \beta A \setminus A; G)$ are Čech homology and cohomology groups based on all finite open covers of $(\beta X \setminus X, \beta A \setminus A)$, respectively (see [17, Ch. IX, p. 237]).

We also investigate the border cohomological coefficient of cyclicity η^{∞}_G, border small and large cohomological dimensions $d^f_{\infty}(X; G)$ and $D^f_{\infty}(X; G)$ and prove the following relations (see Theorem 2.3,

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Theorem 3.2 and Theorem 3.6):

$$\eta^{\infty}_G(X,A)=\eta_G(\beta X\setminus X,\beta A\setminus A),$$
$$d^f_\infty(X;G)=d_f(\beta X\setminus X;G),$$
$$D^f_\infty(X;G)=D_f(\beta X\setminus X;G),$$

where $\eta_G(\beta X\setminus X,\beta A\setminus A)$, $d_f(\beta X\setminus X;G)$ and $D_f(\beta X\setminus X;G)$ are well known cohomological coefficient of cyclicity [10,29], small cohomological dimension and large cohomological dimension [28] of remainders $(\beta X\setminus X,\beta A\setminus A)$ and $\beta X\setminus X$, respectively.

Without any further reference we will use definitions and results from the monographs General Topology [18], Algebraic Topology [17] and Dimension Theory [28].

1. On Čech Border (Co)homology Groups

In this section we give an outline of a generalization of Čech homology theory by replacing the set of all finite open coverings in the definition of Čech (co)homology group ($\hat{H}^n_f(X,A;G)$) $\check{H}^f_n(X,A;G)$ (see [17, Ch. IX, p. 237]) by a set of all finite open families with compact enclosures. For this aim we give the following definitions.

An indexed family of subsets of set X is a function α from an indexed set V_α to the set 2^X of subsets of X. The image $\alpha(v)$ of index $v\in V_\alpha$ is denoted by α_v. Thus the indexed family α is the family $\alpha=\{\alpha_v\}_{v\in V_\alpha}$. If $|V_\alpha|<\aleph_0$, then we say that α family is a finite family.

Let $V^{'}_\alpha$ be a subset of set V_α. A family $\{\alpha_v\}_{v\in V^{'}_\alpha}$ is called a subfamily of family $\{\alpha_v\}_{v\in V_\alpha}$.

By $\alpha=\{\alpha_v\}_{v\in(V_\alpha,V^{'}_\alpha)}$ we denote the family consisting of family $\{\alpha_v\}_{v\in V_\alpha}$ and its subfamily $\{\alpha_v\}_{v\in V^{'}_\alpha}$.

Definition 1.1. (see [33]). A finite family $\alpha=\{\alpha_v\}_{v\in V_\alpha}$ of open subsets of normal space X is called a border cover of X if its enclosure $K_\alpha=X\setminus\bigcup\limits_{v\in V_\alpha}\alpha_v$ is a compact subset of X.

Definition 1.2. (cf. [33]). A finite open family $\alpha=\{\alpha_v\}_{v\in(V_\alpha,V^A_\alpha)}$ is called a border cover of closed pair $(X,A)\in\mathscr{N}^2$ if there exists a compact subset K_α of X such that $X\setminus K_\alpha=\bigcup\limits_{v\in V_\alpha}\alpha_v$ and $A\setminus K_\alpha\subseteq\bigcup\limits_{v\in V^A_\alpha}\alpha_v$.

The set of all border covers of (X,A) is denoted by $\mathrm{cov}_\infty(X,A)$. Let $K^A_\alpha=K_\alpha\cap A$. Then the family $\{\alpha_v\cap A\}_{v\in V^A_\alpha}$ is a border cover of subspace A.

Definition 1.3. Let $\alpha,\beta\in\mathrm{cov}_\infty(X,A)$ be two border covers of (X,A) with indexing pairs (V_α,V^A_α) and (V_β,V^A_β), respectively. We say that the border cover β is a refinement of border cover α if there exists a refinement projection function $p:(V_\beta,V^A_\beta)\to(V_\alpha,V^A_\alpha)$ such that for each index $v\in V_\beta$ $(v\in V^A_\beta)$ $\beta_v\subset\alpha_{p(v)}$.

It is clear that $\mathrm{cov}_\infty(X,A)$ becomes a directed set with the relation $\alpha\le\beta$ whenever β is a refinement of α.

Note that for each $\alpha\in\mathrm{cov}_\infty(X,A)$, $\alpha\le\alpha$, and if for each $\alpha,\beta,\gamma\in\mathrm{cov}_\infty(X,A)$, $\alpha\le\beta$ and $\beta\le\gamma$, then $\alpha\le\gamma$.

Let $\alpha,\beta\in\mathrm{cov}_\infty(X,A)$ be two border covers with indexing pairs (V_α,V^A_α) and (V_β,V^A_β), respectively. Consider a family $\gamma=\{\gamma_v\}_{v\in(V_\gamma,V^A_\gamma)}$, where $V_\gamma=V_\alpha\times V_\beta$ and $V^A_\gamma=V^A_\alpha\times V^A_\beta$. Let $v=(v_1,v_2)$, where $v_1\in V_\alpha$, $v_2\in V_\beta$. Assume that $\gamma_v=\alpha_{v_1}\cap\beta_{v_2}$. The family $\gamma=\{\gamma_v\}_{v\in(V_\gamma,V^A_\gamma)}$ is a border cover of (X,A) and $\gamma\ge\alpha,\beta$.

For each border cover $\alpha\in\mathrm{cov}^\infty(\mathrm{X},\mathrm{A})$ with indexing pair (V_α,V^A_α), by (X_α,A_α) denote the nerve α, where A_α is the subcomplex of simplexes s of complex X_α with vertices of V^A_α such that $\mathrm{Car}_\alpha(\mathrm{s})\cap\mathrm{A}\neq\emptyset$, where $\mathrm{Car}_\alpha(s)$ is the carrier of simplex s (see [17, pp. 234]). The pair (X_α,A_α) is a simplicial pair. Moreover, any two refinement projection functions $p,q:\beta\to\alpha$ induce contiguous simplicial maps of simplicial pairs $p^\beta_\alpha,q^\beta_\alpha:(X_\beta,A_\beta)\to(X_\alpha,A_\alpha)$ (see [17, pp. 234–235]).

Using the construction of formal homology theory of simplicial complexes [17, Ch. VI] we can define the unique homomorphisms

$$p^{\beta}_{\alpha *} : H_n(X_\beta, A_\beta : G) \to H_n(X_\alpha, A_\alpha; G)$$

and

$$p^{\beta *}_{\alpha} : H^n(X_\alpha, A_\alpha : G) \to H_n(X_\beta, A_\beta; G),$$

where G is any abelian coefficient group.

Note that $p^{\alpha}_{\alpha *} = 1_{H_n(X_\alpha, A_\alpha : G)}$ and $p^{\alpha *}_{\alpha} = 1_{H^n(X_\alpha, A_\alpha : G)}$. If $\gamma \geq \beta \geq \alpha$ than

$$p^{\gamma}_{\alpha *} = p^{\beta}_{\alpha *} \cdot p^{\gamma}_{\beta *}$$

and

$$p^{\gamma *}_{\alpha} = p^{\gamma *}_{\beta} \cdot p^{\beta *}_{\alpha}.$$

Thus, the families

$$\{H_n(X_\alpha, A_\alpha; G), p^{\beta}_{\alpha *}, \mathrm{cov}_\infty(X, A)\}$$

and

$$\{H^n(X_\alpha, A_\alpha; G), p^{\beta *}_{\alpha}, \mathrm{cov}_\infty(X, A)\}$$

form inverse and direct systems of groups.

The inverse and direct limit groups of above defined inverse and direct systems are denoted by symbols

$$\check{H}^{\infty}_n(X, A; G) = \varprojlim \{H_n(X_\alpha, A_\alpha; G), p^{\beta}_{\alpha *}, \mathrm{cov}_\infty(X, A)\}$$

and

$$\hat{H}^{n}_\infty(X, A; G) = \varinjlim \{H^n(X_\alpha, A_\alpha; G), p^{\beta *}_{\alpha}, \mathrm{cov}_\infty(X, A)\}$$

and called n-dimensional Čech border homology group and n-dimensional Čech border cohomology group of pair (X, A) with coefficients in abelian group G, respectively.

For a pair $(X, A) \in ob(\mathscr{N}^2_p)$ and a proper map $f : (X, A) \to (Y, B)$ of pairs, the induced homomorphisms

$$f^{\infty}_{*} : \check{H}^{\infty}_n(X, A; G) \to \check{H}^{\infty}_n(Y, B; G)$$

and

$$f^{*}_{\infty} : \hat{H}^{n}_\infty(X, A; G) \to \hat{H}^{n}_\infty(Y, B; G),$$

and the boundary and coboundary homomorphisms

$$\partial^{\infty}_{n} : \check{H}^{\infty}_n(X, A; G) \to \check{H}^{\infty}_{n-1}(A; G)$$

and

$$\delta^{n}_{\infty} : \hat{H}^{n-1}_\infty(A; G) \to \hat{H}^{n}_\infty(X, A; G)$$

are defined. For details of these definitions, see Eilenberg and Steenrod [17].

We have the following theorems.

Theorem 1.4. *There exist the covariant and contravariant functors*

$$\check{\mathrm{H}}^{\infty}_{*}(-,-;G) : \mathscr{N}^2_p \to \mathscr{A}b$$

and

$$\hat{\mathrm{H}}^{*}_{\infty}(-,-;G) : \mathscr{N}^2_p \to \mathscr{A}b$$

given by formulas

$$\check{\mathrm{H}}^{\infty}_{*}(-,-;G)(X, A) = \check{H}^{\infty}_{*}(X, A; G), \quad (X, A) \in ob(\mathscr{N}^2_p)$$
$$\check{\mathrm{H}}^{\infty}_{*}(-,-;G)(f) = f^{\infty}_{*}, \quad f \in \mathrm{Mor}_{\mathscr{N}^2_\mathrm{p}}((X, A), (Y, B))$$

and

$$\hat{\mathrm{H}}^{*}_{\infty}(-,-;G)(X, A) = \hat{H}^{*}_{\infty}(X, A; G), \quad (X, A) \in ob(\mathscr{N}^2_p)$$
$$\hat{\mathrm{H}}^{*}_{\infty}(-,-;G)(f) = f^{*}_{\infty}, \quad f \in \mathrm{Mor}_{\mathscr{N}^2_\mathrm{p}}((X, A), (Y, B)).$$

Theorem 1.5. *Let $f : (X, A) \to (Y, B)$ be a proper map. Then hold the following equalities*

$$(f_{|A})_*^\infty \cdot \partial_n^\infty = \partial_n^\infty \cdot f_*^\infty$$

and

$$\delta_\infty^{n-1} \cdot (f_{|A})_\infty^* = f_\infty^* \cdot \delta_\infty^{n-1}.$$

Theorem 1.6. *Let $(X, A) \in ob(\mathcal{N}_p^2)$ and let $i : A \to X$ and $j : X \to (X, A)$ be the inclusion maps. Then the Čech border cohomology sequence*

$$\cdots \longrightarrow \check{H}_\infty^{n-1}(A; G) \xrightarrow{\delta_\infty^{n-1}} \check{H}_\infty^n(X, A; G) \xrightarrow{j_\infty^*} \check{H}_\infty^n(X; G) \xrightarrow{i_\infty^*} \check{H}_\infty^n(A; G) \longrightarrow \cdots$$

is exact while the Čech border homology sequence

$$\cdots \longleftarrow \hat{H}_{n-1}^\infty(A; G) \xleftarrow{\partial_n^\infty} \hat{H}_n^\infty(X, A; G) \xleftarrow{j_*^\infty} \hat{H}_n^\infty(X; G) \xleftarrow{i_*^\infty} \hat{H}_n^\infty(A; G) \longleftarrow \cdots$$

is partially exact.

Theorem 1.7. *Let $(X, A) \in ob(\mathcal{N}_p^2)$ and G be an abelian group. If U is open in X and $\bar{U} \subset \mathrm{int}A$, then the inclusion map $i : (X \setminus U, A \setminus U) \to (X, A)$ induces isomorphisms*

$$i_*^\infty : \check{H}_n^\infty(X \setminus U, A \setminus U) \to \check{H}_n^\infty(X, A; G)$$

and

$$j_\infty^* : \hat{H}_\infty^n(X, A; G) \to \hat{H}_\infty^n(X \setminus U, A \setminus U)$$

Theorem 1.8. *If X is a compact space, then for each $n \neq 0$,*

$$\check{H}_n^\infty(X; G) = 0 = \check{H}_\infty^n(X; G)$$

and

$$\hat{H}_0^\infty(X; G) = G = \hat{H}_\infty^0(X; G).$$

Theorem 1.9. *Let (X, A, B) be a triple of normal space X and its closed subsets A and B with $B \subset A$. Then the Čech border homology sequence*

$$\cdots \longleftarrow \check{H}_{n-1}^\infty(A, B; G) \xleftarrow{\bar{\partial}_n^\infty} \check{H}_n^\infty(X, A; G) \xleftarrow{\bar{j}_*^\infty} \check{H}_n^\infty(X, B; G) \xleftarrow{\bar{i}_*^\infty} \check{H}_n^\infty(A, B; G) \longleftarrow \cdots$$

and the Čech border cohomology sequence

$$\cdots \longrightarrow \hat{H}_\infty^{n-1}(A, B; G) \xrightarrow{\bar{\delta}_\infty^n} \hat{H}_\infty^n(X, A; G) \xrightarrow{\bar{j}_\infty^*} \hat{H}_\infty^n(X, B; G) \xrightarrow{\bar{i}_\infty^*} \hat{H}_\infty^n(A, B; G) \longrightarrow \cdots$$

are partially exact and exact, respectively. Here $\bar{\partial}_n^\infty = j_{n-1}^{'\infty} \cdot \partial_n^\infty$, $\bar{\delta}_\infty^n = \delta_\infty^n \cdot j_\infty^{'n-1}$ and $j_{n-1}^{'\infty}$, $j_\infty^{'n-1}$, $\bar{j}_\infty^*$, $\bar{j}_*^\infty$, and $\bar{i}_\infty^*$, $\bar{i}_*^\infty$ are the homomorphisms induced by the inclusion maps $j' : A \to (A, B)$, $\bar{i} : (A, B) \to (X, B)$, $\bar{j} : (X, B) \to (X, A)$.

The proofs of formulated theorems are similar to the proofs of corresponding theorems of Eilenberg and Steenrod (see [17], Ch. IX, Theorem 3.4, Theorem 4.3, Theorem 4.4, Theorem 5.1, Theorem 6.1, Theorem 7.6) and hence they will be omitted.

2. On Čech (Co)homology Groups and Coefficients of Cyclicity of Remainders of Stone-Čech Compactifications

Now we are mainly interested in the following problem: how to characterize the Čech homology and cohomology groups, and coefficients of cyclicity of remainders of Stone-Čech compactifications of metrizable spaces.

Our main result about the connection between Čech (co)homology groups of remainders and Čech border (co)homology groups of spaces is the following theorem:

Theorem 2.1. *Let $(X, A) \in ob(\mathscr{M}_p^2)$ and let $(\beta X, \beta A)$ be the pair of Stone-Čech compactifications of X and A. Then*
$$\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G) = \check{H}_n^\infty(X, A; G)$$
and
$$\hat{H}_f^n(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}_\infty^n(X, A; G).$$

Proof. Let $\alpha = \{\alpha_v\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})}$ and $\alpha' = \{\alpha'_w\}_{w \in (W_{\alpha'}, W_{\alpha'}^{\beta A \setminus A})}$ be the closed covers of pairs $(\beta X \setminus X, \beta A \setminus A)$ and $\alpha \geq \alpha'$. By Lemma 4 of [33] there exist open swellings $\beta_1 = \{\beta_v^1\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})}$ and $\beta' = \{\beta'_w\}_{w \in (W_{\alpha'}, W_{\alpha'}^{\beta A \setminus A})}$ of α and α' in βX, respectively. Assume that $\alpha_v \subseteq \alpha'_{w_k}$, $k = 1, 2, \ldots, m_v$. Let
$$\beta_v = \beta_v^1 \cap \Big(\bigcap_{k=1}^{m_v} \beta'_{w_k} \Big), \qquad v \in V_\alpha.$$

Note that $\alpha_v \subset \beta_v \subset \beta_v^1$ for each $v \in V_\alpha$. It is clear that $\beta = \{\beta_v\}_{v \in (V_\alpha, V_\alpha^A)}$ is a swelling of $\alpha = \{\alpha_v\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})}$ and $\beta \geq \beta'$.

The swelling in βX of closed cover α of $(\beta X \setminus X, \beta A \setminus A)$ is denoted by $s(\alpha)$. Let S be the set of all swellings of such kind.

Now define an order $\geq'$ in S. By definition,
$$s(\alpha') \geq' s(\alpha) \Leftrightarrow s(\alpha') \geq s(\alpha) \ \wedge \ \alpha' \geq \alpha.$$

It is clear that S is directed by $\geq'$. Let $((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)})$ be the nerve of $s(\alpha) \in S$ and $p_{s(\alpha)s(\alpha')}$ be the projection simplicial map induced by the refinement $\alpha' \geq \alpha$. Consider an inverse system
$$\{H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha)*}^{s(\alpha')}, S\}$$
and a direct system
$$\{H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha)}^{s(\alpha')*}, S\}.$$

Let $\varphi : S \to \mathrm{cov}_{\mathrm{f}}^{\mathrm{cl}}(\mathrm{\beta X \setminus X, \beta A \setminus A})$ be the function in the set of closed finite covers of pair $(\beta X \setminus X, \beta A \setminus A)$ given by formula
$$\varphi(s(\alpha)) = \alpha, \qquad s(\alpha) \in S.$$

Note that φ is an increasing function and
$$\varphi(S) = \mathrm{cov}_{\mathrm{f}}^{\mathrm{cl}}(\mathrm{\beta X \setminus X, \beta A \setminus A}).$$

For each index $s(\alpha) \in S$, we have
$$H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H_n((\beta X \setminus X)_\alpha, (\beta A \setminus A)_\alpha; G)$$
and
$$H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H^n((\beta X \setminus X)_\alpha, (\beta A \setminus A)_\alpha; G).$$

It is known that for normal spaces the Čech (co)homology groups based on finite open covers and on finite closed covers are isomorphic. By Theorems 3.14 and 4.13 of [17, Ch. VIII] we have
$$\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G) \approx \varprojlim \{H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha)*}^{s(\alpha')}, S\} \tag{2.1}$$
and
$$\hat{H}_f^n(\beta X \setminus X, \beta A \setminus A; G) \approx \varinjlim \{H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p_{s(\alpha)}^{s(\alpha')*}, S\}. \tag{2.2}$$
For each swelling $s(\alpha) = \{s(\alpha)_v\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})} \in S$, the family
$$s(\alpha) \wedge X = \{s(\alpha)_v \cap X\}_{v \in (V_\alpha, V_\alpha^{\beta A \setminus A})}$$
is a border cover of (X, A).

Let $\psi : S \to \mathrm{cov}_\infty(X, A)$ be the function defined by formula
$$\psi(s(\alpha)) = s(\alpha) \wedge X, \qquad s(\alpha) \in S.$$

The function ψ increases and $\psi(S)$ is a cofinal subset of $\operatorname{cov}_\infty(X,A)$. Note that the correspondence

$$((\beta X\setminus X)_{s(\alpha)},(\beta A\setminus A)_{s(\alpha)})\to(X_{s(\alpha)\wedge X},A_{s(\alpha)\wedge X}): s(\alpha)_v\to s(\alpha)_v\cap X,\quad v\in V_\alpha$$

induces an isomorphism of pairs of simplicial complexes. Thus, for each $s(\alpha)\in S$, we have the isomorphisms

$$H_n((\beta X\setminus X)_{s(\alpha)},(\beta A\setminus A)_{s(\alpha)};G)=H_n(X_{s(\alpha)\wedge X},A_{s(\alpha)\wedge X};G)$$

and

$$H^n((\beta X\setminus X)_{s(\alpha)},(\beta A\setminus A)_{s(\alpha)};G)=H^n(X_{s(\alpha)\wedge X},A_{s(\alpha)\wedge X};G).$$

By Theorems 3.15 and 4.13 of [17, Ch.VIII], we have

$$\check{H}^\infty_n(X,A;G)=\varprojlim\{H_n((\beta X\backslash X)_{s(\alpha)},(\beta A\backslash A)_{s(\alpha)};G),p^{s(\alpha')}_{s(\alpha)*},S\}\tag{2.3}$$

and

$$\hat{H}^n_\infty(X,A;G)=\varinjlim\{H_n((\beta X\backslash X,\beta A\backslash A)_{s(\alpha)};G),p^{s(\alpha')*}_{s(\alpha)},S\}.\tag{2.4}$$

From (2.1), (2.2), (2.3) and (2.4) it follows that

$$\check{H}^\infty_n(X,A;G)=\check{H}^f_n(\beta X\setminus X,\beta A\setminus A;G)$$

and

$$\hat{H}^n_\infty(X,A;G)=\hat{H}^n_f(\beta X\setminus X,\beta A\setminus A;G).\qquad\square$$

The cohomological coefficient of cyclicity $\eta_G(X,A)$ of pair (X,A) was defined and investigated by S. Novak [29] and M. F. Bokstein [10].

Now give the following definition and result.

Definition 2.2. Let G be an abelian group and n nonnegative integer. A border cohomological coefficient of cyclicity of pair $(X,A)\in ob(\mathscr{M}^2_p)$ with respect to G denoted by $\eta^\infty_G(X,A)$ is n, if $\hat{H}^m_\infty(X,A;G)=0$ for all $m>n$ and $\hat{H}^n_\infty(X,A;G)\neq 0$.

Finally, $\eta^\infty_G(X,A)=+\infty$ if for every m there is $n\geq m$ with $\hat{H}^n_\infty(X,A;G)\neq 0$.

Theorem 2.3. *For each pair $(X,A)\in ob(\mathscr{M}^2_p)$,*

$$\eta^\infty_G(X,A)=\eta_G(\beta X\setminus X,\beta A\setminus A).$$

Proof. This is an immediate consequence of Theorem 2.1. Indeed, let $\eta_G(\beta X\setminus X,\beta A\setminus A)=n$. Then for each $m>n$, $\hat{H}^m_f(\beta X\setminus X,\beta A\setminus A;G)=0$ and $\hat{H}^n_f(\beta X\setminus X,\beta A\setminus A;G)\neq 0$. From the isomorphism

$$\hat{H}^k_f(\beta X\setminus X,\beta A\setminus A;G)=\hat{H}^k_f(X,A;G)$$

it follows that $\hat{H}^m_\infty(X,A;G)=0$ for each $m>n$, and $\hat{H}^n_\infty(X,A;G)\neq 0$. Thus, $\eta^\infty_G(X,A)=n=\eta_G(\beta X\setminus X,\beta A\setminus A)$. $\square$

3. On Cohomological Dimensions of Remainders of Stone-Čech Compactifications

The theory of cohomological dimension has become an important branch of dimension theory since A. Dranishnikov solved P. S. Alexandrov's problem [16] and he and other authors developed the theory of extension dimension.

Our next aim is to study some questions of theory of cohomological dimension. In particular, we in this section give a description of cohomological dimension of remainder of Stone-Čech compactification of metrizable space.

Following Y. Kodama (see the appendix of [28]) and T. Miyata [26] we give the following definition.

Definition 3.1. The border small cohomological dimension $d^f_\infty(X;G)$ of normal space X with respect to group G is defined to be the smallest integer n such that, whenever $m\geq n$ and A is closed in X, the homomorphism $i^*_{A,\infty}:\hat{H}^m_\infty(X;G)\to\hat{\mathrm{H}}^m_\infty(A;G)$ induced by the inclusion $i:A\to X$ is an epimorphism.

The border small cohomological dimension of X with coefficient group G is a function $d^f_\infty:\mathscr{N}\to\mathrm{N}\cup\{0,+\infty\}:X\to n$, where $d^f_\infty(X;G)=n$ and N is the set of all positive integers.

Theorem 3.2. *Let X be a metrizable space. Then the following equality*

$$d^f_\infty(X;G) = d_f(\beta X \setminus X;G)$$

holds, where $d_f(\beta X \setminus X;G)$ is the small cohomological dimension of $\beta X \setminus X$ (see [28], *p.* 199).

Proof. Let A be a closed subset of X. Assume that $d_f(\beta X \setminus X;G) = n$. Then for each $m \geq n$ the homomorphism $i^*_{\beta X\setminus X,\infty} : \hat{H}^m_f(\beta X \setminus X;G) \to \hat{H}^m_f(\beta A \setminus A;G)$ is an epimorphim. Consider the following commutative diagram

$$\begin{array}{ccc} \hat{H}^m_\infty(X;G) & \approx & \hat{H}^m_f(\beta X \setminus X;G) \\ {\scriptstyle i^*_{A,\infty}}\big\downarrow & & \big\downarrow{\scriptstyle i^*_{\beta A\setminus A}} \\ \hat{H}^m_\infty(A;G) & \approx & \hat{H}^m_f(\beta A \setminus A;G) \end{array} \tag{3.1}$$

It is clear that the homomorphim

$$i^*_{A,\infty} : \hat{H}^m_f(X;G) \to \hat{H}^m_f(A;G)$$

also is an epimorphim for each $m \geq n$. Thus,

$$d^f_\infty(X;G) \leq n = d_f(\beta X \setminus X;G). \tag{3.2}$$

Let $d^f_\infty(X;G) = n$. To see the reverse inequality, let B be a closed subset of $\beta X \setminus X$ and let $m \geq n$.

Consider an open in $\beta X \setminus X$ neighbourhood U of B. There exists an open neighbourhood V of B in $\beta X \setminus X$ such that $\bar{V}^{\beta X\setminus X} \subset U$. By Lemma 5 of [33] we can find an open set W in βX such that $W \cap (\beta X \setminus X) = V$ and $\bar{W}^{\beta X} \cap (\beta X \setminus X) \subseteq U$. Let $A = \bar{W}^{\beta X} \cap X$. It is clear that $\beta A = \bar{A}^{\beta X}$.

We have

$$\bar{W}^{\beta X} = \overline{W \cap X}^{\beta X} \subset \overline{\bar{W}^{\beta X} \cap X}^{\beta X} \subset \overline{\bar{W}^{\beta X}}^{\beta X} = \bar{W}^{\beta X}.$$

Consequently, $\beta A = \overline{\bar{W}^{\beta X} \cap X}^{\beta X} = \bar{W}^{\beta X}$. This shows that

$$B \subset \beta A \cap (\beta X \setminus X) \subset U.$$

Hence, we have

$$B \subset \beta A \setminus A \subset U.$$

Thus, for each closed set B of $\beta X \setminus X$ and its open neighbourhood U in $\beta X \setminus X$ there exists a closed subset A in X such that $B \subset \beta A \setminus A \subset U$.

Let $a \in H^n_f(B;G)$. There is a closed finite cover α of B such that an element $a_\alpha \in H^m(N(\alpha);G)$ represents the element a.

Using Lemma 4 of [33] we can find the swellings $\tilde{\alpha}$ and $\tilde{\tilde{\alpha}}$ of α in B and $\beta X \setminus X$, respectively, such that $\tilde{\tilde{\alpha}}_{|B} = \tilde{\alpha}$. Let U be the union of elements of $\tilde{\tilde{\alpha}}$. There is a closed set A of X with $B \subset \beta A \setminus A \subset U$. The nerves $N(\alpha)$, $N(\tilde{\alpha})$ and $N(\tilde{\tilde{\alpha}}_{|\beta A\setminus A})$ are isomorphic. We can assume that

$$H^n(N(\alpha);G) = H^n(N(\tilde{\alpha});G) = H^n(N(\tilde{\tilde{\alpha}}_{|\beta A\setminus A});G).$$

Hence, the element a_α also belongs to the group $H^n(N(\tilde{\tilde{\alpha}}_{|\beta A\setminus A});G)$. Consequently, it represents some element b of $\hat{H}^n(\beta A \setminus A;G)$.

The inclusion $i_A : A \to X$ induces an epimorphism $i^*_{A,\infty} : \hat{H}^m_\infty(X;G) \to \hat{H}^m(A;G)$. From diagram (3.1) it follows that the homomorphism $i^*_{\beta A\setminus A} : \hat{H}^m(\beta X \setminus X;G) \to \hat{H}^m(\beta A \setminus A;G)$ is an epimorphism. Consequently, there is an element $c \in \hat{H}^m(\beta X \setminus X;G)$ such that $i^*_{\beta A\setminus A}(c) = b$. The homomorphism $j^*_B : \hat{H}^m(\beta A \setminus A;G) \to \check{H}^m(B;G)$ induced by the inclusion $j_B : B \to \beta A \setminus A$ satisfies the condition $j^*_B(b) = a$. From equality $i_{\beta A\setminus A} \cdot j_B = i_B$ it follows that $i^*_B(c) = a$.

Thus the inclusion $i_B : B \to \beta X\setminus X$ also induces an epimorphism $i^*_B : \check{H}^m(\beta X\setminus X;G) \to \check{H}^m(B;G)$. Hence, we obtaine

$$d_f(\beta X \setminus X;G) \leq n = d^f_\infty(X;G). \tag{3.3}$$

From the inequalities (3.2) and (3.3) it follows that

$$d^f_\infty(X;G) = d_f(\beta X \setminus X;G). \qquad \square$$

Theorem 3.3. *Let A be a closed subspace of a normal space X. Then*

$$d^f_\infty(A;G) \le d^f_\infty(X;G).$$

Proof. Let B be an arbitrary closed subset of A and $j_B : B \to A$, $i_A : A \to X$ and $k_B : B \to X$ be the inclusion maps. Note that $k_B = i_A \cdot j_B$. The induced homomorphisms $k^*_{B,\infty} : \hat{H}^n_\infty(X;G) \to \hat{H}^n_\infty(B;G)$, $i^*_{A,\infty} : \hat{H}^n_\infty(X;G) \to \hat{H}^n_\infty(A;G)$ and $j^*_{B,\infty} : \hat{H}^n_\infty(A;G) \to \hat{H}^n_\infty(B;G)$ satisfy the equality $k^*_{B,\infty} = j^*_{B,\infty} \cdot i^*_{A,\infty}$.

Let $n = d^\infty_f(X;G)$. For each $m \ge n$, the homomorphisms $k^*_{B,\infty} : \hat{H}^m_\infty(X;G) \to \hat{H}^m_\infty(B;G)$ and $i^*_{A,\infty} : \hat{H}^m_\infty(X;G) \to \hat{H}^m_\infty(A;G)$ are epimorphisms. Hence, the homomorphism $j^*_{B,\infty} : \hat{H}^m_\infty(A;G) \to \hat{H}^m_\infty(B;G)$ is also an ephimorphism for each $m \ge n$. Thus, $d^\infty_f(A;G) \le n = d^\infty_f(X;G)$. □

Corollary 3.4. *For each closed subspace A of a metrizable space X,*

$$d^f_\infty(A;G) \le d_f(\beta X \setminus X;G).$$

Definition 3.5. The border large cohomological dimension $D^f_\infty(X;G)$ of normal space X with respect to group G is defined to be the largest integer n such that $\hat{H}^n_\infty(X,A;G) \ne 0$ for some closed set A of X.

The border large cohomological dimension of X with coefficient group G is a function $D^f_\infty : \mathscr{N} \to \mathrm{N} \cup \{0,+\infty\} : X \to n$, where $D^f_\infty(X;G) = n$ and N is the set of all positive integers.

Theorem 3.6. *For each metrizable space X, one has*

$$D^f_\infty(X;G) = D_f(\beta X \setminus X;G),$$

where $D_f(\beta X \setminus X;G)$ is the large cohomological dimension of $\beta X \setminus X$ (see [28], *p.* 199).

Proof. Let $D_f(\beta X \setminus X;G) = n$. Consider an arbitrary closed subspace A of X. The remainder $\beta A \setminus A$ is a closed subset of $\beta X \setminus X$. By the assumption, we have $\hat{H}^m(\beta X \setminus X, \beta A \setminus A;G) = 0$ for each $m > n$. Theorem 2.1 implies that $\hat{H}^m_\infty(X,A;G) = 0$ for each $m > n$ and $A \subset X$. Thus,

$$D^f_\infty(X;G) \le n = D_f(\beta X \setminus X;G). \tag{3.4}$$

Let $D^f_\infty(X;G) = n$. Assume that $D_f(\beta X \setminus X;G) = n_1 > n$. Then there is a closed set B in $\beta X \setminus X$ such that $\hat{H}^{n_1}(\beta X \setminus X, B;G) \ne 0$. Using Lemma 4 of [33] and the proof of Theorem 3.2 we can show that there is a closed set A of X such that $B \subset \beta A \setminus A$, and $\hat{H}^{n_1}(\beta X \setminus X, \beta A \setminus A;G) \ne 0$. By Theorem 2.1 $\hat{H}^{n_1}_\infty(X,A;G) \ne 0$. But it is not possible because $D^\infty_f(X;G) = n$. Therefore, $n_1 \le n$. Thus,

$$D_f(\beta X \setminus X;G) \le n = D^f_\infty(X;G). \tag{3.5}$$

The inequalities (3.4) and (3.5) imply

$$D^f_\infty(X;G) = D_f(\beta X \setminus X;G). \qquad \square$$

Theorem 3.7. *If A is a closed subset of normal space X, then*

$$D^f_\infty(A;G) \le D^f_\infty(X;G).$$

Proof. By Theorem 1.9, for each closed set B of A, there is the exact Čech border cohomological sequence

$$\cdots \longrightarrow \hat{H}^{m-1}_\infty(A;G) \xrightarrow{\bar{\delta}^m_\infty} \hat{H}^m_\infty(X,A;G) \xrightarrow{\bar{j}^*_\infty} \hat{H}^m_\infty(X,B;G) \xrightarrow{\bar{i}^*_\infty} \hat{H}^m_\infty(A,B;G) \longrightarrow \cdots$$

It is clear that, if $m > D^f_\infty(X;G)$, then $\hat{H}^m_\infty(X,A;G) = \hat{H}^m_\infty(X,B;G) = 0$. Consequently, $\hat{H}^m_\infty(A,B;G) = 0$. Thus, we have

$$D^f_\infty(A;G) \le D^f_\infty(X;G). \qquad \square$$

Corollary 3.8. *For each closed subspace A of metrizable space X, one has*

$$D^f_\infty(A;G) \le D_f(\beta X \setminus X;G).$$

Theorem 3.9. *If X is a normal space then*

$$d^f_\infty(X;G) \le D^f_\infty(X;G).$$

Proof. Let A be a closed subset of normal space X. Consider the exact Čech border cohomological sequence of pair (X, A)

$$\cdots \longrightarrow \hat{H}^{m-1}_\infty(A,B;G) \xrightarrow{\delta^m_\infty} \hat{H}^m_\infty(X,A;G) \xrightarrow{j^*_\infty} \hat{H}^m_\infty(X;G) \xrightarrow{i^*_\infty} \hat{H}^m_\infty(A;G) \longrightarrow \cdots$$

Let $m > D^f_\infty(X;G)$. Note that $j^*_\infty : \hat{H}^{m-1}_\infty(X;G) \to \hat{H}^{m-1}_\infty(A;G)$ is an epimorphism. Hence,

$$d^f_\infty(X;G) \le D^f_\infty(X;G).$$ □

Corollary 3.10. *For each metrizable space X, one has*

$$d_f(\beta X \setminus X;G) \le D^f_\infty(X;G)$$

and

$$d^f_\infty(X;G) \le D_f(\beta X \setminus X;G).$$

Remark 3.11. The results of this paper also hold for spaces satisfying the compact axiom of countability. Recall that a space X satisfies the compact axiom of countability if for each compact subset $B \subset X$ there exists a compact subset $B' \subset X$ such that $B \subset B'$ and B' has a countable or finite fundamental systems of neighbourhoods (see Definition 4 of [33], p.143). A space X is complete in the sense of Čech if and only if it is G_δ type set in some compact extension. Each locally metrizable spaces, complete in the seance of Čech spaces [15] and locally compact spaces satisfy the compact axiom of countability.

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Department of Mathematics Batumi Shota Rustaveli State University
E-mail address: vbaladze@gmail.com

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RINGS WHOSE ELEMENTS ARE LINEAR EXPRESSIONS OF THREE COMMUTING IDEMPOTENTS

P. DANCHEV

Abstract. We classify up to isomorphism those rings in which all elements are linear expressions over the ring of integers $\mathbb{Z}$ of at most three commuting idempotents. Our results substantially improve on recent publications by the author in Albanian J. Math. (2018), Gulf J. Math. (2018), Mat. Stud. (2018), Bull. Iran. Math. Soc. (2018) and Lobachev. J. Math. (2019) as well as on publications due to Hirano-Tominaga in Bull. Austral. Math. Soc. (1988), Ying et al. in Can. Math. Bull. (2016) and Tang et al. in Lin. & Multilin. Algebra (2019).

1. Introduction and Background

Throughout the text of the paper, all rings R are assumed to be associative, containing the identity element 1 which differs from the zero element 0 of R. The standard terminology and notations are mainly in close agreement with [8]. For instance, $U(R)$ denotes the group of units in R, Id(R) the set of idempotents in R, $Nil(R)$ the set of nilpotents in R and $J(R)$ the Jacobson radical of R. As usual, $\mathbb{Z}$ stands for the ring of integers, and $\mathbb{Z}_k \cong \mathbb{Z}/k\mathbb{Z}$ is its quotient modulo the principal ideal $(k) = k\mathbb{Z}$, where $k \in \mathbb{N}$ is the set of naturals.

About the specific notions, they will be explained below in detail.

The aim of the present work is to describe the isomorphic structure of the following class of rings.

Definition 1.1. We shall say that the ring R is from the class $\mathcal{R}_3$ if, for any $r \in R$, there exist commuting each to other $e_1, e_2, e_3 \in$ Id(R) such that $r = e_1 + e_2 - e_3$ or $r = e_1 - e_2 - e_3$.

It is worthwhile to mention that by substituting $r \to -r$ and an eventual re-numeration of the idempotents, the first equality will yield the second equality, and reversible.

Obvious examples of such rings are the rings $\mathbb{Z}_k$, where $k = 2, 3, 4, 5, 6$. Contrasting with that, the ring $\mathbb{Z}_7$ need not be so.

The most important principally known achievements concerning the subject are as follows: Classically, a ring is said to be *boolean* if each its element is an idempotent – such a ring is known to be a subdirect product of a family of copies of the two element field $\mathbb{F}_2$. A very successful attempt to generalize that concept was made in [7] to the rings whose elements are the sum of two commuting idempotents – in fact, these rings are known to be commutative being a subdirect product of a family of copies of the two and three element fields $\mathbb{F}_2$ and $\mathbb{F}_3$, respectively. In particular, if every element of a ring is an idempotent or minus an idempotent, then this ring is either boolean, or $\mathbb{F}_3$, or the direct product of two such rings.

Further expansions of these notions, in terms of linear expressions over $\mathbb{Z}$ of at most three commuting idempotents, are subsequently given below as follows:

- $\forall r \in R$, $r = e_1 + e_2$ or $r = e_1 - e_2$ for some two commuting $e_1, e_2 \in \mathrm{Id}(R)$ (see [10]).
- $\forall r \in R$, $r = e_1 + e_2$ or $r = -e_1 - e_2$ for some two commuting $e_1, e_2 \in \mathrm{Id}(R)$ (see [5]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [4] and [9]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [2]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = e_1 - e_2$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [4])
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1 - e_2$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [1]).
- $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = -e_1 - e_2 - e_3$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [3]).

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• $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = e_1 + e_2 - e_3$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [6]).
• $\forall r \in R$, $r = e_1 + e_2 + e_3$ or $r = e_1 - e_2 - e_3$ for some three commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ (see [6]).

Actually, the rings from the last two bullets are rings lying in the classes $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively.

In all of the aforementioned variations, the ring is of necessity commutative, which suggest us to state at the end of the article two conjectures which are of some interest and importance.

Our working tactic is somewhat to develop the techniques utilized in [1–6] as well as to build some new methods inspired by the specification of the ring structure. Especially, we shall careful study the rings from the class $\mathcal{R}_3$, stated above in Definitions 1.1, by characterizing them up to an isomorphism.

2. Main Results

We start here with the following useful technicality.

Proposition 2.1. *Any ring R from the class $\mathcal{R}_3$ decomposes as $R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are either zero rings or rings belonging to the class $\mathcal{R}_3$ such that $4 = 0$ in R_1, $3 = 0$ in R_2 and $5 = 0$ in R_3.*

Proof. Let us write $3 = e_1 + e_2 - e_3$. Observing that $e_1 - e_3 = e_1(1 - e_3) - e_3(1 - e_1)$ is a difference of two orthogonal commuting idempotents, we can assume with no harm in generality that $e_1e_3 = 0$. Moreover, since $e_3(1 - e_1)$ remains an idempotent, we may also assume that $e_2e_3 = 0$.

Thus, squaring the equality for 3, one infers that $6 = 2e_1e_2 + 2e_3$ which multiplying by e_3 gives that $4e_3 = 0$. Furthermore, a multiplication of the same equality by e_1e_2 ensures that $4e_1e_2 = 0$ and, finally, the multiplication of the same by 2 riches us that $12 = 0$.

Writing next $3 = e_1 - e_2 - e_3$, as above demonstrated, we can assume without loss of generality that $e_1e_2 = e_1e_3 = 0$. Multiplying the equality of 3 by e_1 leads to $2e_1 = 0$. On the other side, squaring the equality for 3 assures that $12 = 2e_2e_3$ and the multiplication of this with e_2e_3 forces that $10e_2e_3 = 0$. Therefore, $12.5 = 60 = 4.3.5 = 0$, as wanted.

Consequently, the Chinese Remainder Theorem now applies to conclude that $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are either zero or rings again from the class $\mathcal{R}_3$ with characteristics ≤ 4, 3 and 5, respectively. $\square$

The following assertion is pivotal, strengthening [1, Proposition 2.2].

Lemma 2.2. *Suppose that R is a ring of characteristic* 5. *Then the following four conditions are equivalent:*

(i) $x^3 = x$ *or* $x^4 = 1$, $\forall x \in R$.
(ii) $x^3 = -x$ *or* $x^4 = 1$, $\forall x \in R$.
(iii) $x^3 = x$ *or* $x^3 = -x$, $\forall x \in R$.
(iv) *R is isomorphic to the field $\mathbb{Z}_5$.*

Proof. (i) $\Rightarrow$ (iii). For an arbitrary $y \in R$ satisfying $y^4 = 1$ but $y^3 \neq y$, considering the element $y^2 - 1$, it must be that $(y^2 - 1)^4 = 1$ or $(y^2 - 1)^3 = y^2 - 1$. In the first case we receive $y^2 = -1$ and thus $y^3 = -y$, as required, while in the second one we arrive at $y^2 = 1$ and so $y^3 = y$ which is against our initial assumption.

(ii) $\Rightarrow$ (iii). The same trick as that in the previous implication will work, assuming now that $y^3 \neq -y$.

(iii) $\iff$ (iv). Let P be the subring of R generated by 1, and thus note that $P \cong \mathbb{Z}_5$. We claim that $P = R$, so we assume in a way of contradiction that there exists $b \in R \setminus P$. With no loss of generality, we shall also assume that $b^3 = b$ since $b^3 = -b$ obviously implies that $(2b)^3 = 2b$ as $5 = 0$ and $b \notin P \iff 2b \notin P$.

Let us now $(1 + b)^3 = -(1 + b)$. Hence $b = b^3$ along with $5 = 0$ enable us that $b^2 = 1$. This allows us to conclude that $(1 + 2b)^3 \neq \pm(1 + 2b)$, however. In fact, if $(1 + 2b)^3 = 1 + 2b$, then one deduces that $2b = 3 \in P$ which is manifestly untrue. If now $(1 + 2b)^3 = -1 - 2b$, then one infers that $2b = 2 \in P$ which is obviously false. That is why, only $(1 + b)^3 = 1 + b$ holds. This, in turn, guarantees that $b^2 = -b$. Moreover, $b^3 = b$ is equivalent to $(-b)^3 = -b$ as well as $b^3 = -b$ to $(-b)^3 = -(-b)$ and thus, by what we have proved so far applied to $-b \notin P$, it follows that $-b = b^2 = (-b)^2 = -(-b) = b$.

Consequently, $2b = 0 = 6b = b \in P$ because $5 = 0$, which is the wanted contradiction. We thus conclude that $P = R$, as expected.

Conversely, it is trivial that the elements of $\mathbb{Z}_5 = \{0, 1, 2, 3, 4 \mid 5 = 0\}$ are solutions of one of the equations $x^3 = x$ or $x^3 = -x$.

(iv) $\Rightarrow$ (i), (ii). It is self-evident that all elements of $\mathbb{Z}_5 = \{0, 1, 2, 3, 4 \mid 5 = 0\}$ satisfy one of the equations $x^3 = x$ or $x^4 = 1$ as well as one of $x^3 = -x$ or $x^4 = 1$. $\square$

We now come to the following.

Theorem 2.3. *A ring R lies in the class $\mathcal{R}_3$ if, and only if, it is commutative and $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings for which*

(1) *$R_1 = \{0\}$, or $R_1/J(R_1)$ is a boolean factor-ring with nil $J(R_1) = 2\,\mathrm{Id}(R_1)$ such that $4 = 0$;*
(2) *$R_2 = \{0\}$, or R_2 is a subdirect product of a family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$;*
(3) *$R_3 = \{0\}$, or $R_3 \cong \mathbb{Z}_5$.*

Proof. Necessity. Appealing to Proposition 2.1, there is a decomposition $R \cong R_1 \times R_2 \times R_3$, where the direct factors R_1, R_2 and R_3 still belong to the class $\mathcal{R}_3$. What we need to do is to describe explicitly these three rings.

Describing R_1: Here $4 = 0$. Since $2 \in J(R_1)$, we elementarily observe that the quotient-ring $R_1/J(R_1)$ is of characteristic 2 ring from the class $\mathcal{R}_3$. Thus it has to be a boolean ring. What it needs to show is the equality $J(R_1) = 2\,\mathrm{Id}(R_1)$. In fact, given $z \in J(R_1)$, we write $z = e_1+e_2-e_3$ with $e_1e_3 = e_2e_3 = 0$, or $z = e_1 - e_2 - e_3$ with $e_1e_2 = e_1e_3 = 0$, for some three commuting idempotents e_1, e_2, e_3 in R_1. In the first case, $ze_3 = -e_3$ still lies in $J(R_1)$, so that $e_3 = 0$. Hence $z = e_1 + e_2$ implying that $z(1 - e_2) = e_1(1 - e_2) \in J(R_1) \cap \mathrm{Id}(R_1) = \{0\}$ and thus that $e_1 = e_1e_2$. By a reason of symmetry, $e_2 = e_1e_2$ whence $e_1 = e_2$ giving up that $z = 2e_1 \in 2\,\mathrm{Id}(R_1)$, as needed.

In the second case, $ze_1 = e_1 \in J(R_1) \cap \mathrm{Id}(R_1) = \{0\}$ and hence $z = -e_2 - e_3 = -(e_2 + e_3)$. Similarly, as in the previous case, $z = -2e_2 = 2e_2 \in 2\,\mathrm{Id}(R_1)$ because $4 = 0$, as required.

Describing R_2: Here $3 = 0$. In fact, by the same token as in the preceding situation for R_1, we have that $J(R_2) = 2\,\mathrm{Id}(R_2)$ or $J(R_2) = -2\,\mathrm{Id}(R_2)$. If for any $j \in J(R_2)$ we write $j = 2i$ for some $i \in \mathrm{Id}(R_2)$, then $-j + 3i = i \in J(R_2) \cap \mathrm{Id}(R_2) = \{0\}$ whence $i = 0 = j$. Symmetrically, if $j = -2i$, then $j + 3i = i \in J(R_2) \cap \mathrm{Id}(R_2) = \{0\}$ and hence $i = 0 = j$, as required. Furthermore, since $3 = 0$, it easily follows that $x^3 = x$ for all $x \in R_2$ and thus [7] is applicable to get the wanted description of R_2.

Describing R_3: Here $5 = 0$. For any $x \in R_3$ we write that $x = e_1 + e_2 - e_3$ with $e_1e_3 = e_2e_3 = 0$, or $x = e_1 - e_2 - e_3$ with $e_1e_2 = e_1e_3 = 0$. In the first case, squaring the equality for x gives that $x^2-x = 2(e_1e_2+e_3)$ which allows us to deduce that $(x^2-x)^2 = 2(x^2-x)$ since $e_1e_2+e_3$ is obviously an idempotent as e_1e_2 and e_3 are orthogonal idempotents. We, therefore, have that $x^4-2x^3-x^2+2x = 0$. In the second case, again by squaring the equality for x, one derives that $x^2 + x = 2(e_1 + e_2e_3)$ which enables us that $(x^2 + x)^2 = 2(x^2 + x)$ because $e_1 + e_2e_3$ is obviously an idempotent as e_1 and e_2e_3 are orthogonal idempotents. We, consequently, have that $x^4 + 2x^3 - x^2 - 2x = 0$. One also observes that by the substitution $x \to -x$ the first equation will imply the second equation, and vice versa.

Furthermore, replacing $x \to 2x$ and $x \to 3x$ in the equation $x^4 - 2x^3 - x^2 + 2x = 0$, we derive that $x^4 - x^3 + x^2 - x = 0$ and that $x^4 + x^3 + x^2 + x = 0$, respectively. The same replacements in the equation $x^4 + 2x^3 - x^2 - 2x = 0$ lead respectively to $x^4 + x^3 + x^2 + x = 0$ and $x^4 - x^3 + x^2 - x = 0$, which are definitely the same equations in a rotating way, arising from the map $x \to -x$.

The next four main combinations must be considered:

Combination 1. $x^4 - x^3 + x^2 - x = 0$ with $x^4 + x^3 + x^2 + x = 0$ implies that $2x^3 = -2x$, which multiplying it by 3 implies that $x^3 = -x$ because $5 = 0$.

Combination 2. $x^4 - 2x^3 - x^2 + 2x = 0$ with $x^4 + x^3 + x^2 + x = 0$ implies that $3x^3 + 2x^2 - x = 0$.

Combination 3. $x^4 - 2x^3 - x^2 + 2x = 0$ with $x^4 - x^3 + x^2 - x = 0$ implies that $x^3 + 2x^2 - 3x = 0$.

Now, combining $3x^3 + 2x^2 - x = 0$ and $x^3 + 2x^2 - 3x = 0$, we get once again that $2x^3 = -2x$, i.e., $x^3 = -x$.

Similar arguments work for the other initial equation $x^4 + 2x^3 - x^2 - 2x = 0$ getting also that $x^3 = -x$ which as noticed above arisen from $x \to -x$.

Combination 4. $x^4 + 2x^3 - x^2 - 2x = 0$ with $x^4 - 2x^3 - x^2 + 2x = 0$ implies that $4x^3 = 4x$, that is, $x^3 = x$ since $5 = 0$.

After taking into account these four possibilities, one concludes that it must be $x^3 = x$ or $x^3 = -x$ after all. That is why, Lemma 2.2 (iii) finally tells us to obtain the wanted description of R_3 as being isomorphic to $\mathbb{Z}_5$.

Concerning the commutativity of the whole ring R, since R_2 and R_3 are obviously commutative, what remains to show is that this property holds for R_1. This, however, follows by the usage of [4, Theorem 2.2].

Sufficiency. A direct consultation with [7] informs us that every element of R_2 is a sum of two idempotents. Likewise, as in [1,4] or [5], each element in R_1 is a sum of three idempotents. Since R_3 has only five elements, we are, therefore, in a position to exploit the same manipulation as that in the corresponding results from [1,4] and [5] getting that the direct product $R_1 \times R_2 \times R_3$ belongs to the class $\mathcal{R}_3$, as expected. □

It will definitely be somewhat interesting to examine now the equalities $r = e_1 + e_2 - e_3$ and $r = e_1 - e_2 - e_3$ in an arbitrary ring R separately, comparing them with the equation $r = e_1 + e_2 + e_3$ in R which was independently explored in [4] and [9], respectively. Specifically, the latter rings were defined in [4] to be members from the class $\mathcal{K}$. Inspired by this, let we define the rings R for which $r = e_1 + e_2 - e_3$ to lie in the class $\mathcal{K}_1$, and the rings for which $r = e_1 - e_2 - e_3$ in the class $\mathcal{K}_2$.

Proposition 2.4. *Any ring R either from the class $\mathcal{K}_1$ or $\mathcal{K}_2$ decomposes as $R_1 \times R_2$, where R_1, R_2 are rings again from the same ring class such that $2 = 0$ in R_1 and $3 = 0$ in R_2.*

Proof. Firstly, writing $3 = e_1 + e_2 - e_3$, we obtain as in the first part of Proposition 2.1 that $12 = 4.3 = 0$, as asked for.

Secondly, writing $2 = e_1 - e_2 - e_3$, we may assume as in the second part of Proposition 2.1 that $e_1e_2 = e_1e_3 = 0$. Thus $2e_1 = e_1$ yields that $e_1 = 0$. Therefore, $2 = -e_2 - e_3$ implies by squaring that $6 = 2e_2e_3$. As a final step, $2e_2e_3 = -e_2e_3 - e_2e_3$, i.e., $4e_2e_3 = 0$ insuring that $12 = 4.3 = 0$, as pursued.

Furthermore, the Chinese Remainder Theorem is applicable to get the desired decomposition. □

So, we now arrive at the following.

Theorem 2.5. *A ring R is either from the class $\mathcal{K}_1$ or $\mathcal{K}_2$ if, and only if, it is commutative and $R \cong R_1 \times R_2$, where R_1, R_2 are rings for which*

(1) *$R_1 = \{0\}$, or $R_1/J(R_1)$ is a boolean quotient-ring with nil $J(R_1) = 2\,\mathrm{Id}(R_1)$ such that $4 = 0$;*

(2) *$R_2 = \{0\}$, or R_2 is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.*

Proof. Necessity. According to Proposition 2.4, there is a decomposition $R \cong R_1 \times R_2$, where the direct factors R_1 and R_2 still belong to either of classes $\mathcal{K}_1$ or $\mathcal{K}_2$. What we need to do is to describe in an explicit form these two rings.

Describing R_1: Here $4 = 0$. Since $2 \in J(R_1)$, we routinely see that the factor-ring $R_1/J(R_1)$ is of characteristic 2 ring from one of the classes $\mathcal{K}_1$ or $\mathcal{K}_2$. Thus it has to be a boolean ring. What suffices to prove is the equality $J(R_1) = 2\,\mathrm{Id}(R_1)$ which can be handled analogously to the corresponding part of Theorem 2.3.

Describing R_2: Here $3 = 0$. We claim that $J(R_2) = \{0\}$. In fact, as in the preceding case, we have that $J(R_2) = 2\,\mathrm{Id}(R_2)$ or $J(R_2) = -2\,\mathrm{Id}(R_2)$. If for any $j \in J(R_2)$ we write $j = 2i$ for some $i \in \mathrm{Id}(R_2)$, then $-j + 3i = i \in J(R_2) \cap \mathrm{Id}(R_2) = \{0\}$ whence $i = 0 = j$. Symmetrically, if $j = -2i$, then $j + 3i = i \in J(R_2) \cap \mathrm{Id}(R_2) = \{0\}$ and hence $i = 0 = j$, as required. Furthermore, since $3 = 0$, it easily follows that $x^3 = x$ for all $x \in R_2$ and thus [7] is working to get the wanted description of R_2.

The commutativity of the former ring R follows in the same way as in Theorem 2.3 above.

Sufficiency. It follows by adapting the same idea as in the "sufficiency part" of Theorem 2.3. □

Now, to close all possible variations of equalities which depend on idempotents, we shall say that the ring R belongs to the class $\mathcal{P}$, provided that for any r from R the equalities $r = e_1 + e_2 - e_3$ or $r = -e_1 - e_2$ are valid for some commuting idempotents $e_1, e_2, e_3 \in \mathrm{Id}(R)$. This is tantamount to $r = e_1 - e_2 - e_3$ or $r = e_1 + e_2$ via the substitution $r \to -r$ and re-numerating.

One sees that the direct product $\mathbb{Z}_4 \times \mathbb{Z}_5 \notin \mathcal{P}$ by considering the element $(1, 3)$, where $1 = 1+0-0 = 1+1-1$ whereas $3 = -1-1$. Contrastingly, for the element $(2, 3)$ we have $2 = 1 + 1 = -1 - 1$ and $3 = -1 - 1$. However, the ring $\mathbb{Z}_4 \times \mathbb{Z}_5 \in \mathcal{R}_3$ which shows that these two classes are different.

What is currently offer by us is the following slight enlargement of the preceding Theorem 2.5 and of results from [2, 5] and [10].

Theorem 2.6. *The ring R lies in the class $\mathcal{P}$ if, and only if, $R \cong R_1 \times R_2 \times R_3$, where*

(1) *$R_1 = \{0\}$, or $R_1/J(R_1)$ is a boolean ring such that $J(R_1) = 2\,\mathrm{Id}(R_1)$ with $4 = 0$;*
(2) *$R_2 = \{0\}$, or R_2 is a subdirect product of a family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$;*
(3) *$R_3 = \{0\}$ which must be fulfilled when $J(R_1) \neq \{0\}$, or $R_3 \cong \mathbb{Z}_5$.*

Proof. Necessity. We claim that $60 = 4.3.5 = 0$ in R, and thus the Chinese Remainder Theorem applies to infer the wanted decomposition of R into $R_1 \times R_2 \times R_3$ with $R_1, R_2, R_3 \in \mathcal{P}$ such that $4 = 0$ in R_1, $3 = 0$ in R_2 and $5 = 0$ in R_3.

In fact, write $3 = e_1 + e_2 - e_3$ with $e_1e_3 = e_2e_3 = 0$. Therefore, $3e_3 = -e_3$ yields that $4e_3 = 0$. Also, $3e_1e_2 = e_1e_2 + e_1e_2$ gives $e_1e_2 = 0$. On the other hand, squaring the equality for 3 forces that $6 = 2(e_1e_2 + e_3) = 2e_3$. Finally, $6.2 = 4.3 = 0$, as expected. Writing now $3 = -e_1 - e_2$, we obtain $3e_1e_2 = -e_1e_2 - e_1e_2$ amounts to $5e_1e_2 = 0$. The squaring of the equality for 3 ensures that $12 = 2e_1e_2$ whence $12.5 = 4.3.5 = 0$, as promised.

Furthermore, describing separately these three direct factors, one has that:

About R_1: Here $4 = 0$. Since $2 \in J(R_1)$, it is self-evident that the quotient $R_1/J(R_1)$ is a ring of characteristic 2 also belonging to the class $\mathcal{P}$, and thus it is necessarily a boolean ring. As for the equality concerning $J(R_1)$, given $z \in J(R_1)$, we may write that $z = e_1+e_2-e_3$ or that $z = -e_1-e_2$ for some three commuting idempotents $e_1, e_2, e_3 \in R_1$. In the first case, as above demonstrated, we may assume with no harm of generality that $e_1.e_3 = e_2.e_3 = 0$. Hence $-ze_3 = e_3 \in J(R_1) \cap \mathrm{Id}(R_1) = \{0\}$, that is, $e_3 = 0$. Thus the record $z = e_1+e_2$ riches us that $z(1-e_2) = e_1(1-e_2) \in J(R_1) \cap \mathrm{Id}(R_1) = \{0\}$, i.e., $e_1 = e_1e_2$. In a way of similarity $e_2 = e_1e_2$ and, finally, $e_1 = e_2$. Consequently, $z = 2e_1 \in 2\,\mathrm{Id}(R_1)$, as pursued. In the second case, $z = -(e_1 + e_2)$ and processing by the same token as in the former case, one concludes that $z \in -2\,\mathrm{Id}(R_1) = 2\,\mathrm{Id}(R_1)$ since $4 = 0$. This substantiates the desired equality after all.

About R_2: Here $3 = 0$. So, as $R_2 \in \mathcal{P}$, it is plainly checked that each element x in R_2 satisfies the equation $x^3 = x$. Furthermore, a simple consultation with [7] assures that R is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$, as stated.

About R_3: Here $5 = 0$. Writing $x = e_1+e_2-e_3$ or $x = -e_1-e_2$ for some three commuting idempotents $e_1, e_2, e_3 \in R_3$. In the first case, additionally assuming without loss of generality that $e_1.e_3 = e_2.e_3 = 0$, one deduces that $x^2 - x = 2(e_1e_2 + e_3)$. But since the expression in the brackets is obviously an idempotent too being the sum of two orthogonal idempotents, we derive that $(x^2 - x)^2 = 2(x^2 - x)$. This, in turn, yields that $x^4 - 2x^3 - x^2 + 2x = 0$. In the second case, one obtains $x^2 + x = 2e_1e_2$ enabling us that $(x^2 + x)^2 = 2(x^2 + x)$. This, in turn, implies that $x^4 + 2x^3 - x^2 - 2x = 0$. Actually, one easily sees that these two equations arise one from other via the substitution $x \to -x$. Since the equations are the same as in the corresponding part of Theorem 2.3, we may process analogically to finish the conclusion that R_3 is the simple five element field, as formulated.

Sufficiency. Identical arguments to these from the "sufficiency part" of Theorem 2.3 work to deduce the wanted assertion. □

As a concluding discussion, we state:

Remark 2.7. Comparing the results established above with these from [6], it seems that the relationships

$$\mathcal{R}_1 \equiv \mathcal{R}_2 \equiv \mathcal{R}_3,$$

showing the equivalences between the three ring classes $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$, hold. Likewise, these three classes surprisingly coincide with the class of rings from [3] for which each element is the sum or the minus sum of three commuting idempotents.

Besides, concerning the classes $\mathcal{K}$, $\mathcal{K}_1$ and $\mathcal{K}_2$, it seems also by comparison of the already established results with these from [4] that these three classes curiously do coincide.

We close our comments by observing that in the proof of [3, Proposition 2.3], the ring $P[b]$ with $b^4 = b$ is the quotient ring of the ring $P[t]/\langle t(t-1)(t-2)(t-4)\rangle$ which, in its turn, is the direct sum of four copies of the field P. It follows immediately that if $P[b] \neq P$, then the requirements of this proposition do not hold. The same idea can be successfully applied to Case 3 and especially to Case 4 in the proof of necessity of Theorem 2.4 from [3]. Nevertheless, the methodology illustrated in [3], although somewhat elusive, is rather more transparent.

On the other vein, in 'Sufficiency' of the proof of [5, Theorem 2.4] on line 2 the phrase *is also a ring* should be written and read as *in the presence of points* (1)*,* (2) *and* (3) *is also a ring*, which is, definitely, an involuntarily omission.

In ending, we pose the following two conjectures:

Conjecture 1. If every element of a ring is a sum of (a fixed number of) commuting idempotents, then this ring is commutative itself.

Conjecture 2. If each element of a ring is expressed as a linear combination over $\mathbb{Z}$ of (a fixed number of) commuting idempotents, then that ring is necessarily commutative.

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, "Acad. G. Bonchev" Str., bl. 8, 1113 Sofia, Bulgaria

E-mail address: danchev@math.bas.bg; pvdanchev@yahoo.com

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EXISTENCE RESULTS FOR IMPULSIVE STOCHASTIC NEUTRAL INTEGRODIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

M. DIOP[1*], A. GBAGUIDI AMOUSSOU[2], C. OGOUYANDJOU[2], AND M. MOHAMED[1]

Abstract. In this article, we investigate the existence of mild solutions for a class of impulsive neutral stochastic integro-differential equations with state-dependent delay. The results are obtained by using the Krasnoselskii-Schaefer type fixed point theorem combined with theories of resolvent operators. In the end as an application, an example has been presented to illustrate the results obtained.

1. Introduction

The investigation of stochastic differential equations has been picking up much importance and attention of researchers due to its wide applicability in science and engineering. Since arbitrary fluctuations are regular in the real world, scientific (mathematical) models for complex systems are frequently subject to instabilities, for example, indeterminate parameters, fluctuating powers, or random boundary conditions. Also, uncertainties may be created by the absence of knowledge of some chemical, physical or biological systems that are not well known, and in this manner are not suitably represented (or missed totally) in the scientific models. Despite the fact that these fluctuations and unrepresented systems may be extremely little or quick, their long-term effect on the system evolution may be delicate or even meaningful. This kind of delicate effects on the general evolution of dynamical systems has been seen in, for instance, stochastic resonance, stochastic bifurcation and noise-induced pattern development. In this way considering stochastic impacts is of central significance for mathematical modeling of complex systems under uncertainty. Thus, a large number of these systems can be modeled by stochastic differential equations, for example, price processes, exchange rates, and interest rates, among others in finance.

The existence and uniqueness of the mild solutions of stochastic differential equations have been studied by many authors. In [19], author has obtained sufficient conditions for the existence and uniqueness of solution of stochastic differential equations under uniform Lipschitz and the linear growth condition. In [17], author has shown that there exists the unique solution for neutral stochastic functional differential equation under uniform Lipschitz and the linear growth condition. The approximate controllability of nonlocal neutral stochastic fractional differential equations is studied by authors in [7]. In [1], authors have considered an impulsive stochastic semilinear neutral functional differential equations with infinite delays and discussed the existence, uniqueness and stability of mild solutions of considered stochastic differential equations with a Lipschitz condition and without a Lipschitz condition by utilizing the technique of successive approximations. In [35], authors have discussed the existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay. The asymptotic stability of fractional impulsive neutral stochastic partial integrodifferential equations with state-dependent delay is studied by the authors in [36]. The existence and uniqueness of square-mean almost automorphic solutions for some stochastic differential equations have been studied by authors in [8] in which the asymptotic stability of the unique square-mean almost automorphic solution in the square-mean sense has been discussed. In [11], authors have considered an impulsive neutral stochastic functional integro-differential equation with infinite delays in a separable real Hilbert space and established the existence results. In [32],the existence of the mild

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*Corresponding author.

solution nonlinear fractional stochastic differential equation has been studied by the authors by using fixed point theorems and α-resolvent family. For more study on stochastic differential equation, we refer to papers [5, 8, 11, 14, 20, 22, 23, 25, 29–32, 34–36].

Impulsive effects likewise exist in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics and telecommunications, etc. Recently, many interesting and important results on impulsive differential equations have been derived in [2, 33] and the references therein. More recently, in [16, 17], Sakthivel and Luo have discussed the asymptotic stability for mild solution of impulsive stochastic partial differential equations by employing the fixed point theorem; by establishing an impulsive-integral inequality, the exponential stability for mild solution of impulsive stochastic partial differential equations with delays was considered in [6]. Besides, there are some results about the existence and uniqueness for mild solution of impulsive stochastic partial functional differential equations, see [10, 15, 24] and references therein.

On the other hand, there has been intense interest in the study of impulsive neutral stochastic partial differential equations with memory (e.g. delay) and integrodifferential equations with resolvent operators. Since many control systems arising for realistic models depends heavily on histories (that is, effect of infinite delay on the state equations), there is real need to discuss the existence results for impulsive partial stochastic neutral integrodifferential equations with sate-dependent delay. Recently, the problem of the existence of solutions for partial impulsive functional differential equations with state-dependent delay has been investigated in many publications such as [18, 26] and the references therein.

As the motivation of above discussed works, we consider the following neutral stochastic impulsive integrodifferential functional equations with state-dependent

$$\begin{cases} d\Big[x(t) - G\Big(t, x_t, \int_0^t g(t,s,x_s)ds\Big)\Big] = A\Big[x(t) - G\Big(t, x_t, \int_0^t g(t,s,x_s)ds\Big)\Big]dt \\ \qquad + \Big(\int_0^t B(t-s)\Big[x(s) - G\Big(s, x_s, \int_0^s g(s,u,x_u)du\Big)\Big]ds\Big)dt + F\left(t, x_{\rho(t,x_t)}\right)dw(t), \\ \qquad t \in J, J = [0,b], \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1,2,\dots,m\dots \\ x_0(\cdot) = \varphi(\cdot) \in \mathbb{B}, \end{cases} \tag{1}$$

where the state $x(.)$ takes values in a separable real Hilbert space $\mathbb{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{H}$, for $t \geq 0$, $B(t)$ is a closed linear operator with domain $D(A) \subset D(B(t))$; $0 < t_1 < \cdots < t_m < b$, are prefixed points and the symbol $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Let $\mathbb{K}$ be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathbb{K}}$ and norm $\|\cdot\|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given $\mathbb{K}$-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w. We are also employing the same notation $\|\cdot\|$ for the norm $L(\mathbb{K};\mathbb{H})$, where $L(\mathbb{K};\mathbb{H})$ denotes the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. The history $x_t : (-\infty, 0] \to \mathbb{H}$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathbb{B}$ defined axiomatically; the initial data $\{\varphi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathbb{B}$-valued random variable independent of the Wiener process w with finite second moment, and F, G, g, ρ, I_k; $(k = 1,\dots,m)$, are given functions to be specified later.

To the best of the authors's knowledge, there is no results about the existence of mild solutions impulsive neutral partial stochastic functional integrodifferential equations with state-dependent, which is expressed in the form (1). The aim of our paper main is to establish some existence results for the system (1). Our main results concerning (1) rely essentially on techniques using strongly continuous family of operators $\{R(t), t \geq 0\}$, defined on the Hilbert space $\mathbb{H}$ and called their resolvent. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. There is a rich theory for analytic semigroups and we wish to develop theories for (1) which yield analytic resolvent and fractional Brownian motion. However, the resolvent operator does not satisfy semigroup properties (see, for instance [4, 14]) and our objective in the present paper is to apply the theory developed by Grimmer [5], because it is valid for generators of strongly continuous semigroup, not necessarily analytic. The main contribution of this manuscript is that it proposes a framework for studying the mild solution to stochastic integro-differential equation with state-dependent and impulsive conditions.

The structure of this paper is as follows. In Section 2, we recall some necessary preliminaries on stochastic integral and resolvent operator. In Section 3, we discuss the results on existence and uniqueness of mild solutions. Finally in Section 4, an example is presented which illustrates the main results for equation (1).

2. Preliminaries

2.1. **Wiener process.** Throughout this paper, let $\mathbb{H}$ and $\mathbb{K}$ be tow real separabl Hilbert spaces. We denote by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$, $\langle\cdot,\cdot\rangle_{\mathbb{K}}$ their inner products and by $\|\cdot\|_{\mathbb{H}}$, $\|\cdot\|_{\mathbb{K}}$ their vecteur norms, respectively. $\mathcal{L}(\mathbb{K},\mathbb{H})$ denote the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|$ and we abbreviate this notation to $\mathcal{L}(\mathbb{H})$ when $\mathbb{H}=\mathbb{K}$.

In the sequel, we always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises.

Moreover, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

Lets $\{w(t) : t \geq 0\}$ denote a $\mathbb{K}$-valued Wiener process difined on the probability space $(\Omega,\mathcal{F},$ $\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, with covariance operator Q; that is $\mathbb{E}\langle w(t),x\rangle_{\mathbb{K}}\langle w(t),y\rangle_{\mathbb{K}} = (t\wedge s)\langle Qx,y\rangle_{\mathbb{K}}$, for all $x,y\in\mathbb{K}$, where Q is a positive, self-adjoint, trace class operator on $\mathbb{K}$. In particular,we denote W a $\mathbb{K}$-valued Q-wener pocess with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. To define stochastic integrals with respect to the Q-Wiener process with w,we introduce the subspace $\mathbb{K}_0 = Q^{1/2}\mathbb{K}$ of $\mathbb{K}$ endowed with the inner product $\langle u,v\rangle_{\mathbb{K}_0} = \langle Q^{1/2}u, Q^{1/2}v\rangle_{\mathbb{K}}$ as a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}$ in $\mathbb{K}$, a bounded sequence of positive real numbers λ_i such that $Qe_i = \lambda_i e_i$, $i = 1,2,\ldots$, and a sequence $\{\beta_i(t)\}_{i\geq 1}$ of independent standard Brownian motions such that $w(t) = \sum\limits_{i=1}^{+\infty}\sqrt{\lambda_i}\beta_i(t)e_i$ for $t\geq 0$ and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \geq s \geq t\}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{K}_0,\mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\mathbb{K}_0$ to $\mathbb{H}$. It turns out to be a separable Hilbert space equipped whith the norm $\|v\|^2_{\mathcal{L}_2^0} = tr((vQ^{1/2})(vQ^{1/2})^*)$ for any $v\in\mathcal{L}_2^0$. Obviously, for any bounded operator $v\in\mathcal{L}_2^0$, this norm reduces to $\|v\|^2_{\mathcal{L}_2^0} = tr(vQv^*)$.

2.2. **Deterministic integrodifferential equations.** In the present section, we recall some definitions, notations and properties needed in the sequel.

In what follows, $\mathbb{H}$ will denote a Banach space, A and $B(t)$ are closed linear operators on $\mathbb{H}$. Y represents the Banach space $D(A)$, the domain of operator A, equipped with the graph norm

$$\|y\|_Y = \|Ay\| + \|y\|, \qquad y\in Y.$$

The notation $C([0,+\infty[;Y)$ stands for the space of all continuous functions from $[0,+\infty[$ into Y. We then consider the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds, \text{ for } t\geq 0, \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \tag{2}$$

Definition 1 ([5])**.** A resolvent operator of the Eq. (2) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{H})$ for $t \geq 0$, having the following properties:

(1) $R(0) = I$ and $\|R(t)\| \leq \eta e^{\delta t}$ for some constants η and δ.
(2) For each $x \in \mathbb{H}$, $R(t)x$ is strongly continuous for $t \geq 0$.
(3) $R(t) \in \mathcal{L}(\mathbb{H})$. For $x \in Y$, $R(\cdot)x \in C^1([0, +\infty; \mathbb{H}) \cap C([0, +\infty[; Y)$ and

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds = R(t)Ax + \int_0^t R(t-s)B(s)xds, \quad \text{for} \quad t \geq 0.$$

For additional detail on resolvent operators, we refer the reader to [5] and [21]. The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants formula for non-linear systems. For this reason, we need to know when the linear system (2) possesses a resolvent operator. Theorem (1) below provides a satisfactory answer to this problem.

In what follows we suppose the following assumptions:

(H1) A generates a strongly semigroup in Banach space $\mathbb{H}$
(H2) For all $t \geq 0$, $t \mapsto B(t)$ is continuous linear operator from $(Y, \|\cdot\|_Y)$ into $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$. Moreover, there exists an integrable function $c : [0, +\infty[\to \mathbb{R}^+$ such that for any $y \in Y$, $t \mapsto B(t)y$ belongs to $W^{1,1}([0, +\infty[; \mathbb{H})$ and

$$\left\| \frac{d}{dt} B(t)y \right\|_{\mathbb{H}} \leq c(t)\|y\|_Y, \quad \text{for} \quad y \in Y, \quad \text{and} \quad t \geq 0.$$

We recall that $W^{k,p}(\Omega) = \{\tilde{w} \in L^p(\Omega) : D^\alpha \tilde{w} \in L^p(\Omega), \quad \forall |\alpha| \leq k\}$, where $D^\alpha \tilde{w}$ is the weak α-th partial derivative of $\tilde{w}$.

The following theorem gives sufficient conditions of ensuring the existence of resolvent operator for Eq.(2)

Theorem 1 ([5])**.** *Assume that* ***(H1)*** *and* ***(H2)*** *hold. Then there exists a unique resolvent operator for* (2)*.*

In the sequel, we recall some results on the existence of solutions for the following integrodifferential equation:

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds + q(t), \text{ for } t \geq 0, \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \tag{3}$$

where $q : [0, +\infty[\to \mathbb{H}$ is a continuous function.

Definition 2 ([5])**.** A continuous function $v : [0, +\infty[\to \mathbb{H}$ is said to be a strict solution of equation (3) if

(1) $v \in C^1([0, +\infty[, \mathbb{H}) \cap C([0, +\infty[, Y)$,
(2) v satisfies equation Eq. (3) for $t \geq 0$.

Remark 2.1. From this definition we deduce that $v(t) \in D(A)$, the function $B(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0, t[$.

Theorem 2 ([5])**.** *Assume that* ***(H1)****,* ***(H2)*** *hold. If v is a strict solution of the Eq.* (3)*, then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds, \quad \textit{for} \quad t \geq 0. \tag{4}$$

Accordingly, we can establish the following definition.

Definition 3 ([5])**.** A function $v : [0, +\infty[\to \mathbb{H}$ is called a mild solution of equation (3) for $v_0 \in \mathbb{H}$, if v satisfies the variation of constants formula (4).

Theorem 3 ([5])**.** *Let $q \in C^1([0,+\infty[;\mathbb{H})$ and v be defined by* (4)*. If $v_0 \in D(A)$, then v is a strict solution of the equation* (3)*.*

Lemma 2.1 ([14])**.** *Assume that **(H1)** and **(H2)** hold. Then, there exists a constant $L = L(T)$ such that $\|R(t+\varepsilon) - R(t)R(\varepsilon)\| \leq L\varepsilon$ for $0 < \varepsilon \leq t \leq T$.*

Theorem 4 ([14])**.** *Assume that **(H1)** and **(H2)** hold. Let $T(t)$ be a compact for $t > 0$. Then the corresponding resolvent operator $R(t)$ of* (2) *is also compact for $t > 0$.*

In this work, we will employ an axiomatic definition of the phase space $\mathbb{B}$ introduced by Hale and Kato [9].

Definition 4. The phase space $\mathbb{B}(]-\infty,0],\mathbb{H})$ (denoted by $\mathbb{B}$ simply) is the space of continuous functions from $]-\infty,0]$ to $\mathbb{H}$ endowed with seminorm $\|\cdot\|_{\mathbb{B}}$, and $\mathbb{B}$ satisfies the following axioms:

- **(A1)** If $x :]-\infty,T] \to \mathbb{H}$ is continuous on $[t_0,T]$, $0 \leq t_0 \leq T$ and $x_{t_0} \in \mathbb{B}$, then, for every $t \in [t_0,T]$, the following conditions hold:
  - (1) $x_t \in \mathbb{B}$;
  - (2) $\|x_t\|_{\mathbb{B}} \leq \tilde{M}(t-t_0) \sup\limits_{0\leq s\leq t} \|x(s)\|_{\mathbb{H}} + N(t-t_0)\|x_0\|_{\mathbb{B}}$, where $\tilde{M}, N : [0,+\infty[\to [1,+\infty[$, $\tilde{M}$ is continuous and N_1 is locally bounded, $\tilde{M}$, N are independent of $x(\cdot)$.
  - (3) $\|x(t)\|_{\mathbb{H}} \leq \tilde{H}\|x_t\|_{\mathbb{B}}$, where $\tilde{H} > 0$ such that $\tilde{H}$ are independent of $x(\cdot)$.
- **(A2)** For the fonction $x(\cdot)$ in $(A1)$, the function $t \mapsto x_t$ is continuous from $[t_0,T]$ into $\mathbb{B}$.
- **(A3)** The space $\mathbb{B}$ is complete.

The $\mathbb{B}$-valued stochastic process $x_t : \Omega \to \mathbb{B}$, $t \in J$ is defined by setting $x_t = \{x(t+\theta)(w) : \theta \in]-\infty,0]\}$.

The collection of all strongly measurable, square integrable, $\mathbb{H}$-valued random variables, denoted by $L_2(\Omega,\mathbb{H})$ is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = (\mathbb{E}\|x(\cdot,w)\|^2)^{\frac{1}{2}}$, where the expectation, $\mathbb{E}$ is defined by $\mathbb{E}x = \int_\Omega x(w)dP$. Let $C(J,L_2(\Omega,\mathbb{H}))$ be the Banach space of all continuous maps from J into $L_2(\Omega,\mathbb{H})$ satisfying the condition $\sup_{0\leq t\leq T} E\|x(t)\|^2 < \infty$. Let $L_2^0(\Omega,\mathbb{H})$ denote the family of all $\mathcal{F}_0$-measurable, $\mathbb{H}$-valued random variables.

We say that a function $x : [\mu,\tau] \to \mathbb{H}$ is a normalized piecewise continous function on $[\mu,\tau]$, if x is piecewise continuous and left continuous on $(\mu,\tau]$. We denote by $\mathcal{PC}([\mu,\tau],\mathbb{H})$ the space formed by the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable processes from $[\mu,\tau]$ into $\mathbb{H}$. In particular, we introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, $\mathbb{H}$-valued stochastic processes $\{x(t) : t \in [0,T]\}$ such that x is continuous at $t \neq t_k$, $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for $k = 1,2,\ldots,m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm

$$\|x\|_{\mathcal{PC}} = \Big(\sup_{0\leq t\leq T} \mathbb{E}\|x(t)\|^2\Big)^{\frac{1}{2}}.$$

Then $(\mathcal{PC},\|.\|_{\mathcal{PC}})$ is a Banch space.

To simplify the notations, we put $t_0 = 0$, $t_{m+1} = b$ and for $x \in \mathcal{PC}$, we denote by $\hat{x}_k \in C([t_k,t_{k+1}];L_2(\Omega,\mathbb{H}))$, $k = 0,1,\ldots,m$, the function given by

$$\hat{x}_k(t) := \begin{cases} x(t) & \text{for } t \in (t_k,t_{k+1}], \\ x(t_k^+) & \text{for } t = t_k. \end{cases}$$

Moreover, for $B \subset \mathcal{PC}$ we denote by $\hat{B}_k$, $k = 0,1,\ldots,m$, the set $\hat{B}_k = \{\hat{x}_k : x \in B\}$. The notation $B_r(x,\mathbb{H})$ stands for the closed ball with center at x and radius $r > 0$ in $\mathbb{H}$.
Now, we give the definition of mild solution for (1).

Definition 5. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty,b] \to \mathbb{H}$ is said to be a mild solution of the system (1) if $x_0 = \varphi(t)$, $x_{\rho(s,x_s)} \in \mathbb{B}$ satisfying $x_0 \in L_2^0(\Omega,\mathbb{H})$, $x\mid_J \in \mathcal{PC}$, and $\Delta x(t_k) = I_k(x(t_k))$,

$k = 1, \ldots, m$, such that

$$x(t) = R(t)[\varphi(0) - G(0, \varphi, 0)] + G\Big(t, x_t, \int_0^t g(t, s, x_s) ds\Big)$$
$$+ \int_0^t R(t-s) F\left(s, x_{\rho(s,x_s)}\right) dw(s) + \sum_{0<t_k<t} R(t - t_k) I_k(x(t_k)).$$

Lemma 2.2. *A set $B \subset \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, the set $\hat{B}_k$ is relatively compact in $C([t_k, t_{k+1}]; L_2(\Omega, \mathbb{H}))$, for every $k = 0, 1, \ldots, m$.*

The next result is a consequence of the phase space axioms.

Lemma 2.3. *Let $x : (-\infty, b] \to \mathbb{H}$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(\cdot) \in L_2^0(\Omega, \mathbb{B})$ and $x\,|_J \in \mathcal{PC}(J, \mathbb{H})$, then*

$$\|x_s\|_{\mathbb{B}} \le M_b \mathbb{E}\|\varphi\|_{\mathbb{B}} + K_b \sup_{0 \le s \le b} \mathbb{E}\|x(s)\|,$$

where $K_b = \sup\{K(t) : 0 \le t \le b\}$, $M_b = \sup\{\tilde{M}(t) : 0 \le t \le b\}$.

Finally, we end this section by stating the following Krasnoselskii-Schaefer type fixed point theorem appeared in [3] which is our main tool.

Lemma 2.4 ([3])**.** *Let Φ_1, Φ_2 be two operators such that :*

(a) Φ_1 *is a contraction, and*
(b) Φ_2 *is completely continuous.*

Then either:

(i) *the operator equation $x = \Phi_1 x + \Phi_2 x$ has a solution, or*
(ii) *the set $\Lambda = \{x \in X : \lambda \Phi_1(\frac{x}{\lambda}) + \lambda \Phi_2 x = x\}$ is unbounded for $\lambda \in (0, 1)$.*

In the following section, we establish the existence theorem of the mild solution.

3. Main Results

Throughout this paper, for the existence and uniqueness of the mild solution to (1), we shall impose the following assumptions:

(H3) The resolvent operator $R(t)$, $t \ge 0$ is compact and there exists constant M such that $\|R(t)\|^2 \le M$, $t \in J$.

(H4) The function $t \mapsto \varphi_t$ is continous from $\mathcal{R}(\rho^-) = \{\rho(s, \psi) \le 0,\ (s, \psi) \in J \times \mathbb{B}\}$ into $\mathbb{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\| \le J^\varphi(t)\|\phi\|_{\mathbb{B}}$ for each $t \in \mathcal{R}(\rho^-)$.

(H5) The function $F : J \times \mathbb{B} \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$, for each $t \in J$, the function $F(t, .) : \mathbb{B} \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ is continuous and for each $\psi \in \mathbb{B}$, the function $F(., \psi) : J \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ is strongly measurable.

(H6) For each positive number $r > 0$, there exists a positive function $l(r)$ dependent on r such that

$$\sup_{\|\psi\|_{\mathbb{B}}^2 \le r} \mathbb{E}\|F(t, \psi)\|^2 \le l(r),$$

and there exists a constant d such that

$$0 \le \limsup_{\|\psi\|_{\mathbb{B}}^2 \to \infty} \left(\sup_{t \in J} \frac{\mathbb{E}\|F(t, \phi)\|^2}{\|\psi\|_{\mathbb{B}}^2} \right) \le d.$$

(H7) There exists a constant $L_1 > 0$ such that

$$\mathbb{E}\left\| \int_0^t [g(t, s, \psi) - g(t, s, \phi)] ds \right\|^2 \le L_1 \|\psi - \phi\|_{\mathbb{B}}^2$$

for $t,\ s \in J, \psi, \phi \in \mathbb{B}$.

(H8) The function $G : J\times\mathbb{B}\times\mathbb{H}\to\mathbb{H}$ is continous and satisfies the Lipschitz condition, that is, there exists a constant $L_2>0$ such that

$$\mathbb{E}\|G(t_1,\psi_1,\phi_1)-G(t_2,\psi_2,\phi_2)]\|^2\le L_2[|t_1-t_2|+\|\psi_1-\psi_2\|^2_{\mathbb{B}}+\mathbb{E}\|\phi_1-\phi_2\|^2]$$

for $0\le t_1,\ t_2\le b$, $\psi_i\in\mathbb{B}$, $\phi_i\in\mathbb{H}$, $i=1,2,\ldots,m$
with

$$L_0=L_2(1+L_1)K_b^2<1.$$

(H9) $I_k\in C(\mathbb{H},\mathbb{H})$, $k=1,2,\ldots,m$, are completely continuous and there exists constants c_k, $k=1,2,\ldots,m$, such that

$$0\le\limsup_{\|x\|^2\to\infty}\frac{\|I_k(x)\|^2}{\|x\|^2}\le c_k,\quad x\in\mathbb{H}.$$

Lemma 3.1 ([12]). *Let $x:(-\infty,b]\to\mathbb{H}$ such that $x_0=\varphi$. If **(H4)** is satisfied, then $\|x_s\|_{\mathbb{B}}\le(M_b+J_0^{\phi})\|\phi\|_{\mathbb{B}}+K_b\sup\{\|x(\theta)\|;\,\theta\in[0,\max\{0,s\}]\},\quad s\in\mathcal{R}(\rho^-)\cup J$, where $J_0^{\phi}=\sup_{t\in\mathcal{R}(\rho^-)}J^{\phi}(t)$.*

Remark 3.1 ([12, 13]). Let $\varphi\in\mathbb{B}$ and $t\le0$. The notation φ_t represents the function defined by $\varphi_t=\varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom $(A1)$ is such that $x_0=\varphi$, then $x_t=\varphi_t$. We observe that φ_t is well-defined for $t<0$ since the domain of φ is $(-\infty,0]$.

Theorem 5. *Let $\varphi\in L_2^0(\Omega,\mathbb{H})$. If the assumptions **(H1)** – **(H8)** hold and $\rho(t,\psi)\le t$, for every $(t,\ \psi)\in J\times\mathbb{B}$, then there exists a mild solution of equation Eq. (1) provided that*

$$8\Big[L_2(1+L_1)K_b^2+Mm\sum_{k=1}^{m}c_k\Big]\le1. \tag{5}$$

Proof. Consider the space $\mathbb{Y}=\{x\in\mathcal{PC}:x(0)=\varphi(0)=0\}$ endowed with the uniform convergence topology ($\|\cdot\|_\infty$) and define the mapping Φ on $\mathbb{Y}$ by

$$\Phi(x)(t)=\begin{cases}0, & t\in]-\infty,0],\\ R(t)[\varphi(0)-G(0,\varphi,0)]+G\left(t,\bar{x}_t,\int_0^t g(t,s,\bar{x}_s)ds\right)+\displaystyle\int_0^t R(t-s)F\left(s,x_{\rho(s,\bar{x}_s)}\right)dw(s) & \\ +\displaystyle\sum_{0<t_k<t}R(t-t_k)I_k(\bar{x}(t_k))\quad\text{for}\,t\in J, & \end{cases}$$

where $\bar{x}:(-\infty,0]\to\mathbb{H}$ is such that $\bar{x}_0=\varphi$ and $\bar{x}=x$ on J.

Then it is clear that to prove the existence of mild solutions of the problem (1) is equivalent to find a fixed point for the operator Φ. First we show that $\Phi(\mathcal{PC})\subset\mathcal{PC}$.

From **(H5)**, **(H7)** and **(H8)**, it follows that the function G,F and I_k, $k=1,2,\ldots,m$ are continuous,which enables us to conclude that Φ is well-defined operator from $\mathbb{Y}$ into $\mathbb{Y}$. We show that Φ has a fixed point, which in turn is a mild solution of the problem (1).

Let $\bar{\varphi}:(-\infty,T)\to\mathbb{H}$ be the extension of $(-\infty,0]$ such that $\bar{\varphi}(\theta)=\varphi(0)=0$ on J and $J_0^{\varphi}=\sup\{J^{\varphi}(s)\ :\ \ s\in\mathcal{R}(\rho^-)\}$. Now, we decompose Φ as $\Phi_1+\Phi_2$, where

$$(\Phi_1x)(t)=-R(t)G(0,\varphi,0)+G\bigg(t,\bar{x}_t,\int_0^t g(t,s,\bar{x}_s)ds\bigg),\quad t\in J$$

$$(\Phi_2x)(t)=R(t)\varphi(0)+\int_0^t R(t-s)F\left(s,\bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)+\sum_{0<t_k<t}R(t-t_k)I_k(\bar{x}(t_k)),\quad t\in J.$$

The proof is divided into the following five steps.
Step 1. Φ_1 is a contraction on $\mathbb{Y}$.

Let $t \in J$ and $y^1, y^2 \in \mathbb{Y}$. Then, by using (**H7**) and (**H8**), we have

$$\begin{aligned}
\mathbb{E}\|(\Phi_1\overline{y^1})(t) - (\Phi_1\overline{y^2})(t)\|^2 &\le \mathbb{E}\left\| G\left(t, \overline{y^1}_t, \int_0^t g(t,\tau,\overline{y^1}_\tau)d\tau\right) - G\left(t, \overline{y^2}_t, \int_0^t g(t,\tau,\overline{y^2}_\tau)d\tau\right)\right\|^2 \\
&\le L_2\left(\|\overline{y^1}_t - \overline{y^2}_t\|^2_{\mathbb{B}} + L_1\|\overline{y^1}_t - \overline{y^2}_t\|^2_{\mathbb{B}}\right) \\
&\le L_2(1+L_1)\|\overline{y^1}_t - \overline{y^2}_t\|^2_{\mathbb{B}} \\
&\le L_2(1+L_1)K_b^2 \sup_{s\in J} \mathbb{E}\|y^1(s) - y^2(s)\|^2_{\mathbb{B}}
\end{aligned}$$

by using $\overline{y} = y$ on J.
Taking supremum over t,

$$\|\Phi_1 y^1 - \Phi_1 y^2\|^2_{PC} \le L_0\ \|y^1 - y^2\|^2_{PC},$$

where $L_0 = L_2(1+L_1)K_b^2 < 1$. Thus Φ_1 is a contraction on $\mathbb{Y}$.

Step 2. Φ_2 maps bounded sets into bounded sets in $\mathbb{Y}$.
For each $r > 0$, let

$$B_r(0,\mathbb{Y}) := \{x \in \mathbb{Y} : \mathbb{E}\|x\|^2 \le r\}.$$

Then, for each $r, B_r(0,\mathbb{Y})$ is a bounded closed convex subset in $\mathbb{Y}$. Indeed, it is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $x \in B_r(0,\mathbb{Y})$ one has $\mathbb{E}\|\Phi_2 x\|^2 \le \mathcal{L}$. Now, for $t \in J$ we have

$$(\Phi_2 x)(t) = R(t)\varphi(0) + \int_0^t R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s) + \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k)), \quad t \in J. \tag{6}$$

In view of **(H6)** and **(H9)**, there exist positive constants ϵ, ϵ_k $(k = 1,\dots,m)$, γ and $\bar{\gamma}$ such that, for all $\|\psi\|^2_{\mathcal{B}} > \gamma$, $\|\phi\|^2 > \bar{\gamma}$,

$$\begin{aligned}
\|F(t,\psi)\|^2 &\le (d+\epsilon)\|\psi\|^2_{\mathbb{B}}, \\
\|I_k(\phi)\|^2 &\le (c_k+\epsilon_k)\|\phi\|^2,
\end{aligned}$$

and

$$8[L_2(1+L_1)K_b^2 + Mm\sum_{k=1}^m(\epsilon_k + c_k)] \le 1. \tag{7}$$

Let

$$\begin{aligned}
F_1 &= \{\psi : \|\psi\|^2_{\mathbb{B}} \le \gamma\}, \quad F_2 = \{\psi : \|\psi\|^2_{\mathbb{B}} > \gamma\}, \\
G_1 &= \{\phi : \|\phi\|^2 \le \bar{\gamma}\}, \quad G_2 = \{\phi : \|\phi\|^2 > \bar{\gamma}\}, \\
C_1 &= \max\{\|I_k(\phi)\|^2, \phi \in G_1\}.
\end{aligned}$$

Thus

$$\|F(t,\psi)\|^2 \le l(\gamma) + (d+\epsilon)\|\psi\|^2_{\mathbb{B}}, \tag{8}$$

$$\|I_k(\phi)\|^2 \le C_1 + (c_k+\epsilon_k)\|\phi\|^2. \tag{9}$$

If $x \in B_r(0,\mathbb{Y})$, from Lemma 2.3 and 3.1, it follows that

$$\|\bar{x}_{\rho(s,\bar{x}_s)}\|^2_{\mathcal{B}} \le 2\big[(M_b + \bar{J}_0^{\varphi})\|\varphi\|_{\mathcal{B}}\big]^2 + 2K_b^2 r := r^*.$$

By (8), (9), from (6) we have for $t \in J$

$$\mathbb{E}\|(\Phi_2 x)(t)\|^2 \le 3\mathbb{E}\|R(t)\varphi(0)\|^2 + 3\mathbb{E}\left\|\int_0^t R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)\right\|^2$$

$$+ 3\mathbb{E}\bigg\|\sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k))\bigg\| \leq 3M\mathbb{E}\|\varphi(0)\|^2 + 3Tr(Q)M$$
$$\times \int_0^t \big[l(\gamma) + (d+\epsilon)\|\bar{x}_{\rho(s,\bar{x}_s)}\|^2_{\mathcal{B}}\big]ds$$
$$+ 3Mm\sum_{k=1}^{m}\big[C_1 + (c_k+\epsilon_k)\mathbb{E}\|\bar{x}(t_k)\|^2\big]$$
$$\leq 3M\tilde{H}^2\|\varphi\|^2_{\mathcal{B}} + 3Tr(Q)MT[l(\gamma) + (d+\epsilon)r^*]$$
$$+ 3Mm\sum_{k=1}^{m}\big[C_1 + (c_k+\epsilon_k)r\big] := \mathcal{L}.$$

Then for each $x \in B_r(0,\mathbb{Y})$, we have $\mathbb{E}\|\Phi_2 x\|^2 \leq \mathcal{L}$.
Step 3. We show that the operator Φ_2 is completely continuous.
For this purpose, we decompose Φ_2 as $\Psi_1 + \Psi_2$, Ψ_1, Ψ_2 are the operators on $B_r(0,\mathbb{Y})$ defined respectively by

$$(\Psi_1 x)(t) = R(t)\varphi(0) + \int_0^t R(t-s)F[s,\bar{x}_{\rho(s,\bar{x}_s)}]dw(s), \quad t \in J,$$
$$(\Psi_2 x)(t) = \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k)) \quad t \in J.$$

We first show that Ψ_1 is completely continuous.

(i) $\Psi_1(B_r(0,\mathbb{Y}))$ is equicontinuous.

Let $0 < \tau_1 < \tau_2 \leq T$ and $\varepsilon > 0$ be small. For each $x \in B_r(0,\mathbb{Y})$, we have

$$\mathbb{E}\bigg\|(\Psi_1 x)(\tau_2) - (\Psi_1 x)(\tau_1)\bigg\|^2 \leq 4\mathbb{E}\bigg\|[R(\tau_2) - R(\tau_1)]\varphi(0)\bigg\|^2$$
$$+ 4\mathbb{E}\bigg\|\int_0^{\tau_1-\epsilon} \big[R(\tau_2-s) - R(\tau_1-s)\big]F\left(s,\bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)\bigg\|^2$$
$$+ 4\mathbb{E}\bigg\|\int_{\tau_1-\epsilon}^{\tau_1} \big[R(\tau_2-s) - R(\tau_1-s)\big]F\left(s,\bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)\bigg\|^2$$
$$+ 4\mathbb{E}\bigg\|\int_{\tau_1}^{\tau_2} R(\tau_2-s)F\left(s,\bar{x}_{\rho(s,\bar{x}_s)}\right)dw(s)\bigg\|^2$$
$$\leq 4\mathbb{E}\bigg\|[R(\tau_2) - R(\tau_1)]\varphi(0)\bigg\|^2$$
$$+ 4Tr(Q)\big[l(\gamma) + (d+\epsilon)r^*\big]\int_0^{\tau_1-\epsilon} \|R(\tau_2-s) - R(\tau_1-s)\|^2_{\mathcal{L}(\mathbb{H})}ds$$
$$+ 4Tr(Q)\big[l(\gamma) + (d+\epsilon)r^*\big]\int_{\tau_1-\epsilon}^{\tau_1} \|R(\tau_2-s) - R(\tau_1-s)\|^2_{\mathcal{L}(\mathbb{H})}ds$$
$$+ 4Tr(Q)\big[l(\gamma) + (d+\epsilon)r^*\big]\int_{\tau_1}^{\tau_2} \|R(\tau_2-s)\|^2_{\mathcal{L}(\mathbb{H})}ds.$$

From the above inequalities, we see that the right-hand side of $\mathbb{E}\|(\Psi_1 x)(\tau_2)-(\Psi_1 x)(\tau_1)\|^2$ tends to zero independent of $x \in B_r(0,\mathbb{Y})$ as $\tau_2-\tau_1 \to 0$ with ε sufficiently small, since the compactness of $R(t)$ forb $t>0$ implies the continuity in the uniform operator topology. Thus the set $\{\Psi_1 x : x \in B_r(0,\mathbb{Y})\}$ is equicontinuous. The equicontinuities for the other cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2 \leq T$ are very simple.

(ii) The set $\Psi_1(B_r(0,\mathbb{Y}))(t)$ is precompact in $\mathbb{H}$ for each $t \in J$.
Let $0 < t \leq s \leq b$ fixed and let ε be a real number satisfying $\varepsilon \in (0,t)$. For $x \in B_r(0,\mathbb{Y})$, we define the operators

$$(\Psi_1^{*\varepsilon} x)(t) = R(t)\varphi(0) + R(\varepsilon)\int_0^{t-\varepsilon} R(t-s-\varepsilon)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s),$$

$$(\tilde{\Psi}_1^{\varepsilon} x)(t) = R(t)\varphi(0) + \int_0^{t-\varepsilon} R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s).$$

By the compactness of the operators $R(t)$, the set $V_\varepsilon^*(t) = \{(\Psi_1^{*\varepsilon} x)(t); x \in B_r(0,\mathbb{Y})\}$ is relatively compact in $\mathbb{H}$, for every ε, $\varepsilon \in (0,t)$.

Moreover, also by Lemma 2.1 and assumption **(H6)** we have

$$\begin{aligned}
\mathbb{E}\|(\Psi_1^{*\varepsilon} x)(t) - (\tilde{\Psi}_1^{\varepsilon} x)(t)\|^2 &\leq \mathbb{E}\Bigg\| \int_0^{t-\varepsilon} R(\varepsilon)R(t-s-\varepsilon)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s) \\
&\quad - \int_0^{t-\varepsilon} R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s)\Bigg\|^2 \\
&\leq \mathbb{E}\Big\| \int_0^{t-\varepsilon} \big[R(\varepsilon)R(t-s-\varepsilon) - R(t-s)\big]F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s)\Big\|^2 \\
&\leq Tr(Q)M\big[l(\gamma) + (d+\epsilon)r^*\big]\int_0^{t-\varepsilon} \|R(\varepsilon)R(t-s-\varepsilon) - R(t-s)\|^2 ds.
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $\varepsilon \to 0$. So the set $\tilde{V}_\varepsilon(t) = \{(\tilde{\Psi}_1^{\varepsilon} x)(t);$ $x \in B_r(0,\mathbb{Y})\}$ is precompact in $\mathbb{H}$ by using the total Boundedness.

Applying the idea again, we obtain

$$\begin{aligned}
\mathbb{E}\|(\Psi_1 x)(t) - (\tilde{\Psi}_1^{\varepsilon} x)(t)\|^2 &\leq \mathbb{E}\Bigg\| \int_{t-\varepsilon}^{t} R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s)\Bigg\|^2 \\
&\leq Tr(Q)M\int_{t-\varepsilon}^{t} \big[l(\gamma) + (d+\epsilon)r^*\big] ds \\
&\leq Tr(Q)M\big[l(\gamma) + (d+\epsilon)r^*\big]\varepsilon.
\end{aligned}$$

The right hand side of the above inequality tends to zero as $\varepsilon \to 0$. Since there are precompact sets arbitrarily close to the set $U(t) = \{(\Psi_1 x)(t) : x \in B_r(0,\mathbb{Y})\}$. Hence the set U(t) is precompact in $\mathbb{H}$. By Arzelá-Ascoli theorem, we conclude that Ψ_1 maps $B_r(0,\mathbb{Y})$ into a precompact set in $\mathbb{H}$.

Next, it remains to verify that $\Psi_2(B_r(0,\mathbb{Y}))$ is also completely continuous.

We begin by showing $\Psi_2(B_r(0,\mathbb{Y}))$ is equicontinuous. For any $\varepsilon > 0$ and $0 < t < b$. Since the functions $I_k, k = 1,2,\dots,m$, are completely continuous in $\mathbb{H}$, we can choose $\xi > 0$ such that

$$\mathbb{E}\|[R(t+h) - R(t)]I_k(x)\|^2 < \frac{\varepsilon}{Mm}; \quad \mathbb{E}\|x\|^2 \leq r$$

when $|h| < \xi$. For each $x \in B_r(0, \mathbb{Y}), t \in (0, T]$ be fixed, $t \in [t_i, t_{i+1}]$, and $t + \xi \in [t_i, t_{i+1}]$, such that

$$\big[(\widehat{\Psi_2 x})\big]_i(t) = \sum_{0<t_k<t} R(t - t_k) I_k(\bar{x}(t_k)),$$

then we have

$$\begin{aligned}\mathbb{E}\|\big[(\widehat{\Psi_2 x})\big]_i(t+h) - \big[(\widehat{\Psi_2 x})\big]_i(t)\|^2 &\leq \mathbb{E}\|\sum_{0<t_k<t} \big[R(t+h-t_k) - R(t-t_k)\big] I_k(\bar{x}(t_k))\|^2 \\ &\leq m \sum_{k=1}^{m} \mathbb{E}\|\big[R(t+h-t_k) - R(t-t_k)\big] I_k(\bar{x}(t_k))\|^2.\end{aligned}$$

As $h \to 0$ and ε sufficiently small, the right-hand side of the above inequality tends to zero independently of x, so $[\Psi_2(\widehat{B_r(0, \mathbb{Y})}]_i$ $i = 1, 2, \ldots, m$, are equicontinuous.

Now we prove that $[\Psi_2(\widehat{B_r(0, \mathbb{Y})})]_i(t)$ $i = 1, 2, \ldots, m$, is precompact for every $t \in J$.

From the following relations

$$\big[(\widehat{\Psi_2 x})\big]_i(t) = \sum_{0<t_k<t} R(t - t_k) I_k(\bar{x}(t_k)) \in \sum_{k=1}^{m} R(t - t_k) I_k(B_r(0, \mathbb{H})).$$

We conclude that $[\Psi_2(\widehat{B_r(0, \mathbb{Y})})]_i(t)$ $i = 1, 2, \ldots, m$, is precompact for every $t \in [t_i; t_{i+1}]$. By Lemma 2.2, we infer that $\Psi_2(B_r(0, \mathbb{Y}))$ is precompact. As an application of the Arzelá-Ascoli theorem, Ψ_2 is completely continuous.

Step 4. $\Phi_2 : \mathbb{Y} \to \mathbb{Y}$ is continuous.

Let $\{x^n\} \subseteq B_r(0, \mathbb{Y})$ with $x^n \to x (n \to \infty)$ in $\mathbb{Y}$. From Axiom (A1), $(A2)$ and $(A3)$, it is easy to see that $(\overline{x^n}_s) \to (\bar{x}_s)$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. By assumption (H4)–(H5), we have

$$F\left(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}\right) \to F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) \text{ as } \quad n \to \infty$$

for each $s \in [0, t]$, and since

$$\|F\left(s, (\overline{x^n})_{\rho(s,(\overline{x^n})_s)}\right) - F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)\| \leq 2[l(\gamma) + (d + \epsilon) r^*].$$

Then by the continuity of I_k $(k = 1, 2, \ldots, m)$ and the dominated convergence theorem we have

$$\begin{aligned}\|\Phi_2 x^n - \Phi_2 x\|^2_{\mathcal{PC}} \leq & \sup_{t\in[0,T]} \Bigg\| \int_0^t R(t-s)\left[F\left(s, \overline{x^n}_{\rho(s,(\overline{x^n})_s)}\right) - F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)\right] ds \\ & + \sum_{0<t_k<t} R(t-t_k)\big[I_k(\overline{x^n}(t_k)) - I_k(\bar{x}(t_k))\big] \Bigg\|^2 \\ & \leq 2Tr(Q)M \int_0^t \mathbb{E}\|F\left(s, (\overline{x^n})_{\rho(s,(\overline{x^n})_s)}\right) - F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right)\|^2 ds \\ & + 2Mm \sum_{0<t_k<t} \|I_k(\overline{x^n}(t_k)) - I_k(\bar{x}(t_k))\|^2 \to 0 \quad as\, n \to \infty.\end{aligned}$$

Therefore, Φ_2 is continuous.

Step 5. We shall show the set $\Lambda = \{x \in \mathbb{Y} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2(x) = x, \text{ for some } \lambda \in (0, 1)\}$ is bounded on J.

To do this, we consider the following nonlinear operator equation

$$x(t) = \lambda\Phi x(t),\, 0 < \lambda < 1, \tag{10}$$

where Φ is already defined. Next we gives a priori estimate for the solution of the above equation. Indeed, let $x \in \mathbb{Y}$ be a possible solution of $x = \lambda\Phi(x)$ for some $0 < \lambda < 1$. This implies by (10) that for each $t \in J$ we have

$$x(t) = \lambda R(t)\big[\varphi(0) - G(0,\varphi,0)\big] + \lambda G\Big(t, \bar{x}_t, \int_0^t g(t,s,\bar{x}_s)ds\Big)$$
$$+ \lambda \int_0^t R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) ds + \lambda \sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k)), \quad t \in J. \tag{11}$$

By **(H6)**, (8), (9), from (11) we have for $t \in J$

$$\mathbb{E}\|x(t)\|^2 \le 4\mathbb{E}\|R(t)\big[\varphi(0) - G(0,\varphi,0)\big]\|^2 + 4\mathbb{E}\Big\|G\Big(t, \bar{x}_t, \int_0^t g(t,s,\bar{x}_s)ds\Big)\Big\|^2$$
$$+ 4\mathbb{E}\Big\|\int_0^t R(t-s)F\left(s, \bar{x}_{\rho(s,\bar{x}_s)}\right) dw(s)\Big\|^2 + 4\mathbb{E}\Big\|\sum_{0<t_k<t} R(t-t_k)I_k(\bar{x}(t_k))\Big\|^2$$
$$\le 4M\big[\tilde{H}^2\|\varphi\|^2_{\mathbb{B}} + (L_2(M_T + J_0^{\varphi})^2\|\varphi\|^2_{\mathbb{B}} + l_2)\big] + 4M\big[L_2(\|\bar{x}_t\|^2_{\mathbb{B}} + L_1\|\bar{x}_t\|^2_{\mathbb{B}} + l_1) + l_2\big]$$
$$+ 4Tr(Q)M\int_0^t \big[l(\lambda) + (d+\epsilon)\|\bar{x}_{\rho(s,\bar{x}_s)}\|^2_{\mathbb{B}}\big]ds + Mm\sum_{k=1}^m \big[C_1 + (c_k + \epsilon_k)\mathbb{E}\|\bar{x}(t_k)\|^2\big],$$

where $l_1 = \sup\limits_{t_0 \le t \le T} \|G(t,0,0)\|^2$ and $l_2 = \sup\limits_{t_0 \le s \le t \le b} \|g(t,s,0)\|^2$. By Lemmas 2.3 and 2.4, it follows that $\rho(s,\bar{x}_s) \le s$, $s \in [0,t]$, $t \in J$ and

$$\|\bar{x}_{\rho(s,\bar{x}_s)}\|^2_{\mathbb{B}} \le 2\big[(M_b + J_0^{\varphi})\|\varphi\|_{\mathbb{B}}\big]^2 + 2K_b^2 \sup_{0\le s\le T}\mathbb{E}\|x(s)\|^2.$$

For each $t \in J$, we have

$$\mathbb{E}\|x(t)\|^2 \le M_* + 8L_2(1+L_1)K_b^2\sup_{t\in J}\mathbb{E}\|x(t)\|^2$$
$$+ 8Tr(Q)M(d+\epsilon)K_b^2\int_0^t \sup_{\tau\in[0,s]}\mathbb{E}\|x(\tau)\|^2 ds + 8Mm\sum_{k=1}^m (c_k+\epsilon_k)\sup_{t\in J}\mathbb{E}\|x(t)\|^2,$$

where

$$M_* = 8M[\tilde{H}^2\|\varphi\|^2_{\mathbb{B}} + (L_2c_1^* + l_2)] + 8 \times \{L_2[(1+L_1)c_1^* + l_1] + l_2\}$$
$$+ 8Tr(Q)MT[l(\lambda) + (d+\epsilon)c_1^*] + 10Mm^2C_1,$$

$$c_1^* = \big[(M_b + J_0^{\varphi})\|\varphi\|_{\mathbb{B}}\big]^2.$$

Since $L_* = 8L_2(1+L_1)K_T^2 + 8Mm\sum_{k=1}^m (c_k+\epsilon_k) < 1$, we have

$$\sup_{t\in[0,T]}\mathbb{E}\|x(t)\|^2 \le \frac{M_*}{1-L_*} + P_2\int_0^b \sup_{\tau\in[0,s]}\mathbb{E}\|x(\tau)\|^2 ds,$$

where $P_2 = \frac{1}{1-L_*}8Tr(Q)M(d+\epsilon)K_T^2$.
Applying Gronwall's inequality in the above expression, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}\|x(t)\|^2 \le \frac{M_*}{1-L_*}\exp\{P_2T\} := \bar{K}.$$

Then for any $x \in \Lambda(\Phi)$, we get that $\|x\|^2_{\mathcal{PC}} \le \bar{K}$. This implies that Λ is bounded on J. Consequently, by Lemma 2.4, we deduce that Λ has a fixed point $x \in \mathbb{Y}$, which is a mild solution of problem (1). The proof is complete. □

4. Application

Consider the following impulsive neutral stochastic partial integrodifferential equations of the form

$$\begin{cases} \frac{\partial}{\partial t}\left[z(t,x)-\mu_1\left(t,z(t-\tau,x),\int_0^s\mu_2(t,s,z(s-\tau,x))ds\right)\right]=-\frac{\partial^2}{\partial x^2}\left[z(t,x)\right. \\ \left.\mu_1\left(t,z(t-\tau,x),\int_0^s\mu_2(t,s,z(s-\tau,x))ds\right)\right] \\ +\int_0^t b(t-s)\frac{\partial^2}{\partial\xi^2}\left[z(s,x)-\mu_1\left(s,z(s-\tau,x),\int_0^s\mu_2(s,u,z(u-\tau,x))du\right)\right]ds \\ +\mu_3[t,z(s-\rho_1(\tau)\rho_2(\|z(\tau)\|),x)]dw(t), \quad 0\le t\le b; \quad \tau>0, \quad 0\le x\le\pi \\ z(t,0)=z(t,\pi)=0, \quad 0\le t\le T, \\ z(t_k^+,x)-z(t_k^-,x)=I_k(z(t_k,x)), \quad k=1,\ldots,m, \\ z(t,x)=\varphi(t,x), \quad -\infty\le t\le 0, \quad 0\le x\le\pi, \end{cases} \tag{12}$$

where φ is continuous and $I_k\in C(\mathbb{R},\mathbb{R}), w(t)$ denotes a standard cylindrical Wiener process in $\mathbb{H}$ defined on a stochastic space $(\Omega,\mathcal{F},P)$ and $\mathbb{H}=L^2([0,\pi])$ with the norm $\|.\|$ and define the operators $A:\mathbb{H}\to\mathbb{H}$ by $Aw=w^{''}$ with the domain
$D(A):=\{w\in\mathbb{H}: w,w^{'}$ are absolutely continuous, $w^{''}\in\mathbb{H}, w(0)=w(\pi)=0\}$.
Then

$$Aw=\sum_{n=1}^{\infty}n^2\langle w,w_n\rangle w_n, \quad w\in D(A),$$

where $w_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n=1,2,\ldots$ is the orthogonal set eigenvectors of A. It is well known that A is the infintesimal generator of an analytic semigroup $T(t)$, $t\ge 0$ in H and is given by

$$T(t)w=\sum_{n=1}^{\infty}\exp(-n^2t)(w,w_n)w_n, \quad w\in\mathbb{H}.$$

Let $B:D(A)\subset\mathbb{H}\to\mathbb{H}$ be the operator defined by

$$B(t)(y)=b(t)Ay \quad t\ge 0 \quad y\in D(A).$$

Let $\sigma>0$, define the phase space

$$\mathbb{B}=\{\phi\in C((-\infty,0],\mathbb{H}):\lim_{\theta\to\infty}e^{\sigma\theta}\phi(\theta) \quad \text{exists in } \mathbb{H}\},$$

and let $\|\phi\|_{\mathbb{B}}=\sup_{-\infty<\theta<0}\{e^{\sigma\theta}\|\phi(\theta)\|\}$. Then $(\mathbb{B},\|\phi\|_{\mathbb{B}})$ is a Banach space which satisfies (A1)–(A3) with $\tilde{H}=1$, $K(t)=\max\{1,e^{-\sigma t}\}$, $M(t)=e^{-\sigma t}$. Hence for $(t,\phi)\in[0,T]\times\mathbb{B}$, where $\phi(\theta)(x)=\phi(\theta,x)$, $(\theta,x)\in(-\infty,0]\times[0,\pi]$, let $z(t)(x)=z(t,x)$,

$$\begin{aligned} G\left(t,\phi,\int_0^s\mu_1(t,s,\phi)ds\right)(x)&=\mu_2\left(t,\phi(\theta,x),\int_0^s\mu_2(t,s,\phi)ds\right), \\ g(t,s,\phi)(x)&=\mu_2-t,s,\phi(\theta,x)), \\ F(t,\phi)(x)&=\mu_3(t,\phi(\theta,x)), \\ \rho(t,\phi)&=\rho_1(t)\rho_2(\|\phi(0)\|). \end{aligned}$$

Then the problem (12) can be writen as (1). Moreover, if b is bounded and C^1 function such that b' is bounded and uniformly continuous, then **(H1)** and **(H2)** are satisfied, and hence, by Theorem 1, 2.1 has a resolvent operator $(R(t))_{t\ge 0}$ on $\mathbb{H}$. Thus, under appropriate conditions on the functions G,g,F, and I_k as those in **(H1)**–**(H7)**, the problem (12) has a mild solution on J.

5. Conclusion

In this paper, we have studied the existence results for impulsive stochastic neutral integrodifferential systems with state-dependent delay conditions in a Hilbert space by utilizing the stochastic analysis theory, resolvent operator, and the Krasnoselskii fixed point theorem. To validate the obtained theoretical results, we analyze one example. The impulsive stochastic neutral integrodifferential systems with state-dependent delay are very efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and controllability. There are two direct issues that require further study. First, we will investigate the controllability of neutral stochastic integrodifferential systems with state-dependent delay in the case of nonlocal conditions. Second, we will study the approximate controllability of a new class of impulsive stochastic integrodifferential equations with state-dependent delay and noninstantaneous impulses.

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[1]Université Gaston Berger de Saint-Louis, UFR SAT, Département de Mathématiques, 234, Saint-Louis, Sénégal

E-mail address: ordydiop@gmail.com

E-mail address: mmedsalem93@gmail.com

[2]Institut de Mathématiques et de Sciences Physiques, Porto Novo, Benin

E-mail address: amour.gbaguidi@imsp-uac.org

E-mail address: ogouyandjou@imsp-uac.org

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ON THE DOUBLE LIMIT ASSOCIATED WITH RIEMANN'S SUMMATION METHOD

O. DZAGNIDZE[1] AND I. TSIVTSIVADZE[2]

Abstract. By Riemann's first theorem the convergence of any series $\sum_{k=0}^{\infty} a_k$ to a finite value s implies the existence of the limit $\lim\limits_{h\to 0} \sum_{k=0}^{\infty} a_k \left(\frac{\sin kh}{kh}\right)^2$, i.e. the existence of the repeated limit $\lim\limits_{h\to 0} \lim\limits_{n\to\infty} \sum_{k=0}^{n} a_k \left(\frac{\sin kh}{kh}\right)^2$ with the value s, but the converse statement does not hold. In the article it is proved the following theorem: *A numerical series* $\sum_{k=0}^{\infty} a_k$ *converges to a finite number s if and only if there exists the double limit* $\lim\limits_{\substack{h\to 0\\ n\to\infty}} \sum_{k=0}^{n} a_k \left(\frac{\sin kh}{kh}\right)^2$ *and the limit is equal to s.* The proof is based on Toeplitz's condition on the uniform boundedness for summation (see, relation (12) in the article) and Moore–Osgood's double limit theorem. An application of the theorem to trigonometric Fourier series is given.

Along with an arbitrary series

$$\sum_{k=0}^{\infty} a_k, \tag{1}$$

no matter whether it is converging or not, we will consider the series

$$\sum_{k=0}^{\infty} a_k \left(\frac{\sin kh}{kh}\right)^2 \tag{2}$$

which depends on the variable h under the assumption that this series converges for sufficiently small $h \neq 0$ and $\dfrac{\sin 0}{0} = 1$.

In particular, the series (2) will be converging for any $h \neq 0$ if the sequence $|a_k|$, $k = 0, 1, \dots$ is bounded by some number $M > 0$. Indeed, we have

$$\left|\sum_{k=0}^{\infty} a_k \left(\frac{\sin kh}{kh}\right)^2\right| \le |a_0| + Mh^{-2} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

If under the above assumption the finite limit

$$\lim_{h\to 0} \sum_{k=0}^{\infty} a_k \left(\frac{\sin kh}{kh}\right)^2 = \sigma \tag{3}$$

exists, then the series (1) is called Riemann-summable (or, briefly, R-summable) to the value σ.

It is obvious that the equality (3) can be written in the following form

$$\lim_{h\to 0} \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left(\frac{\sin kh}{kh}\right)^2 = \sigma,$$

i.e. in the form of the repeated limit

$$\lim_{h\to 0} \lim_{n\to\infty} A_n(h) = \sigma, \tag{4}$$

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Key words and phrases. Riemann's summation method; Repeated limit; Double limit.

where it is assumed that

$$A_n(h)=\sum_{k=0}^{n}a_k\Big(\frac{\sin kh}{kh}\Big)^2.$$

Therefore the fulfillment of the equality (4) is equivalent to the R-summability of the series (1) to the value σ.

The existence of another repeated limit with the finite value ω

$$\lim_{n\to\infty}\lim_{h\to 0}A_n(h)=\omega, \tag{5}$$

implies the equality

$$\sum_{k=0}^{\infty}a_k=\omega \tag{6}$$

and vice versa: from the equality (6) there follows the equality (5). Hence we have the following

Proposition. *The convergence of the series* (1) *to the value* ω *is the necessary and sufficient condition for the fulfillment of the equality* (5).

We establish the relationship between the convergence of the series (1) and the existence of the double limit

$$\lim_{\substack{h\to 0\\ n\to\infty}}A_n(h). \tag{7}$$

As to this relationship we have the following statement.

Theorem. *The convergence of the series* (1) *to the finite value* s

$$\sum_{k=0}^{\infty}a_k=s \tag{8}$$

is the necessary and sufficient condition for the fulfillment of the equality

$$\lim_{\substack{h\to 0\\ n\to\infty}}A_n(h)=s. \tag{9}$$

Sufficiency. By virtue of the above Proposition, from the equality (8) we obtain the equality (5) where ω is replaced by s. Therefore the limit

$$\lim_{h\to 0}A_n(h) \tag{10}$$

is finite for any n.

Furthermore, from the equality (8) there follows the equality

$$\lim_{h\to 0}\lim_{n\to\infty}A_n(h)=s \tag{11}$$

by virtue of Riemann's first theorem [5, p. 319].

Along with this, during the proof of this Riemann's first theorem an important fact is established that consists in that the series

$$\sum_{k=0}^{\infty}a_k\Big(\frac{\sin kh}{kh}\Big)^2$$

converges uniformly with respect to h, i.e. the limit

$$\lim_{n\to\infty}A_n(h) \tag{12}$$

exists uniformly with respect to h ([3, Ch. XIII, §13.8.2], [4, Ch. 9, §9.62], [5, Ch. IX, §2, inequality (2.6), which holds uniformly with respect to the family $\{(h_i)\}$ of all sequences (h_i) tending to zero as $i\to\infty$]).

Therefore by virtue of the Moore–Osgood Double Limit Theorem [2, p. 180] modified for the continuous parameter h, the equalities

$$\lim_{n\to\infty}\lim_{h\to 0}A_n(h)=\lim_{h\to 0}\lim_{n\to\infty}A_n(h)=\lim_{\substack{h\to 0\\ n\to\infty}}A_n(h)=s$$

are fulfilled.

Thus the convergence of the series (1) to the value s implies the existence of the limit (7) and the equality (9).

Necessity. If the double limit

$$\lim_{\substack{h\to 0\\ n\to\infty}} A_n(h) = s$$

exists, then there exists the partial limit s equal to $\lim\limits_{n\to\infty} A_n(0)$. But $\lim\limits_{n\to\infty} A_n(0) = \sum\limits_{k=0}^{\infty} a_k$.

Therefore the equality (8) is fulfilled. The theorem is proved.

Finally, we give an application of the above theorem to trigonometric Fourier series. It is well known that there is the summable Kolmogorov function $K(x)$ on $[-\pi,\pi]$, whose Fourier series

$$K \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{13}$$

diverges at every point $x \in [-\pi,\pi]$ [5, p. 310].

However, the series (13) is R-summable at almost all points $x \in [-\pi,\pi]$ to values $K(x)$ [1, Ch. I, paragraph 69]. From the theorem that is proved above it follows.

Corollary. *For the series* (13) *the following statements are true:*

1. *The equality*

$$\lim_{h\to 0}\lim_{n\to\infty}\left[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\Big(\frac{\sin kh}{kh}\Big)^2\right] = K(x)$$

is fulfilled for almost all points $x \in [-\pi,\pi]$;

2. *There is not a point $x \in [-\pi,\pi]$ at which the double limit*

$$\lim_{\substack{h\to 0\\ n\to\infty}}\left[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\Big(\frac{\sin kh}{kh}\Big)^2\right]$$

would exist.

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[1]A. Razmadze Mathematical Institute I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

E-mail address: omar.dzagnidze@tsu.ge

[2]Akaki Tsereteli Kutaisi State University, Kutaisi 4600, Georgia

E-mail address: irmatsiv@gmail.com

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE BOUNDEDNESS OF THE GEGENBAUER–RIESZ POTENTIAL IN MODIFIED MORREY SPACES

V. S. GULIYEV[1] AND E. J. IBRAHIMOV[2]

Abstract. In this paper we study the Gegenbauer–Riesz potential I_G^α (G-Riesz potential) generated by Gegenbauer differential operator $G_\lambda = (x^2-1)^{1/2-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+1/2}\frac{d}{dx}$. We prove that the operator I_G^α is bounded from the modified Morrey space $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+)$ to the weak modified Morrey space $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+)$ if and only if $\frac{\alpha}{2\lambda+1} \le 1-\frac{1}{q} \le \frac{\alpha}{2\lambda+1-\gamma}$ for $1<q<\infty$ and from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+)$ if and only if $\frac{\alpha}{2\lambda+1} \le \frac{1}{p}-\frac{1}{q} \le \frac{\alpha}{2\lambda+1-\gamma}$ for $1<p<q<\infty$. Obtained results are the analogue of the results taken in [6].

1. Definitions and Auxiliary Results

The study of boundedness of the Riesz potential, singular integrals and commutators were studied by lots of researchers in the last decades. Morrey estimates of such kind of operators is a more recent problem and is still very popular. Just as an example we recall the study made in [1,2,8,12]. Our aim is to continue this research focusing in necessary and sufficient conditions in suitable Morrey estimates of some kind of Riesz potential. The Gegenbauer differential operator was introduced in [3]. About properties of Gegenbauer differential operator we reference detail in [7].

In this paper, we consider the following generalized shift operator

$$A_{ch\,t} f(ch\,x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi f(ch\,x ch\,t - sh\,x sh\,t\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi$$

generated by the Gegenbauer differential operator

$$G_\lambda = (x^2-1)^{1/2-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+1/2}\frac{d}{dx},\ \ x\in(1,\infty),\ \ \lambda\in\big(0,1/2\big).$$

Let $H(x,r)=(x-r,x+r)\cap[0,\infty), r\in(0,\infty), x\in[0,\infty)$. For all measurable set $E\subset[0,\infty), \mu E \equiv |E|_\lambda = \int_E sh^{2\lambda}\,tdt$. In [10] the Gegenbaur maximal function (G-maximal function) is defined as follows:

$$M_G f(ch\,x) = \sup_{r>0}\frac{1}{|(0,r)|_\lambda}\int_0^r A_{ch\,t}|f(ch\,x)| sh^{2\lambda} tdt.$$

Also we consider the following maximal function

$$M_\mu f(ch\,x) = \sup_{r>0}\frac{1}{|H(x,r)|_\lambda}\int_{H(x,r)} |f(ch\,t)| sh^{2\lambda} tdt.$$

Symbol $A\lesssim B$ denote that there exists a constant $C>0$ with that $0<A\le CB$ and C can depends of some parameters. If $A\lesssim B$ and $B\lesssim A$ we write $A\approx B$.

Note that, the following inequality is valid (see [10, Theorem 2.1]):

$$M_G f(ch\,x) \lesssim M_\mu f(ch\,x). \tag{1.1}$$

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Key words and phrases. Gegenbauer differential operator; Gegenbauer–Riesz potential; Modified Morrey space; Hardy-Littlewood-Sobolev inequality.

In what follows we need the following theorems to prove our main results (see [11, Theorem 2.1 and Theorem 2.2]).

Theorem A ([11, Theorem 2.1]). *For all non-negative function $g \in L^{loc}_{1,\lambda}(\mathbb{R}_+)$ and $1 \le p < \infty$ the inequality*

$$\int\limits_{H(x,r)} (M_\mu f(ch\, y))^p g(ch\, y) sh^{2\lambda} y dy \le \int\limits_{H(x,r)} |f(ch\, y)|^p M_\mu(g(ch\, y)) sh^{2\lambda} y dy$$

is valid.

Theorem B ([11, Theorem 2.2]). *For all $\alpha > 0$ the following Chebyshev type inequality*

$$|\{y \in H(x,r) : M_\mu f(ch\, y) > \alpha\}|_\gamma \lesssim \frac{1}{\alpha} \int\limits_{H(x,r)} M_\mu f(ch\, y) sh^{2\lambda} y dy$$

is valid.

For $1 \le p \le \infty$ let $L_p([0,\infty), G) \equiv L_{p,\lambda}[0,\infty)$ be the space of functions measurable on $[0,\infty)$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \Bigg(\int\limits_0^\infty |f(ch\, t)|^p sh^{2\lambda}\ t dt\Bigg)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\|_{\infty,\lambda} = \operatorname*{ess\,sup}_{t\in[0,\infty)} |f(ch\, t)|, \quad p = \infty.$$

The following theorem was proved in [10].

Theorem C. a) *If $f \in L_{1,\lambda}[0,\infty)$, then for all $\alpha > 0$ the inequality*

$$|\{x : M_\mu f(ch\, x) > \alpha\}|_\lambda \le \frac{c_\lambda}{\alpha} \|f\|_{L_{1,\lambda}[0,\infty)}$$

holds, where $c_\lambda > 0$ depends only on λ.

b) *If $f \in L_{p,\lambda}[0,\infty)$, $1 < p \le \infty$, then $M_\mu f(ch\, x) \in L_{p,\lambda}[0,\infty)$ and*

$$\|M_\mu f\|_{L_{p,\lambda}[0,\infty)} \le c_{p,\lambda} \|f\|_{L_{p,\lambda}[0,\infty)}.$$

Corollary A. *If $f \in L_{p,\lambda}[0,\infty)$, $1 \le p \le \infty$, then*

$$\lim_{r\to 0} \frac{1}{|(0,r)|_\lambda} \int\limits_{(0,r)} A^\lambda_{ch\, t} f(ch\, x) sh^{2\lambda}\ t dt = f(ch\, x)$$

for a.e., $x \in [0,\infty)$.

2. Some Embeddings into the G-Morrey and Modified G-Morrey Spaces

We introduce the following notation analogously in [4–6].

Definition 2.1. Let $1 \le p < \infty$, $0 < \lambda < \frac{1}{2}$, $0 \le \gamma \le 2\lambda + 1$, $[r]_1 = \min\{1, r\}$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$, $\mathbb{R}_+ = [0,\infty)$, the G-Morrey space, and by $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ the modified G-Morrey space, as the set of locally integrable functions $f(ch\, x)$, $x \in \mathbb{R}_+ = [0,\infty)$, with the finite norms

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \Bigg(r^{-\gamma} \int\limits_{H(x,r)} |f(ch\, t)|^p sh^{2\lambda} t dt\Bigg)^{\frac{1}{p}},$$

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \Bigg([r]_1^{-\gamma} \int\limits_{H(x,r)} |f(ch\, t)|^p sh^{2\lambda} t dt\Bigg)^{\frac{1}{p}},$$

respectively.

Note that $\widetilde{L}_{p,\lambda,0}(\mathbb{R}_+,G) = L_{p,\lambda,0}(\mathbb{R}_+,G) = L_{p,\lambda}(\mathbb{R}_+,G)$. $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G) \subset L_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ $\cap L_{p,\lambda}(\mathbb{R}_+,G)$ and $\max\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\lambda}}\} \leq \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$ and if $\gamma < 0$ or $\gamma > 2\lambda+1$, then $L_{p,\lambda,\gamma}(\mathbb{R}_+,G) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}_+$.

Definition 2.2. Let $1 \leq p < \infty$, $0 < \lambda < \frac{1}{2}$, $0 \leq \gamma \leq 1+2\lambda$. We denote by $WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ the weak G-Morrey space and by $W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ the modified weak G-Morrey space as the set of locally integrable functions $f(ch\, x)$, $x \in \mathbb{R}_+$ with finite norms

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}_+} \left(t^{-\gamma} |\{y \in H(x,t) : |f(ch\, y)| > r\}|_\gamma\right)^{\frac{1}{p}},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}_+} \left([t]_1^{-\gamma} |\{y \in H(x,t) : |f(ch\, y)| > r\}|_\gamma\right)^{\frac{1}{p}}$$

respectively.

Note that $WL_{p,\lambda}(\mathbb{R}_+,G) = WL_{p,\lambda,0}(\mathbb{R}_+,G) = W\widetilde{L}_{p,\lambda,0}(\mathbb{R}_+,G)$, $L_{p,\lambda,\gamma}(\mathbb{R}_+,G) \subset WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ and $\|f\|_{WL_{p,\lambda,\gamma}} \leq \|f\|_{L_{p,\lambda,\gamma}}$, $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G) \subset WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ and $\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} \leq \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$.

We note that

$$L_{p,\lambda,0}(\mathbb{R}_+,G) = L_{p,\lambda}(\mathbb{R}_+,G),$$

and if $\gamma < 0$ or $\gamma > 1+2\lambda$, then $L_{p,\lambda,\gamma}(\mathbb{R}_+,G) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}_+$.

Lemma 2.1. *Let $1 \leq p < \infty$, $0 < \lambda < \frac{1}{2}$. Then*

$$L_{p,\lambda,1+2\lambda}(\mathbb{R}_+,G) = L_\infty(\mathbb{R}_+)$$

and

$$c_\lambda^{-1/p} \|f\|_{L_\infty} \leq \|f\|_{L_{p,\lambda,1+2\lambda}} \leq \|f\|_{L_\infty},$$

where $c_\lambda = \frac{2^{\frac{1}{2}-\lambda}}{(1+2\lambda)(1+ch\, 1)^{\frac{1}{2}-\lambda}}$.

Proof. Let $f \in L_\infty(\mathbb{R}_+)$. Then

$$\left(\frac{1}{|(0,r)|_\lambda} \int\limits_{(0,r)} A^\lambda_{ch\, t} f(ch\, x) sh^{2\lambda}\, t dt\right)^{1/p} \leq \|f\|_{L_\infty}.$$

Therefore $f \in L_{p,\lambda,1+2\lambda}(\mathbb{R}_+,G)$ and

$$\|f\|_{L_{p,\lambda,1+2\lambda}} \leq \|f\|_{L_\infty}.$$

Let $f \in L_{p,\lambda,1+2\lambda}(\mathbb{R}_+,G)$. By the Lebesgue's Theorem we have (see Section 1, Corollary A)

$$\lim_{r\to 0} \frac{1}{|(0,r)|_\lambda} \int\limits_{(0,r)} A^\lambda_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\, t dt = |f(ch\, x)|^p.$$

Then

$$|f(ch\, x)| = \left(\lim_{r\to 0} \frac{1}{|(0,r)|_\lambda} \int\limits_{(0,r)} A^\lambda_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\, t dt\right)^{1/p}$$
$$\leq \sup_{0<r<1} \left(\frac{r^{1+2\lambda}}{|(0,r)|_\lambda}\right)^{1/p} \|f\|_{L_{p,\lambda,1+2\lambda}}.$$

From the proof of the Lemma 1.1 in [10] for $0 < r < 1$ we have

$$|(0,r)|_\lambda \geq \frac{2^{\lambda+\frac{3}{2}}}{(1+2\lambda)(1+ch\, 1)^{\frac{1}{2}-\lambda}} \left(sh\, \frac{r}{2}\right)^{1+2\lambda} \geq \frac{2^{\frac{1}{2}-\lambda}}{(1+2\lambda)(1+ch\, 1)^{\frac{1}{2}-\lambda}}\, r^{1+2\lambda}.$$

Therefore $f \in L_\infty(\mathbb{R}_+)$ and

$$\|f\|_{L_\infty} \leq c_\lambda^{1/p} \|f\|_{L_{p,\lambda,1+2\lambda}}. \qquad \square$$

Lemma 2.2. *Let $1 \le p < \infty$, $0 < \lambda < \frac{1}{2}$, $0 \le \gamma \le 1 + 2\lambda$. Then*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G) = L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\lambda}}\right\}.$$

Proof. Let $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$. Then

$$\begin{aligned}\|f\|_{L_{p,\lambda}} &= \sup_{x\in\mathbb{R}_+, r>0} \left(\int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\ tdt \right)^{1/p} \\ &\le \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\ tdt \right)^{1/p} \\ &= \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}\end{aligned}$$

and

$$\begin{aligned}\|f\|_{L_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\ tdt \right)^{1/p} \\ &\le \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\ tdt \right)^{1/p} \\ &= \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.\end{aligned}$$

Therefore, $f \in L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$ and the embedding

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$$

is valid.

Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$. Then

$$\begin{aligned}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\ tdt \right)^{1/p} \\ &= \max\left\{ \sup_{x\in\mathbb{R}_+, 0<r\le 1} \left(r^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\ tdt \right)^{1/p}, \right. \\ &\left. \sup_{x\in\mathbb{R}_+, r>1} \left(\int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\ tdt \right)^{1/p} \right\} \le \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\lambda}}\right\}.\end{aligned}$$

Therefore, $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and the embedding $L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G) \subset\!\!\succ \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ is valid.

Thus $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G) = L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$.

Let now $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$. Then

$$\begin{aligned}
\|f\|_{L_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{(0,r)} A^{\gamma}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\; tdt \right)^{1/p} \\
&= \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-1}[r]_1 \right)^{\frac{\gamma}{p}} \left([r]_1^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\; tdt \right)^{1/p} \\
&= \sup_{x\in\mathbb{R}_+, r>0} \left([r]_1^{-\gamma} \int\limits_{(0,r)} A^{\lambda}_{ch\, t} |f(ch\, x)|^p sh^{2\lambda}\; tdt \right)^{1/p} \\
&= \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \qquad \square
\end{aligned}$$

3. Hardy-Littlewood-Sobolev Inequality in Modified G-Morrey Spaces

In this section we study the $\widetilde{L}_{p,\lambda,\gamma}$-boundedness of the G-maximal operator M_μ.

Theorem 3.1. 1) *If $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$, $0 \le \gamma < 1 + 2\lambda$, then $M_\mu f \in W\widetilde{L}_{1,\lambda,\gamma}$ and*

$$\|M_\mu f\|_{W\widetilde{L}_{1,\lambda,\gamma}} \lesssim \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}.$$

2) *If $f \in \widetilde{L}_{p,\lambda,\gamma}$, $1 < p < \infty$, $0 \le \gamma < 1 + 2\lambda$, then $M_\mu f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and*

$$\|M_\mu f\|_{\widetilde{L}_{p,\lambda,\gamma}} \lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Proof. 1) From the definition of weak modified Morrey spaces

$$\|M_\mu f\|_{W\widetilde{L}_{1,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}_+} \left([t]_1^{-\gamma} |\{y \in H(x,t) : M_\mu f(ch\, y) > r\}|_\gamma \right)^{\frac{1}{p}}.$$

Using the Theorem B and also Theorem A at $p = 1$ and $g(ch\, y) \equiv 1$ we obtain

$$\|M_\mu f\|_{W\widetilde{L}_{1,\lambda,\gamma}} \lesssim \sup_{t>0,\, x\in\mathbb{R}_+} \left([t]_1^{-\gamma} \int\limits_{H(x,t)} |f(ch\, y)| sh^{2\lambda} y dy \right) = \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}.$$

Assertion 2) follows from Theorem A at $g(ch\, y) \equiv 1$. $\square$

We consider of the Gegenbauer–Riesz potential (G - Riesz potential) (see [10])

$$I^\alpha_G f(ch\, x) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int\limits_0^\infty \left(\int\limits_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\, t) dr \right) A_{ch\, t} f(ch\, x) sh^{2\lambda} t dt,$$

where

$$h_r(ch\, t) = \int\limits_1^\infty e^{-\gamma(\gamma+2\lambda)r} P^\lambda_\gamma(ch\, t)(\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma$$

and P^λ_γ is eigenfunction of operator G_λ.

The following Hardy–Littlewood–Sobolev inequality in modified G-Morrey spaces is valid.

Theorem 3.2. *Let $0 \le \alpha < 1 + 2\lambda$, $0 \le \gamma < 2\lambda + 1 - \alpha$ and $1 \le p < \frac{2\lambda+1-\gamma}{\alpha}$.*

1) *If $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$, then the condition $\frac{\alpha}{2\lambda+1} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{2\lambda+1-\gamma}$ is necessary and sufficient for the boundedness of the operator I^α_G from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.*

2) *If $p = 1 < \frac{2\lambda+1-\gamma}{\alpha}$, then the condition $\frac{\alpha}{2\lambda+1} \le 1 - \frac{1}{q} \le \frac{\alpha}{2\lambda+1-\gamma}$ is necessary and sufficient for the boundedness of the operator I^α_G from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.*

Proof. 1) *Sufficiency.* Let $0 \le \alpha < 1+2\lambda$, $0 \le \gamma < 2\lambda+1-\gamma$, $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$.
For I_G^α take place the following estimate (see [10, the proof of Corollary 3.1])

$$\begin{aligned} |I_G^\alpha f(ch\, x)| &\lesssim \int_0^\infty A_{ch\, t}|f(ch\, x)|\,(sh\, x)^{\alpha-2\lambda-1}\, sh^{2\lambda} t dt \\ &= \int_0^\infty A_{ch\, t}(sh\, x)^{\alpha-2\lambda-1}|f(ch\, t)| sh^{2\lambda} t dt. \end{aligned} \tag{3.1}$$

From (3.1) we have

$$\begin{aligned} |I_G^\alpha f(ch\, x)| &\lesssim \Big(\int_0^r + \int_r^\infty\Big) A_{ch\, t}(sh\, x)^{\alpha-2\lambda-1}|f(ch\, t)| sh^{2\lambda} t dt \\ &= A_1(x,r) + A_2(x,r). \end{aligned}$$

We consider $A_1(x,r)$. Let $0 < r < 2$, then by (1.1) we obtain

$$\begin{aligned} |A_1(x,r)| &\lesssim \int_0^r \frac{A_{ch\, t}|f(ch\, x)| sh^{2\lambda} t}{(sh\, t)^{2\lambda+1-\alpha}} dt \lesssim \sum_{j=0}^\infty \int_{2^{-j-1}r}^{2^{-j}r} \frac{A_{ch\, t}|f(ch\, x)| sh^{2\lambda} t}{(sh\, t)^{2\lambda+1-\alpha}} dt \\ &\lesssim \sum_{j=0}^\infty \Big(sh \frac{r}{2^{j+1}}\Big)^\alpha \Big(sh \frac{r}{2^{j+1}}\Big)^{-2\lambda-1} \int_0^{2^{-j}r} A_{ch\, t}|f(ch\, x)| sh^{2\lambda} t dt \\ &\lesssim (sh\, r)^\alpha M_G f(ch\, x) \Big(\sum_{j=0}^\infty 2^{-(j+1)\alpha}\Big) \lesssim (sh\, r)^\alpha M_\mu f(ch\, x). \end{aligned} \tag{3.2}$$

Let $2 \le r < \infty$ and $0 < \alpha < 4\lambda$. Then (see [10, the proof of Corollary 3.1])

$$\begin{aligned} A_1(x,r) &\lesssim \int_0^r \frac{A^\lambda_{ch\, t}|f(ch\, x)| sh^{2\lambda} t dt}{(ch\, t)^{2\lambda+1-\alpha}} \\ &\le \int_0^r \frac{A^\lambda_{ch\, t}|f(ch\, x)| sh^{2\lambda} t dt}{(ch\, t)^{4\lambda-\alpha}} \le \int_0^r \frac{A^\lambda_{ch\, t}|f(ch\, x)| sh^{2\lambda} t dt}{(sh\, t)^{4\lambda-\alpha}} \\ &\le \sum_{j=0}^\infty \int_{2^{-j-1}r}^{2^{-j}r} \frac{A^\lambda_{ch\, t}|f(ch\, x)| sh^{2\lambda} t dt}{(sh\, t)^{4\lambda-\alpha}} \\ &\le \sum_{j=0}^\infty \Big(sh \frac{r}{2^{j+1}}\Big)^\alpha \Big(sh \frac{r}{2^{j+1}}\Big)^{-4\lambda} \int_0^{2^{-j}r} A^\lambda_{ch\, t}|f(ch\, x)| sh^{2\lambda} t dt \\ &\lesssim M_G f(ch\, x) \sum_{j=0}^\infty \Big(sh \frac{r}{2^{j+1}}\Big)^\alpha \le (sh\, r)^\alpha M_\mu f(ch\, x) \sum_{j=0}^\infty 2^{-(j+1)\alpha} \\ &\lesssim (sh\, r)^\alpha M_\mu f(ch\, x), \quad 0 < \alpha < 4\lambda. \end{aligned}$$

Now let $4\lambda \le \alpha < 2\lambda + 1$. From the proof of Corollary 3.1 and [10] it follows that $|I_G^\alpha f(ch\,x)| \lesssim 1$, then we have

$$\begin{aligned}|A_1(x,r)| &\lesssim \int\limits_0^r A_{ch\,t}^\lambda |f(ch\,x)| sh^{2\lambda} t dt = \frac{\left(sh\frac{r}{2}\right)^{4\lambda}}{\left(sh\frac{r}{2}\right)^{4\lambda}} \int\limits_0^r A_{ch\,t}^\lambda |f(ch\,x)| sh^{2\lambda} t dt \\ &\le \left(sh\frac{r}{2}\right)^{4\lambda} M_G f(ch\,x) \lesssim (sh\,r)^\alpha M_\mu f(ch\,x), \quad 4\lambda \le \alpha < 2\lambda+1.\end{aligned}$$

Thus for $0 < r < \infty$ we have

$$A_1(x,r) \lesssim (sh\,r)^\alpha M_\mu f(ch\,x), \quad 0 < \alpha < 2\lambda + 1. \tag{3.3}$$

We consider $A_2(x,r)$. From (3.1) and Hölder's inequality we get

$$\begin{aligned}A_2(x,r) &\lesssim \Big(\int\limits_r^\infty (A_{ch\,t}|f(ch\,x)|)^p (sh\,t)^{-\beta} sh^{2\lambda} t dt\Big)^{\frac{1}{p}} \\ &\times \Big(\int\limits_r^\infty (sh\,t)^{\left(\frac{\beta}{p}+\alpha-2\lambda-1\right)p'} sh^{2\lambda} t dt\Big)^{\frac{1}{p'}} = A_{21} \cdot A_{22}.\end{aligned} \tag{3.4}$$

Let $\gamma < \beta < 2\lambda + 1 - p\alpha$. Taking into account the inequality (see [9, Lemma 2])

$$\|A_{ch\,t} f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$

we obtain

$$\begin{aligned}A_{21} &\lesssim \Big(\sum_{j=0}^\infty \int\limits_{2^j r}^{2^{j+1} r} (A_{ch\,t}|f(ch\,x)|)^p (sh\,t)^{-\beta} sh^{2\lambda} t dt\Big)^{\frac{1}{p}} \\ &\lesssim \|A_{ch\,t} f\|_{\widetilde{L}_{p,\lambda,\gamma}} \Big(\sum_{j=0}^\infty \frac{[2^{j+1} r]_1^\gamma}{(sh\,2^j r)^\beta}\Big)^{\frac{1}{p}} \\ &\lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \begin{cases} \Big((2r)^\gamma \sum\limits_{j=0}^{\left[\log_2 \frac{1}{2r}\right]} 2^{(\gamma-\beta)j} + \sum\limits_{j=\left[\log_2 \frac{1}{2r}\right]+1}^\infty 2^{-\beta j}\Big)^{\frac{1}{p}}, & 0 < r < \frac{1}{2}, \\ \Big(\sum\limits_{j=0}^\infty 2^{-\beta j}\Big)^{\frac{1}{p}}, & r \ge \frac{1}{2} \end{cases} \\ &\lesssim (sh\,r)^{-\frac{\beta}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \begin{cases} (r^\gamma + r^\beta), & 0 < r < \frac{1}{2}, \\ 1, & r \ge \frac{1}{2} \end{cases} \\ &\lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \begin{cases} (r^{\frac{\gamma}{p}} (sh\,r)^{-\frac{\beta}{p}}, & 0 < r < \frac{1}{2}, \\ (sh\,r)^{-\beta}, & r \ge \frac{1}{2} \end{cases} \\ &\lesssim [2r]_1^{\frac{\gamma}{p}} (sh\,r)^{-\frac{\beta}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.\end{aligned} \tag{3.5}$$

For A_{22} we have

$$
\begin{aligned}
A_{22} &= \Big(\int_r^\infty (sh\, t)^{\left(\frac{\beta}{p}+\alpha-2\lambda-1\right)p'} sh^{2\lambda} t dt \Big)^{\frac{1}{p'}} \\
&\le \Big(\int_r^\infty (sh\, t)^{\left(\frac{\beta}{p}+\alpha-2\lambda-1\right)p'} sh^{2\lambda} t d(sh\, t) \Big)^{\frac{1}{p'}} \\
&\lesssim (sh\, r)^{\frac{\beta}{p}+\alpha-2\lambda-1+\frac{2\lambda+1}{p'}} \lesssim (sh\, r)^{\frac{\beta}{p}+\alpha-2\lambda-1+(2\lambda+1)(1-\frac{1}{p})} \\
&\lesssim (sh\, r)^{\frac{\beta}{p}+\alpha-\frac{2\lambda+1}{p}}. \qquad (3.6)
\end{aligned}
$$

Taking into account (3.5) and (3.6) on (3.4) we obtain

$$A_2(x,r) \lesssim [2r]_1^{\frac{\gamma}{p}} (sh\, r)^{\alpha-\frac{2\lambda+1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \qquad (3.7)$$

Thus from (3.3) and (3.7) we get

$$
\begin{aligned}
|I_G^\alpha f(ch\, x)| &\lesssim \Big([r]_1^{\frac{\gamma}{p}} (sh\, r)^{\alpha-\frac{2\lambda+1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} + (sh\, r)^\alpha M_\mu f(ch\, x) \Big) \\
&\lesssim \min \Big\{ (sh\, r)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} + (sh\, r)^\alpha M_\mu f(ch\, x), \\
&\quad (sh\, r)^{\alpha-\frac{2\lambda+1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} + (sh\, r)^\alpha M_\mu f(ch\, x) \Big\}, \quad r>0. \qquad (3.8)
\end{aligned}
$$

The right-hand side attains its minimum at

$$sh\, r = \Big(\frac{2\lambda+1-p\alpha}{p\alpha} \frac{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}}{M_\mu f(ch\, x)} \Big)^{\frac{p}{2\lambda+1}} \qquad (3.9)$$

and

$$sh\, r = \Big(\frac{2\lambda+1-\gamma-p\alpha}{p\alpha} \frac{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}}{M_\mu f(ch\, x)} \Big)^{\frac{p}{2\lambda+1-\gamma}}. \qquad (3.10)$$

Taking into account (3.9) and (3.10) in (3.8) we obtain

$$|I_G^\alpha f(ch\, x)| \lesssim \min \left\{ \Big(\frac{M_\mu f(ch\, x)}{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}} \Big)^{1-\frac{p\alpha}{2\lambda+1}}, \Big(\frac{M_\mu f(ch\, x)}{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}} \Big)^{1-\frac{p\alpha}{2\lambda+1-\gamma}} \right\} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Then

$$|I_G^\alpha f(ch\, x)| \lesssim (M_\mu f(ch\, x))^{\frac{p}{q}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{1-\frac{p}{q}}.$$

Hence, by Theorem 3.1, we have

$$
\begin{aligned}
\int_{H(x,r)} |I_G^\alpha f(ch\, x)|^q sh^{2\lambda} t dt &\lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^{q-p} \int_{H(x,r)} (M_\mu f(ch\, t))^p sh^{2\lambda} t dt \\
&\lesssim [r]_1^\gamma \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}^q.
\end{aligned}
$$

From this it follows that

$$\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

i.e., I_G^α is bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

Necessity. Let $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$, $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and I_G^α be bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

Let the function $f(ch\,x)$ be non-negative and monotonically increasing on $\mathbb{R}_+$. The delates function $f_t(ch\,x)$ is defined as follows

$$\begin{cases} f(ch(th\,t)\,x) \le f_t(ch\,x) \le f(ch(cth\,t)\,x), & 0 < t < 1, \\ f(ch(th\,t)\,x) \le f_t(ch\,x) \le f(ch(sh\,t)\,x), & 1 \le t < \infty. \end{cases} \tag{3.11}$$

We suppose $[t]_{1,+} = \max\{1, t\}$.

From (3.11) we have at $0 < t < 1$

$$\begin{aligned}
\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \Big([r]_1^{-\gamma} \int\limits_{H(x,r)} |f_t(ch\,y)|^p sh^{2\lambda} y dy \Big)^{\frac{1}{p}} \\
&\le \sup_{x\in\mathbb{R}_+, r>0} \Big([r]_1^{-\gamma} \int\limits_{H(x,r)} |f(ch(cth\,t)\,y)|^p sh^{2\lambda} y dy \Big)^{\frac{1}{p}} \quad [(cth\,t)y = u, \quad dy = (th\,t)du] \\
&= (th\,t)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Big([r]_1^{-\gamma} \int\limits_{H(xcth\,t, rcth\,t)} |f(ch\,u)|^p sh^{2\lambda} (th\,t) u du \Big)^{\frac{1}{p}} \\
&\le (th\,t)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Big([r]_1^{-\gamma} \int\limits_{H(xcth\,t, rcth\,t)} |f(ch\,u)|^p sh^{2\lambda} u du \Big)^{\frac{1}{p}} \\
&= (sh\,t)^{\frac{2\lambda+1}{p}} \sup_{r>0} \Big(\frac{[r\ cth\,t]_1}{[r]_1} \Big)^{\frac{\gamma}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Big([r\ cth\,t]_1^{-\gamma} \int\limits_{H(xcth\,t, rcth\,t)} |f(ch\,u)|^p sh^{2\lambda} u du \Big)^{\frac{1}{p}} \\
&= (th\,t)^{\frac{2\lambda+1}{p}} [cth\,t]_{1,+}^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le (th\,t)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&= \Big(\frac{sh\,t}{ch\,t} \Big)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \lesssim \frac{1}{(ch\,t)^{\frac{2\lambda+1-\gamma}{p} - \alpha}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&\lesssim (sh\,t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.
\end{aligned} \tag{3.12}$$

On the other hand

$$\begin{aligned}
\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \Big([r]_1^{-\gamma} \int\limits_{H(x,r)} |f_t(ch\,y)|^p sh^{2\lambda} y dy \Big)^{\frac{1}{p}} \\
&\ge \sup_{x\in\mathbb{R}_+, r>0} \Big([r]_1^{-\gamma} \int\limits_{H(x,r)} |f(ch(th\,t)y)|^p sh^{2\lambda} y dy \Big)^{\frac{1}{p}} \quad [(th\,t)y = u, \quad dy = (cth\,t)du] \\
&= (cth\,t)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Big([r]_1^{-\gamma} \int\limits_{H(xth\,t, rth\,t)} |f(ch\,u)|^p sh^{2\lambda} (cth\,t) u du \Big)^{\frac{1}{p}} \\
&\ge (cth\,t)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \Big([r]_1^{-\gamma} \int\limits_{H(xth\,t, rth\,t)} |f(ch\,u)|^p sh^{2\lambda} u du \Big)^{\frac{1}{p}} \\
&= (cth\,t)^{\frac{2\lambda+1}{p}} \Big(\sup_{r>0} \frac{[rth\,t]_1}{[r]_1} \Big)^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&= (cth\,t)^{\frac{2\lambda+1}{p}} [th\,t]_1^{\frac{\gamma}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \ge (cth\,t)^{\frac{2\lambda+1}{p} - \frac{\gamma}{p} - \alpha} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&= (cth\,t)^{\frac{2\lambda+1-\gamma}{p} - \alpha} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \ge (sh\,t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.
\end{aligned} \tag{3.13}$$

Now, let $1 \le t < \infty$, then from (3.11) we obtain

$$\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|f_t(ch\,y)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$\ge \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|f(ch(cth\,t)\,y)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}} \quad [(th\,t)y=u, \quad dy=(cth\,t)du]$$

$$= (cth\,t)^{\frac{1}{p}}\sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(xth\,t, rth\,t)}|f(ch\,u)|^p sh^{2\lambda}(cth\,t)udu\right)^{\frac{1}{p}}$$

$$\ge (cth\,t)^{\frac{2\lambda+1}{p}}\sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(xth\,t, rth\,t)}|f(ch\,u)|^p sh^{2\lambda}udu\right)^{\frac{1}{p}}$$

$$= (cth\,t)^{\frac{2\lambda+1}{p}}\sup_{r>0}\left(\frac{[rth\,t]_1}{[r]_1}\right)^{\frac{\gamma}{p}}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$

$$= (cth\,t)^{\frac{2\lambda+1}{p}}[th\,t]_1^{\gamma}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \ge (cth\,t)^{\frac{2\lambda+1-\gamma}{p}-\alpha}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$

$$\ge (sh\,t)^{\alpha+\frac{\gamma-2\lambda-1}{p}}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.14}$$

On the other hand

$$\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|f_t(ch\,y)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}}$$

$$\le \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|f(ch(sh\,t)y)|^p sh^{2\lambda}ydy\right)^{\frac{1}{p}} \quad \left[(sh\,t)y=u, \quad dy=\frac{du}{sh\,t}\right]$$

$$= (sh\,t)^{-\frac{1}{p}}\sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(xsh\,t, rsh\,t)}|f(ch\,u)|^p sh^{2\lambda}\frac{u}{sh\,t}du\right)^{\frac{1}{p}}$$

$$\le (sh\,t)^{-\frac{2\lambda+1}{p}}\sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(xsh\,t, rsh\,t)}|f(ch\,u)|^p sh^{2\lambda}udu\right)^{\frac{1}{p}}$$

$$= (sh\,t)^{-\frac{2\lambda+1}{p}}\left(\sup_{r>0}\frac{[rsh\,t]_1}{[r]_1}\right)^{\frac{\gamma}{p}}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$

$$= (sh\,t)^{-\frac{2\lambda+1}{p}}[sh\,t]_{1,+}^{\frac{\gamma}{p}}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le (sh\,t)^{\alpha+\frac{\gamma-2\lambda-1}{p}}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.15}$$

From (3.12)–(3.15) for all $0 < t < \infty$ we obtain

$$\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} \approx (sh\,t)^{\alpha+\frac{\gamma-2\lambda-1}{p}}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{3.16}$$

According to the define of G-potential we can write

$$I_G^{\alpha}f_t(ch\,x) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^{\infty}\left(\int\limits_0^{\infty}u^{\frac{\alpha}{2}-1}h_u(chv)du\right)A_{chv}f_t(ch\,x)sh^{2\lambda}vdv.$$

From this and (3.11) for $0 < t < 1$ we have

$$\|I_G^{\alpha}f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)}|I_G^{\alpha}f_t(ch\,y)|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\le \sup_{x\in\mathbb{R}_+,r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)} |I_G^{\alpha}f(ch(cth\,t)\,y)|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$[(cth\,t)y = z, \quad dy = (th\,t)dz]$

$$= (th\,t)^{\frac{1}{q}}\sup_{x\in\mathbb{R}_+,r>0}\left([r]_1^{-\gamma}\int\limits_{H(xcth\,t,rcth\,t)} |I_G^{\alpha}f(ch\,z)|^q sh^{2\lambda}(th\,t)zdz\right)^{\frac{1}{q}}$$

$$\le (th\,t)^{\frac{2\lambda+1}{q}}\sup_{x\in\mathbb{R}_+,r>0}\left([r]_1^{-\gamma}\int\limits_{H(xcth\,t,rcth\,t)} |I_G^{\alpha}f(ch\,z)|^q sh^{2\lambda}zdz\right)^{\frac{1}{q}}$$

$$= (th\,t)^{\frac{2\lambda+1}{q}}\left(\sup_{r>0}\frac{[rcth\,t]_1}{[r]_1}\right)^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$= (th\,t)^{\frac{2\lambda+1}{q}}[cth\,t]_1^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\le (cth\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}} \le (sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.17}$$

On the other hand

$$\|I_G^{\alpha}f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+,r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)} |I_G^{\alpha}f_t(ch\,y)|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\ge \sup_{x\in\mathbb{R}_+,r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)} |I_G^{\alpha}f(ch(th\,t)y)|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$[(th\,t)y = z, \quad dy = (cth\,t)dz]$

$$= (cth\,t)^{\frac{1}{q}}\sup_{x\in\mathbb{R}_+,r>0}\left([r]_1^{-\gamma}\int\limits_{H(xth\,t,rth\,t)} |I_G^{\alpha}f(ch\,z)|^q sh^{2\lambda}(cth\,t)zdz\right)^{\frac{1}{q}}$$

$$\ge (cth\,t)^{\frac{2\lambda+1}{q}}\left(\sup_{r>0}\frac{[rth\,t]_1}{[r]_1}\right)^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\ge (cth\,t)^{\frac{2\lambda+1}{q}}[th\,t]_1^{\gamma}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\ge \left(\frac{ch\,t}{sh\,t}\right)^{\frac{2\lambda+1-\gamma}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}} \ge (sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.18}$$

Combining (3.17) and (3.18) we obtain

$$\|I_G^{\alpha}f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \approx (sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}, \quad 0 < t < 1. \tag{3.19}$$

Now we consider the case, then $1 \le t < \infty$. From (3.11) we have

$$\|I_G^{\alpha}f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \ge \sup_{x\in\mathbb{R}_+,r>0}\left([r]_1^{-\gamma}\int\limits_{H(x,r)} |I_G^{\alpha}f(ch((th\,t)y))|^q sh^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$[(th\,t)y = z, \quad dy = (cth\,t)dz]$

$$= (cth\,t)^{\frac{1}{q}}\sup_{x\in\mathbb{R}_+,r>0}\left([r]_1^{-\gamma}\int\limits_{H(xth\,t,rth\,t)} |I_G^{\alpha}f(ch\,z)|^q sh^{2\lambda}(cth\,t)zdz\right)^{\frac{1}{q}}$$

$$\ge (cth\,t)^{\frac{2\lambda+1}{q}}\left(\sup_{r>0}\frac{[r\,th\,t]_1}{[r]_1}\right)^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\ge (cth\,t)^{\frac{2\lambda+1}{q}}[th\,t]_1^{\gamma}\|I_G^{\alpha}f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\geq \left(\frac{ch\,t}{sh\,t}\right)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \geq (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.20}$$

On the other hand

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \leq \sup_{x\in\mathbb{R}_+,r>0} \left([r]_1^{-\gamma} \int\limits_{H(x,r)} |I_G^\alpha f(ch(sh\,t)y)|^q sh^{2\lambda} y dy\right)^{\frac{1}{q}}$$

$$\left[(sh\,t)y = z, \quad dy = \frac{dz}{sh\,t}\right]$$

$$= (sh\,t)^{-\frac{1}{q}} \sup_{x\in\mathbb{R}_+,r>0} \left([r]_1^{-\gamma} \int\limits_{H(xsh\,t,rsh\,t)} |I_G^\alpha f(ch\,z)|^q sh^{2\lambda}\left(\frac{z}{sh\,t}\right) dz\right)^{\frac{1}{q}}$$

$$= (sh\,t)^{-\frac{2\lambda+1}{q}} \sup_{x\in\mathbb{R}_+,r>0} \left([r]_1^{-\gamma} \int\limits_{H(xsh\,t,r\,sh\,t)} |I_G^\alpha f(ch\,z)|^q sh^{2\lambda} dz\right)^{\frac{1}{q}}$$

$$= (sh\,t)^{-\frac{2\lambda+1}{q}} \left(\sup_{r>0} \frac{[r\,sh\,t]_1}{[r]_1}\right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\leq (sh\,t)^{-\frac{2\lambda+1}{q}} [th\,t]_{1,+}^\gamma \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}$$

$$\leq (sh\,t)^{-\frac{2\lambda+1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \leq (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.21}$$

From (3.20) and (3.21) it follows that

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \approx (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}, \quad 1 \leq t < \infty. \tag{3.22}$$

Now from (3.19) and (3.22) we have

$$\|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \approx (sh\,t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}}, \quad 0 < t < \infty. \tag{3.23}$$

Since I_G^α is bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$, i.e.

$$\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} \lesssim \|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

then taking into account (3.23) and (3.16) we obtain

$$\begin{aligned}
\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} &\approx (sh\,t)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} \lesssim (sh\,t)^{\frac{2\lambda+1-\gamma}{q}} \|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&\lesssim (sh\,t)^{\alpha+(\gamma-2\lambda-1)(\frac{1}{p}-\frac{1}{q})} \|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} \\
&\lesssim \begin{cases} (sh\,t)^{\alpha-(2\lambda+1)(\frac{1}{p}-\frac{1}{q})}, & 0 < t < 1, \\ (sh\,t)^{\alpha+(\gamma-2\lambda-1)(\frac{1}{p}-\frac{1}{q})}, & 0 \leq t < \infty. \end{cases}
\end{aligned}$$

If $\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{2\lambda+1}$, then in the case $t \to 0$ we have $\|I_G^\alpha f\|_{\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.
As well as if $\frac{1}{p} - \frac{1}{q} > \frac{\alpha}{2\lambda+1-\gamma}$, then $t \to \infty$ we obtain $\|I_G^\alpha f\|_{\widetilde{L}_{p,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$.
Therefore $\frac{\alpha}{2\lambda+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{2\lambda+1-\gamma}$.

2) *Sufficiency.* Let $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$, then

$$\begin{aligned}
&|\{y \in H(x,r) : |I_G^\alpha f(ch\,y)| > 2\beta\}|_\gamma \\
&\leq |\{y \in H(x,r) : A_1(y,r) > \beta\}|_\gamma + |\{y \in H(x,r) : A_2(y,r) > \beta\}|_\gamma.
\end{aligned}$$

Also

$$A_2(y,r) = \int\limits_r^\infty A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1} |f(ch\,t)| sh^{2\lambda} t dt$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} (A_{ch\,t}|f(ch\,x)|)(sh\,t)^{\alpha-2\lambda-1} sh^{2\lambda} t dt \\
&\leq \|A_{ch\,t} f\|_{\widetilde{L}_{1,\lambda,\gamma}} \sum_{j=0}^{\infty} \frac{[2^{j+1}r]_1^{\gamma}}{(2^j r)^{2\lambda+1-\alpha}} \\
&= r^{\alpha-2\lambda-1} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \begin{cases} (2r)^{\gamma} \sum\limits_{j=0}^{\left[\log_2 \frac{1}{2r}\right]} 2^{(\alpha+\gamma-2\lambda-1)j} + \sum\limits_{j=\left[\log_2 \frac{1}{2r}\right]+1}^{\infty} 2^{(\alpha-2\lambda-1)j}, & 0<r<\frac{1}{2}, \\ \sum\limits_{j=0}^{\infty} 2^{(\alpha-2\lambda-1)j}, & r \geq \frac{1}{2} \end{cases} \\
&\lesssim r^{\alpha-2\lambda-1} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \begin{cases} r^{\gamma} + r^{2\lambda+1-\alpha}, & 0<r<\frac{1}{2}, \\ 1, & r \geq \frac{1}{2} \end{cases} \\
&\lesssim \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \begin{cases} r^{\alpha+\gamma-2\lambda-1}, & 0<r<\frac{1}{2}, \\ r^{\alpha-2\lambda-1}, & r \geq \frac{1}{2}. \end{cases} \lesssim [2r]^{\gamma} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}. \qquad (3.24)
\end{aligned}
$$

Taking into account the inequality (3.2) and Theorem B we obtain at $0<r<1$

$$
\begin{aligned}
&\left|\left\{y \in H(x,r) : A_1(y,r) > \beta\right\}\right|_{\gamma} \\
&\lesssim \left|\left\{y \in H(x,r) : M_{\mu} f(ch\,y) > \frac{\beta}{C\ sh^{\alpha} r}\right\}\right|_{\gamma} \lesssim \frac{sh^{\alpha} r}{\beta} [r]_1^{\gamma} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}. \qquad (3.25)
\end{aligned}
$$

And from (3.3) and Theorem B we have at $1 \leq r < \infty$

$$
\begin{aligned}
&\left|\left\{y \in H(x,r) : A_1(y,r) > \beta\right\}\right|_{\gamma} \\
&\lesssim \left|\left\{y \in H(x,r) : M_{\mu} f(ch\,y) > \frac{\beta}{C(sh\,r)^{\alpha}}\right\}\right|_{\gamma} \lesssim \frac{(sh\,r)^{\alpha}}{\beta} [r]_1^{\gamma} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}. \qquad (3.26)
\end{aligned}
$$

From (3.25) and (3.26) we obtain, that for all $0<r<\infty$

$$
\left|\left\{y \in H(x,r) : A_1(y,r) > \beta\right\}\right|_{\gamma} \lesssim \frac{(sh\,r)^{\alpha}}{\beta} [r]_1^{\gamma} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}. \qquad (3.27)
$$

If $[2r]_1^{\gamma}(sh\,r)^{\alpha-2\lambda-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}} = \beta$, then from (3.24) we obtain that $|A_2(y,r)| \lesssim \beta$ and consequently, $|\{y \in H(x,r) : A_2(y,r) > \beta\}|_{\gamma} = 0$. Then by $2r<1$, $\beta = (sh\,r)^{\gamma+\alpha-2\lambda-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}$ and from (3.27) we have

$$
\begin{aligned}
&\left|\left\{y \in H(x,r) : |I_G^{\alpha} f(ch\,y)| > 2\beta\right\}\right|_{\gamma} \lesssim \frac{(sh\,r)^{\alpha}}{\beta} [r]_1^{\gamma} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \\
&= (sh\,r)^{2\lambda-1-\gamma} [r]_1^{\gamma} = \left(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1-\gamma}{2\lambda+1-\gamma-\alpha}} [r]_1^{\gamma}. \qquad (3.28)
\end{aligned}
$$

And for $2r \geq 1$, $\beta = (sh\,r)^{\alpha-2\lambda-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}$ and from (3.26) we have

$$
\begin{aligned}
&\left|\left\{y \in H(x,r) : |I_G^{\alpha} f(ch\,y)| > 2\beta\right\}\right|_{\gamma} \lesssim \frac{(sh\,r)^{\alpha}}{\beta} [r]_1^{\gamma} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \\
&= [r]_1^{\gamma} (sh\,r)^{2\lambda+1} = \left(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1}{2\lambda+1-\alpha}} [r]_1^{\gamma}. \qquad (3.29)
\end{aligned}
$$

Finally from (3.28) and (3.29) we have

$$
\begin{aligned}
&\left|\left\{y \in H(x,r) : |I_G^{\alpha} f(ch\,y)| > 2\beta\right\}\right|_{\gamma} \\
&\lesssim [r]_1^{\gamma} \min\left\{\left(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1}{2\lambda+1-\alpha}}, \left(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1-\gamma}{2\lambda+1-\gamma-\alpha}}\right\}
\end{aligned}
$$

$$\lesssim [r]_1^{\gamma}\Big(\beta^{-1}\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}\Big)^q,$$

where by condition of the theorem

$$\frac{2\lambda+1}{2\lambda+1-\alpha}\le q\le\frac{2\lambda+1-\gamma}{2\lambda+1-\gamma-\alpha}\Leftrightarrow\frac{\alpha}{2\lambda+1}\le 1-\frac{1}{q}\le\frac{\alpha}{2\lambda+1-\gamma}.$$

Necessity. Preliminarily we established the estimates for $\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$. From (3.11) for $0<t<1$ we have

$$\|I_G^{\alpha}f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}}\ge\sup_{r>0}r\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(x,u):|I_G^{\alpha}f(ch(th\,t)y)|>r\}}sh^{2\lambda}zdz\Big)^{\frac{1}{q}}$$

$$[(th\,t)y=z,\ dy=(cth\,t)dz]$$

$$=(cth\,t)^{\frac{1}{q}}\sup_{r>0}r\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(xth\,t,uth\,t):|I_G^{\alpha}f(ch\,z)|>rth\,t\}}sh^{2\lambda}(cth\,t)zdz\Big)^{\frac{1}{q}}$$

$$=(cth\,t)^{\frac{1}{q}}\sup_{u>0}\Big(\frac{[uth\,t]_1}{[u]_1}\Big)^{\frac{\gamma}{q}}\sup_{r>0}rth\,t$$

$$\times\sup_{x\in\mathbb{R}_+,u>0}\Big([uth\,t]_1^{-\gamma}\int\limits_{\{y\in H(xth\,t,uth\,t):|I_G^{\alpha}f(ch\,z)|>rth\,t\}}sh^{2\lambda}(cth\,t)zdz\Big)^{\frac{1}{q}}$$

$$\ge(cth\,t)^{\frac{2\lambda+1}{q}}[th\,t]_1^{\frac{\gamma}{q}}$$

$$\times\sup_{r>0}rth\,t\sup_{x\in\mathbb{R}_+,u>0}\Big([uth\,t]_1^{1-\gamma-2\lambda}\int\limits_{\{y\in H(xth\,t,uth\,t):|I_G^{\alpha}f(ch\,z)|>rth\,t\}}sh^{2\lambda}sh^{2\lambda}zdz\Big)^{\frac{1}{q}}$$

$$\ge(th\,t)^{-\frac{2\lambda+1}{q}}[th\,t]_1^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$\ge(th\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}\ge(sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}.$$

On the other hand from (3.11) we have

$$\|I_G^{\alpha}f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}}\le\sup_{r>0}\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(x,u):|I_G^{\alpha}f(ch(cth\,t)\,y)|>r\}}sh^{2\lambda}ydy\Big)^{\frac{1}{q}}$$

$$[(cth\,t)y=z,\ dy=(th\,t)dz]$$

$$=(th\,t)^{\frac{1}{q}}\sup_{r>0}\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(xcth\,t,u\,cth\,t):|I_G^{\alpha}f(ch\,z)|>r\}}sh^{2\lambda}(th\,t)zdz\Big)^{\frac{1}{q}}$$

$$\le(th\,t)^{\frac{2\lambda+1}{q}}\sup_{u>0}\Big(\frac{[u\,cth\,t]_1}{[u]_1}\Big)^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$=(th\,t)^{\frac{2\lambda+1}{q}}[cth\,t]_{1,+}^{\frac{\gamma}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$\le(th\,t)^{\frac{2\lambda+1-\gamma}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}\le(sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.30}$$

From (3.34) and (3.30) it follows that

$$\|I_G^{\alpha}f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}}\lesssim(sh\,t)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^{\alpha}f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.31}$$

Now we consider the case then $1\le t<\infty$. From (3.11) we have

$$\|I_G^{\alpha}f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}}\ge\sup_{r>0}r\sup_{x\in\mathbb{R}_+,u>0}\Big([u]_1^{-\gamma}\int\limits_{\{y\in H(x,u):|I_G^{\alpha}f(ch(th\,t)y)|>r\}}sh^{2\lambda}ydy\Big)^{\frac{1}{q}}$$

$$[(th\,t)y=z,\ dy=(cth\,t)dz]$$

$$= (cth\, t)^{\frac{1}{q}} \sup_{r>0} \sup_{x\in\mathbb{R}_+, u>0} \Bigg([u]_1^{-\gamma} \int\limits_{\{y\in H(x\, th\, t, u\, th\, t): |I_G^\alpha f(ch\, z)|>r\}} sh^{2\lambda}(cth\, t) z dz \Bigg)^{\frac{1}{q}}$$

$$= (cth\, t)^{\frac{2\lambda+1}{q}} \sup_{r>0} r\, th\, t \sup_{x\in\mathbb{R}_+, u>0} \Bigg([u]_1^{-\gamma} \int\limits_{\{y\in H(x\, th\, t, u\, th\, t): |I_G^\alpha f(ch\, z)|>r\, th\, t\}} sh^{2\lambda} z dz \Bigg)^{\frac{1}{q}}$$

$$= (cth\, t)^{\frac{2\lambda+1}{q}} \sup_{r>0} \Big(\frac{[u\, th\, t]_1}{[u]_1} \Big)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$= (cth\, t)^{\frac{2\lambda+1}{q}} [th\, t]_1^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$\geq (th\, t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} \geq (sh\, t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}.$$

On the other hand from (3.11) we get

$$\|I_G^\alpha f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}} \leq \sup_{r>0} \sup_{x\in\mathbb{R}_+, u>0} \Bigg([u]_1^{-\gamma} \int\limits_{\{y\in H(x,u): |I_G^\alpha f(ch(sh\, t)y)|>r\}} sh^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\Big[(sh\, t) y = z,\ dy = \frac{dz}{sh\, t}\ \Big]$$

$$= (sh\, t)^{-\frac{1}{q}} \sup_{r>0} r \sup_{x\in\mathbb{R}_+, u>0} \Bigg([u]_1^{1-\gamma-2\lambda} \int\limits_{\{y\in H(xsh\, t, ush\, t): |I_G^\alpha f(ch\, z)|>r\}} sh^{2\lambda} \frac{z}{sh\, t} dz \Bigg)^{\frac{1}{q}}$$

$$\leq (sh\, t)^{-\frac{2\lambda+1}{q}} \sup_{r>0} r\, sh\, t \sup_{x\in\mathbb{R}_+, u>0} \Bigg([u]_1^{1-\gamma-2\lambda} \int\limits_{\{y\in H(xsh\, t, ush\, t): |I_G^\alpha f(ch\, z)|>r\, sh\, t\}} sh^{2\lambda} z dz \Bigg)^{\frac{1}{q}}$$

$$= (sh\, t)^{-\frac{2\lambda+1}{q}} \sup_{u>0} \Big(\frac{[ush\, t]_1}{[u]_1} \Big)^{\frac{\gamma+2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$\leq (sh\, t)^{-\frac{2\lambda+1}{q}} [sh\, t]_{1,+}^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}$$

$$\leq (sh\, t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.32}$$

From (3.31) and (3.32) for $1 \leq t < \infty$ we have

$$\|I_G^\alpha f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}} \lesssim (sh\, t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.33}$$

Combining (3.31) and (3.33) for all $0 < t < \infty$ we obtain

$$\|I_G^\alpha f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}} \approx (sh\, t)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}}. \tag{3.34}$$

From the boundedness I_G^α from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ to $W\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ and from (3.16) and (3.34) we have

$$\begin{aligned}
\|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} &\lesssim (sh\, t)^{\frac{2\lambda+1-\gamma}{q}} (sh\, t)^{\alpha+\gamma-2\lambda-1} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \\
&\lesssim (sh\, t)^{\alpha+(2\lambda-1)(1-\frac{1}{q})} (sh\, t)^{\gamma(1-\frac{1}{q})} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \\
&\lesssim \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} \begin{cases} (sh\, t)^{\alpha-(2\lambda+1)(1-\frac{1}{q})}, & 0 < t < 1, \\ (sh\, t)^{\alpha+(\gamma-2\lambda-1)(1-\frac{1}{q})}, & 1 \leq t < \infty. \end{cases}
\end{aligned}$$

If $1 - \frac{1}{q} < \frac{\alpha}{2\lambda+1}$, then for $t \to 0$ we have $\|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$.

Similarly, if $1 - \frac{1}{q} > \frac{\alpha}{2\lambda+1}$, then for $t \to \infty$ we obtain $\|I_G^\alpha f\|_{W\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$.

Therefore, $\frac{\alpha}{2\lambda+1} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{2\lambda+1-\gamma}$. □

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[1,2]Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail address: vagif@guliyev.com

[1]Dumlupinar University, Department of Mathematics, 43100 Kutahya, Turkey
E-mail address: elmanibrahimov@yahoo.com

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ASYMPTOTIC AND QUALITATIVE ANALYSIS OF THE MULTIDIMENSIONAL GKP AND 3-DNLS EQUATION SOLUTIONS FOR THEIR CLASSIFICATION

O. A. KHARSHILADZE[1] AND V. YU. BELASHOV[2]

Abstract. In this paper, basing on our results obtained by us earlier, we consider an approach to study of structure of possible multidimensional solutions of the Belashov-Karpman (BK) system which includes as partial cases the generalized Kadomtsev–Petviashvili (GKP) equation and the 3D derivative nonlinear Schrödinger (3-DNLS) equation. For the GKP equation with due account of the arbitrary nonlinearity exponent we study the solutions asymptotes along the direction of the wave propagation. The problem of the asymptotic behavior of the solutions of the 3-DNLS equation along the direction of the wave propagation is more simple one because we can write at once its exact solutions in the explicit form in one-dimensional approximation on the basis of the results known earlier. We also present some considerations on constructing of the phase-plane portraits in the 8-dimensional phase space for the GKP equation on the basis of the results of qualitatively analysis of the generalized equations of the KdV-class.

1. Basic Equations

In this paper we study the types and structure of possible multidimensional solitary waves forming on the low-frequency branch of oscillations in fluids and plasma which are described by the Belashov-Karpman (BK) class of equations [4]

$$\partial_t u + A(t,u)u = f, \quad f = \kappa \int\limits_{-\infty}^{x} \Delta_\perp u dx, \quad \Delta_\perp = \partial_y^2 + \partial_z^2 \tag{1}$$

which with the operator

$$A(t,u) = \alpha u \partial_x - \partial_x^2(\mu - \beta\partial_x - \delta\partial_x^2 - \gamma\partial_x^3) \tag{2}$$

turns into the generalized Kadomtsev-Petviashvili (GKP) equation [10], and in the case, when operator

$$A(t,u) = 3s|p|^2 u^2 \partial_x - \partial_x^2(i\lambda + \nu) \tag{3}$$

eq. (1) turns into the 3-dimensional derivative nonlinear Schrödinger (3-DNLS) equation, where $p = (1 + ie)$, and e is “an eccentricity” of the polarization ellipse of the wave [4, 10]. The upper and lower signs of $\lambda = \pm 1$ correspond to the right and left circularly polarized wave, respectively; sign of nonlinearity is accounted by coefficient $s = \mathrm{sgn}(1 - p) = \pm 1$ in nonlinear term.

The sets of equations (1), (2) and (1), (3) are not completely integrable ones, and a problem of existence of multidimensional stable soliton solutions and their structure requires especial investigation. In [6] we studied the problem of stability of possible multidimensional solutions for two particular cases of the BK system mentioned above. Here, we investigate the structure of possible solutions of the sets of equations (1), (2) and (1), (3) using the methods of both qualitative and asymptotic analysis basing on the results for the generalized equations of the KdV-class obtained in [9].

Consider at first the GKP equation and then discuss the analogous problem for the 3-DNLS equation. Let us write the GKP equation in form [10]:

$$\partial_\eta(\partial_t u + \alpha u \partial_\eta u - \mu\partial_\eta^2 u + \beta\partial_\eta^3 u + \delta\partial_\eta^4 u + \gamma\partial_\eta^5 u) = \kappa\Delta_\perp u \tag{4}$$
$$\Delta_\perp = \partial_{\zeta_1}^2 + \partial_{\zeta_2}^2,$$

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where ζ_1 and ζ_2 are the transverse coordinates. At $\mu = \delta = \gamma = 0$ equation (4) is the classic KP equation which is the completely integrable Hamiltonian system and has in case $\Delta_\perp = \partial^2_{\zeta_1}$ the solutions in form of the 1-dimensional (for $\beta\kappa < 0$) or 2-dimensional (for $\beta\kappa > 0$) solitons (see [10]). The structure and the dynamics of the solutions of model (4) nonintegrable analytically in case $\delta = 0$ has been investigated in detail in [2, 7] where it was shown that at $\mu = 0$ in dependence on the signs of coefficients β, γ and κ the 2-dimensional and 3-dimensional soliton type solutions with the monotonous or oscillatory asymptotics can take place which at the presence of the "viscous-type" dissipation in the medium ($\mu > 0$) lose their symmetry and damp with evolution (see [10] for details). In [9] with use of the methods of the both asymptotic and qualitative analysis the asymptotics of the one-dimensional analogue of equation (4) were studied in detail and the sufficiently complete classification of its solutions including the solutions of the both soliton and non-soliton type were constructed.

Now, our purpose is the generalization of the results obtained in [9] with due account of the results presented in [5, 7, 8] to the multidimensional cases.

For the avoidance of the unhandiness of obtaining expressions let us consider the equation (4) in the 2-dimensional form supposing that $\Delta_\perp = \delta^2_{\zeta_1}$. Generalization of using technique and the results obtaining at that to a case $\Delta_\perp = \delta^2_{\zeta_1} + \delta^2_{\zeta_2}$ is rather trivial [3]. Let us assume that $\zeta_1 \equiv \zeta$ and take for the distinctness that $\alpha = 6$ (that can be obtained easily using the scaling transformation $u \to (6/\alpha)u$ in the equation).

Let us introduce new variables, $\bar\eta = \eta + \zeta$, $\bar\zeta = \eta - \zeta$. As a result, changing the variables η and ζ in equation (4) at first by way of $\bar\eta$ and then by way of $\bar\zeta$, we obtain the pair of one-dimensional equations:

$$\begin{aligned}\partial_{\bar\eta}(\partial_t u + 6u\partial_{\bar\eta} u - \mu\partial^2_{\bar\eta} u + \beta\partial^3_{\bar\eta} u + \delta\partial^4_{\bar\eta} u + \gamma\partial^5_{\bar\eta} u) &= \kappa\partial^2_{\bar\eta} u,\\ \partial_{\bar\zeta}(\partial_t u + 6u\partial_{\bar\zeta} u - \mu\partial^2_{\bar\zeta} u + \beta\partial^3_{\bar\zeta} u + \delta\partial^4_{\bar\zeta} u + \gamma\partial^5_{\bar\zeta} u) &= \kappa\partial^2_{\bar\zeta} u\end{aligned} \tag{5}$$

writing in the co-ordinates with axes $\bar\eta$ and $-\bar\zeta$ rotated relative to axes η and ζ at the angle $+45°$. Representation (5) means in fact that the starting equation (4) admit two types of the 1-dimensional solutions, $u(\bar\eta, t)$ and $u(\bar\zeta, t)$, satisfying the first and second equations of set (5), respectively. But, at this, it is necessary to have in view that "1-dimensionality" of these solutions nevertheless implicity assumes the linear dependence of each either of the new variables $\bar\eta$ and $\bar\zeta$ on both coordinates, η and ζ.

Integrating equations (5) on $\bar\eta$, $-\bar\zeta$, respectively, we obtain two equiform the generalized KdV equations

$$\begin{aligned}\partial_t u + (-\kappa + 6u)\partial_{\bar\eta} u - \mu\partial^2_{\bar\eta} u + \beta\partial^3_{\bar\eta} u + \delta\partial^4_{\bar\eta} u + \gamma\partial^5_{\bar\eta} u &= 0,\\ \partial_t u + (-\kappa + 6u)\partial_{\bar\zeta} u - \mu\partial^2_{\bar\zeta} u + \beta\partial^3_{\bar\zeta} u + \delta\partial^4_{\bar\zeta} u + \gamma\partial^5_{\bar\zeta} u &= 0\end{aligned} \tag{6}$$

connected with each other by way of the change of the coordinates made above. Now, passing into the coordinates moving along the corresponding axis with velocity $-k$, i.e. making the change $\eta' = \bar\eta + \kappa t$, $\zeta' = \bar\zeta + \kappa t$ in equations (6) and leaving out the strokes, let us write equations (6) in the standard form:

$$\begin{aligned}\partial_t u + 6u\partial_\eta u - \mu\partial^2_\eta u + \beta\partial^3_\eta u + \delta\partial^4_\eta u + \gamma\partial^5_\eta u &= 0,\\ \partial_t u + 6u\partial_\zeta u - \mu\partial^2_\zeta u + \beta\partial^3_\zeta u + \delta\partial^4_\zeta u + \gamma\partial^5_\zeta u &= 0.\end{aligned} \tag{7}$$

So, we can now conduct the analysis for only one generalized equation of the set (6), and then, fulfilling the inverse change of the variables, extend the results to the 2-dimensional solutions $u(\eta, \zeta, t)$ of equation (4) with $\Delta_\perp = \partial^2_\zeta$.

As to the 3-DNLS equation, write it, at first, in the differential form:

$$\partial_\eta[\partial_t h + s\partial_\eta(|h|^2 h) - i\lambda\partial^2_\eta h - \nu\partial^2_\eta h] = \sigma\Delta_\perp h, \quad \Delta_\perp = \partial^2_{\zeta_1} + \partial^2_{\zeta_2}, \tag{8}$$

then supposing for simplification of the statement that $\Delta_\perp = \partial^2_\zeta$ (it is clear that generalization to 3-dimensional case is trivial) and introducing, by analogy with the GKP equation, new variables

$\bar{\eta} = \eta + \zeta$, $\bar{\zeta} = \eta - \zeta$, we also obtain the pair of one-dimensional equations:

$$\partial_{\bar{\eta}}[\partial_t h + s\partial_{\bar{\eta}}(|h|^2 h) - i\lambda\partial^2_{\bar{\eta}} h - \nu\partial^2_{\bar{\eta}} h] = \sigma\partial^2_{\bar{\eta}} h,$$
$$\partial_{\bar{\zeta}}[\partial_t h + s\partial_{\bar{\zeta}}(|h|^2 h) - i\lambda\partial^2_{\bar{\eta}} h - \nu\partial^2_{\bar{\zeta}} h] = \sigma\partial^2_{\bar{\zeta}} h$$

writtten in the coordinates with axes $\bar{\eta}$ and $-\bar{\zeta}$ rotated relative to axes η and ζ at the angle $+45°$. Further transformations give us the set

$$\begin{aligned} &\partial_t h + s\partial_{\eta'}(|h|^2 h) - i\lambda\partial^2_{\eta'} h - \nu\partial^2_{\eta'} h = 0,\\ &\partial_t h + s\partial_{\zeta'}(|h|^2 h) - i\lambda\partial^2_{\zeta'} h - \nu\partial^2_{\zeta'} h = 0 \end{aligned} \tag{9}$$

written in the coordinates $\eta' = \bar{\eta} + \sigma t$, $\zeta' = \bar{\zeta} + \sigma t$ i.e. moving along the corresponding axis with velocity $-\sigma$.

So, as in case of the GKP equation, we can conduct the analysis for only one equation of the set (9) and then, fulfilling the inverse change of the variables, extend the results to the 2-dimensional solutions $h(\eta, \zeta, t)$ of equation (8) with $\Delta_\perp = \partial^2_\zeta$.

2. Generalization of Earlier Obtained Results to Multidimensional Cases

At first, let us consider the generalization of the results obtained in [10] to the equations of the GKP class (4). Following the results presented in ref. [5], consider more general case when the equations (7) were expended by introducing of the arbitrary positive nonlinearity exponent p and, for example, first equation of the set (7) takes the form

$$\partial_t u + 6u^p\partial_\eta u - \mu\partial^2_\eta u + \beta\partial^3_\eta u + \delta\partial^4_\eta u + \gamma\partial^5_\eta u = 0 \tag{10}$$

(see [10] for detail). Remind that in case $\mu = \delta = \gamma = 0$ it is the known KdV equation if $p = 1$, and the modified KdV equation (the MKdV equation) if $p = 2$. Note also that, analogously to the 1-dimensional case, the cases, when in equation (4) with the nonlinear term $6u^p\partial_\eta u$ the nonlinearity exponent $p = 1, 2$, are interesting from physical point of view, and the applications with $p > 2$ are unknown today. But, similarly to the generalized KdV equation considered in [9], in view of that the equations with arbitrary integer $p > 0$ display very largely similar mathematical characteristics, for the elucidation of dependence of the solution parameters on the value of the nonlinearity exponent we will consider the general case for $p > 0$.

With due account of the coefficients' signs, $\mu > 0$, $\delta > 0$ (in accordance with the physical sense of proper terms - see [4, 10] for detail), assuming without a loss of generality as in [5, 7, 8] that $\gamma > 0$, $\beta = \pm 1$ and making substitution $u = Vw$ (where V is a velocity of the wave propagation relatively coordinate axis η and ζ for the first and second equation of set (7), respectively), we can generalize the results obtained in [9] for different signs of V and β to the equations (5) and, accordingly, to equation (4) with $p \geq 1$ in the following way.

1. The value of the nonlinearity exponent p defines a character of dependence $V = f(u)$, namely: for $p > 1$ such dependence for equation (4), as in the 1-dimensional case (see ref. [9]), becomes nonlinear unlike of the known linear one for $p = 1$ (for example, in case of the KP equation). Moreover, for even p the solutions of equation (4) may have both positive and negative pulse direction ($u \gtrless 0$ for either sign of V).

2. In case of the **conservative equations** of class (4) (the cases when $\mu = \delta = 0$) the solutions asymptotics are defined by the following relations:

a) for the cases $V > 0$, $\beta = -1$ and $V < 0$, $\beta = -1$ (upper and lower signs, respectively):

$$w = A_1 \exp\Big\{(2\gamma)^{-1/2}\Big[C^2 + \sqrt{C^4 \pm 4\gamma}\Big]^{1/2}\chi\Big\}, \tag{11}$$

b) for case $V < 0$, $\beta = 1$:

$$\begin{aligned} w = A_2 &\exp\{(2C^{-1}\gamma^{-1/2})^{-1}(2C^{-2}\gamma^{1/2} - 1)^{1/2}\chi\}\\ &\times\cos\{(2C^{-1}\gamma^{-1/2})^{-1}(2C^{-2}\gamma^{1/2} + 1)\chi + \Theta\}, \end{aligned} \tag{12}$$

where A_1, A_2 and Θ are the arbitrary constants, $C = |V|^{-1/4}$, $\chi = (\eta \pm \zeta + (\kappa - V)t)$ (here the signs plus and minus relate to the first and second equations of the set (5), respectively). As one can see from expressions (11), (12)[1], in the solutions $u(\eta, \zeta, t)$ of equation (4) at $\mu = \delta = 0$ the solitons with both monotonous and oscillating asymptotics can take a place dependently on the signs of V and β. (Note that at $\beta = 0$ and any value of $\gamma > 0$ the solutions of the equations (5) with $\mu = \delta = 0$ have form $w = (A_1 + A_2 C^{-1}\chi)\exp(\gamma^{-1/4} C^{-1}\chi)$ and, consequently, also describe the soliton with monotonous asymptotics [7].) Fig. 1 shows the results of numerical simulation of equation (4) for $\mu = \delta = 0$ with the initial condition $u = u_0 \exp(-x^2/l_1^2 - y^2/l_2^2)$, that confirm the results of our asymptotic analysis.

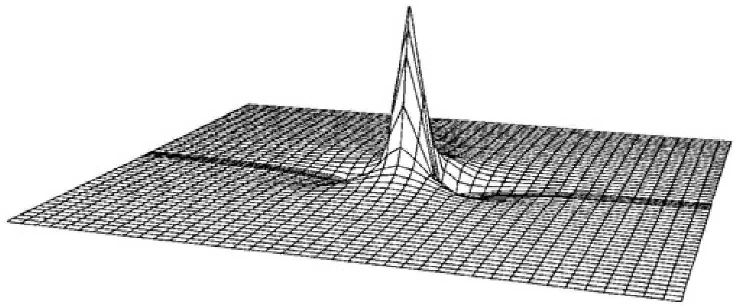

FIGURE 1. General view of a 2-dimensional soliton of eq. (4) with $\Delta_\perp = \partial_y^2$ for $\mu = \delta = 0$, $p = 1$, $\gamma = 1$, $\beta = -0.8$ at $t = 0.2$.

3. In case of the **dissipative equations** of class (4) with the instability (the cases when $\beta = \gamma = 0$) the solutions asymptotics are defined by the following relations:

a) for $\delta > (4/27)\mu^3 C^8$

$$\begin{aligned} w = A_1 \exp[(2\delta C)^{-1/3} Q_1^+ \chi] + \exp[-(16^{\delta C})^{-1/3} Q_1^+ \chi] \\ \times \Big\{ A_2 \cos[\sqrt{3}(16^{\delta C})^{-1/3} Q_1^- \chi + \Theta_1] + A_3 \sin[\sqrt{3}(16^{\delta C})^{-1/3} Q_1^- \chi + \Theta_2] \Big\}, \end{aligned} \tag{13}$$

b) for $\delta = (4/27)\mu^3 C^8$

$$w = A_1 \exp[(\delta C/4)^{-1/3} \chi] + A_2(1 + A_3 \chi) \exp[-(2\delta C)^{-1/3} \chi], \tag{14}$$

c) for $\delta < (4/27)\mu^3 C^8$

$$\begin{aligned} w = A_1 \exp[(\delta C/4)^{-1/3} \operatorname{Re}(Q^\pm)\chi] \\ + A_2 \exp\Big\{ -(2\delta C)^{-1/3} \chi \big[\operatorname{Re}(Q^\pm) - \sqrt{3} |\operatorname{Im}(Q^\pm)| \big] \Big\} \\ + A_3 \exp\Big\{ -(2\delta C)^{-1/3} \chi \big[\operatorname{Re}(Q^\pm) + \sqrt{3} |\operatorname{Im}(Q^\pm)| \big] \Big\} \end{aligned} \tag{15}$$

where A_1, A_2, A_3, Θ_1 and Θ_2 are the arbitrary constants, $Q_1^\pm = Q^+ \pm Q^-$,

$$Q^\pm = \Big[1 \pm \sqrt{1 - 4\mu^3 C^8 / 27\delta} \Big]^{1/3}$$

and $Q^\pm$ is real in the cases (a) and (b) and complex in case (c).

It is easy to see from formulae (13) – (15) that the solutions $u(\eta, \zeta, t)$ of equation (4) have the oscillating asymptotics in case (a) and the exponential ones in the cases (b) and (c). Fig. 2 shows the numerical solutions of equation (4) corresponding to the cases (c) and (a), respectively, obtained for the initial condition $u = u_0 \exp(-x^2/l_1^2 - y^2/l_2^2)$.

4. As to the proper transformation of the phase portraits and "binding" them for the 2-dimensional equation, as here, the fact is that in case $\mu = \delta = 0$ the phase space is 8-dimensional, and in case $\beta = \gamma = 0$ it is 6-dimensional, we are obliged to the results obtained in ref. [5] binding the characteristics of each singular point of each equation of set (5) accordingly in the 8-dimensional

[1]The cases of another correlation of signs of V and β are not realized (see ref. [5]).

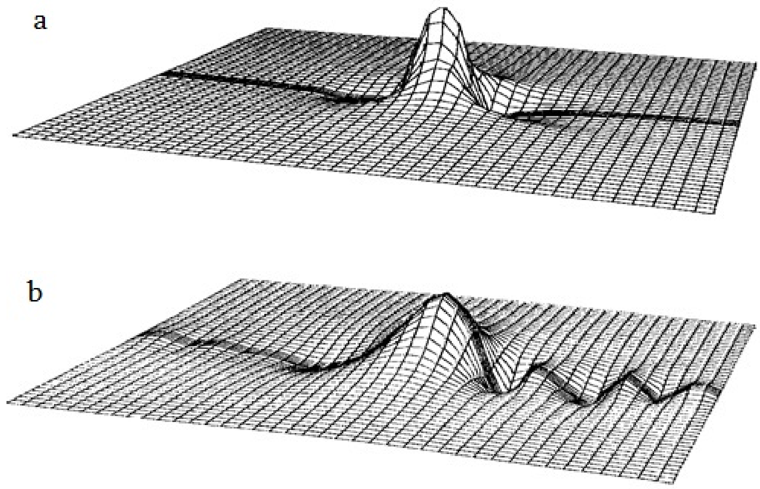

FIGURE 2. General view of a 2-dimensional soliton described by eq. (4) with $\Delta_\perp = \delta_y^2$ for $\beta = \gamma = 0$, $V > 0$ at $t = 0.3$: (a) $\mu = 1$, $\delta = 1 \times 10^{-6}$ [$\delta \leq (4/27)\mu^3 C^8$-case (c)]; (b) $\mu = 1$, $\delta = 1$ [$\delta > (4/27)\mu^3 C^8$-case (a)].

and 6-dimensional spaces. Thus the type of singularities in either of 4-dimensional or 3-dimensional subspaces (see ref. [5]) under the inverse transform of the coordinates, $\eta = (\bar{\eta} + \bar{\zeta})/2$, $\zeta = (\bar{\eta} - \bar{\zeta})/2$, will not be changed, and only those parameters of the phase portraits change which correspond for the solutions of the same class to changing of such parameters as the amplitude, the fronts steepness, frequency of the oscillations etc.

Now, let us make some our observations concerning the 3-DNLS equation (8) with $\Delta_\perp = \partial_\zeta^2$. Because, as it was shown in [10], equation (8) may be represented in form of set (9) , and, as it is known from [4, 11], exact solution of the 1-dimensional DNLS equation may be represented in form

$$h(x,t) = (A/2)^{1/2}[\exp(-Ax) + i\exp(Ax)]\exp(-iA^2 t)\cosh^{-2}(2Ax), \tag{16}$$

where A is the amplitude of the wave (see [4, 10] for detail), we can fulfilling the inverse change of the variables, $\eta = (\bar{\eta}+\bar{\zeta})/2$, $\zeta = (\bar{\eta}-\bar{\zeta})/2$ and extending solution (16) to the 2-dimensional case (equation (8) with $\Delta_\perp = \partial_\zeta^2$) write at once for $\nu = 0$

$$h(\eta,\zeta,t) = (A/2)^{1/2}[\exp(-A\chi) + i\exp(A\chi)]\exp(-iA^2 t)\cosh^{-2}(2A\chi), \tag{17}$$

where, as for the GKP equation, $\chi = (\eta \pm \zeta + (\sigma - V)t)$, and V is a velocity of the wave propagation relatively coordinate axis η and ζ for the first and second equations of set (9), respectively. Fig. 3 shows a character of solution for the first equation of set (9) with $\nu = 0$.

The dependence of the form of solution on dissipation and its dynamical characteristics for $\nu > 0$ have considered in [4, 10] in detail.

We think that there is no need to discuss here the problem of the qualitative analysis of solutions of the 3-DNLS equation, because unlike the GKP equation (4) and the corresponding set (7) the exact solution of either equation of set (9) is known, and there is no need to construct any special classification of its solutions in phase space.

3. Concluding Remarks

In conclusion, note that in this chapter for the GKP equation we have considered the special cases when $\mu = \delta = 0$ and $\beta = \gamma = 0$ in equation (4), and for other values of the coefficients more complicated wave structures resulting from the presence of all considered effects in the whole may be observed. So, the results obtained numerically in [1] (see also [4, 10]) show that for β, μ, $\delta \neq 0$

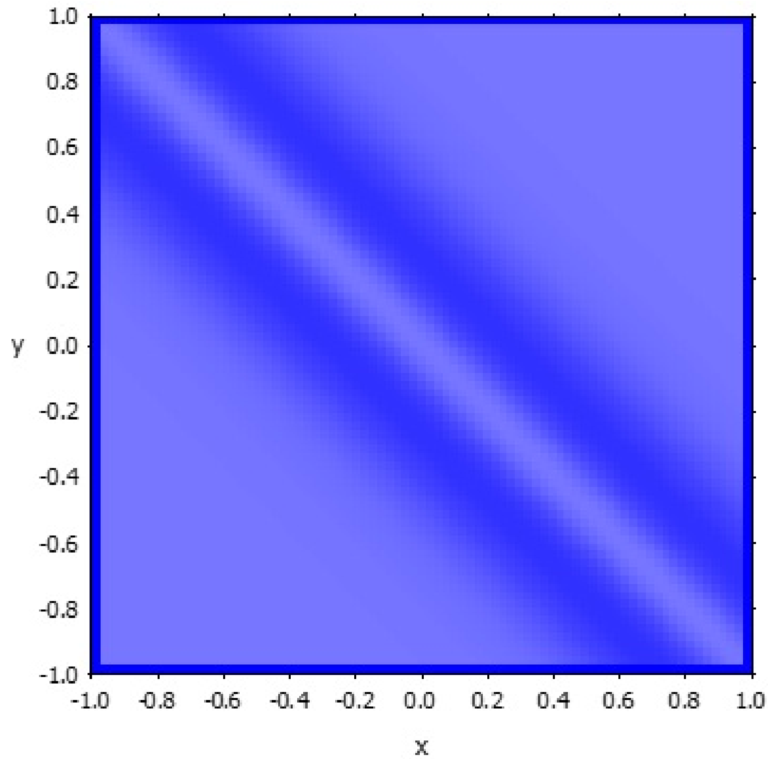

FIGURE 3. General view of solution $|h^2|$ of the first equation of set (9) for $A = 1$, $t = 0$.

at the time evolution in the presence of the Gaussian random fluctuations of the wave field for the harmonic initial conditions and the initial conditions in form of the solitary pulse the stable wave structures of the soliton type can be formed too. Furthermore, the stable soliton structures may be formed also at $\gamma \neq 0$. However, the analytical study of such cases is highly complicate, though the approach considered above can be also used. Note also that the results obtained in [5] and presented here for the GKP equation may be highly useful when studying the solutions and interpreting the multidimensional phase portraits of more complicated multidimensional model equations (see, for example, [8]).

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[1]Iv. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail address: oleg.kharshiladze@gmail.com

[2]Kazan Federal University, Kazan, Russia
E-mail address: vybelashov@yahoo.com

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ON ONE CLASS OF ELLIPTIC EQUATIONS CONNECTED WITH THE NONLINEAR WAVES

N. KHATIASHVILI

Abstract. Nonlinear elliptic equation connected with the nonlinear waves in the infinite area is considered. The non-smooth effective solutions exponentially vanishing at infinity are obtained. By means of such solutions the exact and approximate solutions of different nonlinear elliptic equations are derived. The profiles of nonlinear waves and symmetric solitary waves connected with those solutions are plotted by using "Maple".

1. Introduction

Nonlinear elliptic equations describe wide range of physical phenomena and those equations with the different kind of nonlinearity were considered by numerous authors [4–6, 8, 10, 14, 23, 24, 28–31, 36–45, 47–49, 52, 53].

In this paper we focus on the nonlinear elliptic equation connected with the different nonlinear waves. Particular case of this equation is the cubic nonlinear Schrödinger equation (cNLS).

The equation is considered in the infinite area. The effective solutions exponentially vanishing at infinity and having peaks at some lines are obtained. Non-smooth solitary waves connected with those solutions in a specific class of functions are constructed. Also the bounded solutions are given.

2. Statement of the Problem

In R^3 let us consider the following equation

$$P_1(\psi)\Delta\psi + P_2(\psi)(\nabla\psi)^2 + P_3(\psi) = 0, \tag{1}$$

where $\psi(x,y,z)$ is unknown function, $P_1(\xi)$, $P_2(\xi)$, $P_3(\xi)$ are the polynomials with respect to ξ, $P_1(\xi) = \sum\limits_{i=0}^{n} a_i\xi^i$, $P_2(\xi) = \sum\limits_{i=0}^{n} b_i\xi^i$, $P_3(\xi) = \sum\limits_{i=1}^{n+2} c_i\xi^i$, a_0, b_0, a_i, b_i, c_i, c_{n+1}, c_{n+2}, n, $i = 1,\dots,n$; are some constants.

The particular case of the equation (1) is the following equation

$$\left(1 - \frac{\psi_2^2}{2}\right)\Delta\psi_2 - \psi_2\big(\nabla\psi_2\big)^2 + \lambda_0 R^2\psi_2^3 - A_0\left(\psi_2 - \frac{\psi_2^3}{6}\right) = 0, \tag{2}$$

where λ_0, R, A_0 are the definite constants, ψ_2 is unknown function. When the function ψ_2 has negligible fifth degree value, the equation (2) is the approximation of the cubic nonlinear Schrödinger equation [27]. The solution of this equation was obtained in [27] in the specific class of functions.

Let us consider the following problem

Problem 1. In the space R^3 to find piecewise smooth continuous function ψ vanishing at infinity exponentially, satisfying the equation (1), having second order continuous derivatives $\frac{\partial^2\psi}{\partial x^2}$, $\frac{\partial^2\psi}{\partial y^2}$, $\frac{\partial^2\psi}{\partial z^2}$

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and first order derivatives with the jump planes $x=0$, $y=0$, $z=0$ satisfying the conditions

$$\begin{aligned} \Big(\frac{\partial\psi}{\partial x}\Big)^{-}\Big|_{x=0} &= -\Big(\frac{\partial\psi}{\partial x}\Big)^{+}\Big|_{x=0}, \\ \Big(\frac{\partial\psi}{\partial y}\Big)^{-}\Big|_{y=0} &= -\Big(\frac{\partial\psi}{\partial y}\Big)^{+}\Big|_{y=0}, \\ \Big(\frac{\partial\psi}{\partial z}\Big)^{-}\Big|_{z=0} &= -\Big(\frac{\partial\psi}{\partial z}\Big)^{+}\Big|_{z=0}. \end{aligned} \tag{3}$$

Note 1. Here $\left(\frac{\partial\psi}{\partial x}\right)^{-}\big|_{x=0}$ and $\left(\frac{\partial\psi}{\partial x}\right)^{+}\big|_{x=0}$ means

$$\lim_{x\longrightarrow 0^-}\frac{\partial\psi}{\partial x}, \quad \lim_{x\longrightarrow 0^+}\frac{\partial\psi}{\partial x}$$

respectively.

3. Solution of the Problem

Let us consider the function

$$\psi_0 = \exp[-\alpha|x| - \beta|y| - \gamma|z| - D], \;\; \alpha,\, \beta,\, \gamma > 0, \tag{4}$$

where α, β, γ are some non-negative constants, D is an arbitrary parameter.

The function (4) vanishes at infinity exponentially and satisfies the conditions (3). By direct verification we obtain, that it will be the solution of the equation (1) if the constants a_0, b_0, a_i, b_i, c_i, c_{n+1}, c_{n+2}, $i=1,\dots,n$; satisfy the following conditions

$$\begin{cases} d^2a_0 + c_1 = 0, \\ d^2(a_1+b_0) + c_2 = 0, \\ d^2(a_2+b_1) + c_3 = 0, \\ d^2(a_3+b_2) + c_4 = 0, \\ \dots\dots\dots\dots\dots\dots \\ d^2(a_{n-1}+b_{n-2}) + c_n = 0, \\ d^2(a_n+b_{n-1}) + c_{n+1} = 0, \\ d^2b_n + c_{n+2} = 0, \end{cases} \tag{5}$$

where $\alpha^2+\beta^2+\gamma^2 = d^2$.

Hence, we conclude, that the following theorem is valid

Thoerem 1. *If the coefficients a_0, b_0, a_i, b_i, c_i, c_{n+1}, c_{n+2}, $i=1,\dots,n$; of the equation* (1) *satisfy the system* (5)*, then the function given by the formula* (4) *is the solution of the Problem* 1.

Also, it is easy to see, that the following theorem is true

Thoerem 2. *If the coefficients a_0, b_0, a_i, b_i, c_i, c_{n+1}, c_{n+2}, $i=1,\dots,n$; of the equation* (1) *satisfy the system* (5) *and $\alpha=0 \vee \beta=0, \vee \gamma=0$ then the function given by the formula* (4) *is the solution of the equation* (1) *bounded in R^3 and satisfying the condition* (3).

Note 2. Here we do not discuss the uniqueness of the solutions of the Problem 1, as the function (4) depends on an arbitrary parameters α, β, γ, D.

In the next chapter we consider some particular cases of (1). By means of the function (4) we will construct exact and approximate solutions of these equations.

4. Examples

Let us consider several cases.

1) In case of $a_0 = c_1 = 0$, $a_1 = 1$, $b_0 = -1$, $b_1 = a_i = b_i = c_i = c_{n+1} = c_{n+2} = 0$, $i = 2, \dots, n$, the equation (1) takes the form

$$\psi\Delta\psi - (\nabla\psi)^2 = 0. \tag{6}$$

According to the Theorem 1 the solution of the equation (6) satisfies the conditions (3) will be given by the formula

$$\psi = R\psi_0, \tag{7}$$

where R is an arbitrary constant and ψ_0 is given by (4).

In Figure 1 the graphic of (7) vanishing at infinity is given for some parameters and in Figure 2 the graphic of (7) bounded at infinity is given. The graphics are constructed by using “Maple”.

Note 3. The Dirichlet problem for the equation (6) was studied in [5, 6].

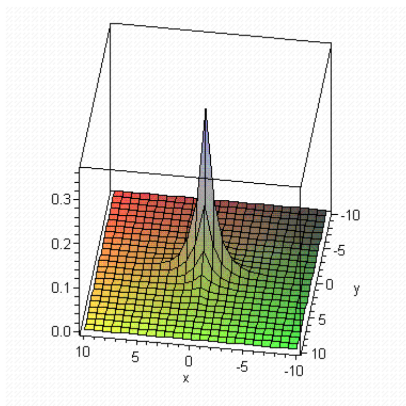

Figure 1. The graphic of (7) in case of $D = 1$; $R = 1$; $\alpha = \beta = \gamma = 1$; $z = 0$;

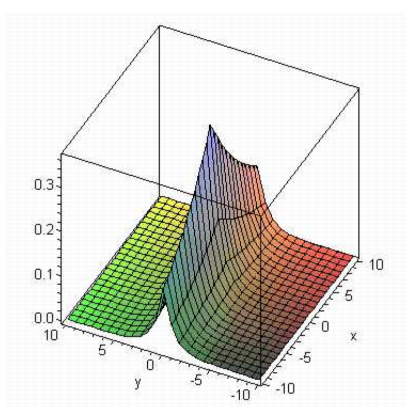

Figure 2. The graphic of (7) in case of $D = 1$; $R = 1$; $\alpha = 0.1$; $\beta = 1$; $\gamma = 0$;

2) In case of $a_0 = 1$, $c_1 = -A_0 + \lambda_0$, $a_1 = b_1 = a_i = b_i = c_i = c_{n+1} = c_{n+2} = 0$, $i = 2, \dots, n$, the equation (1) represents the well-known Helmholtz equation

$$\Delta\psi - (A_0 - \lambda_0)\psi = 0, \tag{8}$$

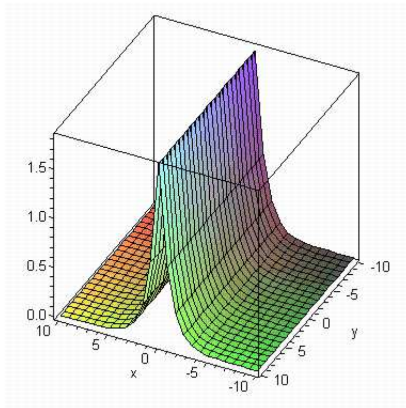

FIGURE 3. The linear wave. The graphic of (9) in case of $m = 1$; $D_1 = 4$; $R_1 = 100$; $\alpha = 1$; $\beta = \gamma = 0$; $A_0 - \lambda_0 = 1$.

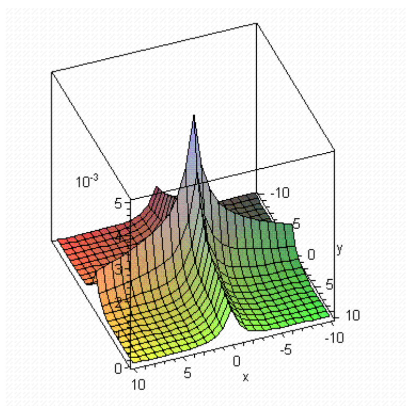

FIGURE 4. Superposition of linear waves. The graphic of (9) in case of $m = 2$; $D_1 = D_2 = 5$; $R_1 = R_2 = 1$; $z = 1$; $\alpha_1 = \beta_2 = 0.1$; $A_0 - \lambda_0 = 2.01$; $\beta_1 = \alpha_2 = \gamma_1 = \gamma_2 = 1$.

By the Theorem 1, the solution of the equation (8) for the Problem 1 is given by (7), where R is an arbitrary constant, α, β, γ satisfy the conditions

$$\alpha^2 + \beta^2 + \gamma^2 = A_0 - \lambda_0, \quad A_0 > \lambda_0.$$

The function (7) represents some class of stationary non-smooth linear waves. Their superposition is also the solution of (8) and is given by the sum

$$\psi = \sum_1^m R_k \exp\big[-\alpha_k|x| - \beta_k|y| - \gamma_k|z| - D_k\big], \quad D_k > 0, \tag{9}$$

where R_k are an arbitrary constants and $\alpha_k^2 + \beta_k^2 + \gamma_k^2 = A_0 - \lambda_0$, $A_0 > \lambda_0$, m is an arbitrary natural number.

The graphics of (9) are given in Figures 3, 4 for the different parameters.

In case of $A_0 = \lambda_0$ the equation (8) will be reduced to the Laplace equation

$$\Delta\psi = 0. \tag{10}$$

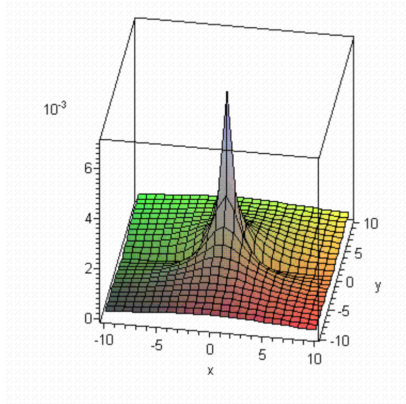

FIGURE 5. The linear wave. The graphic of (11) in case of $D = 5$; $R_1 = 1$; $a = 1$; $R_2 = R_3 = R_4 = R_5 = R_6 = 0$.

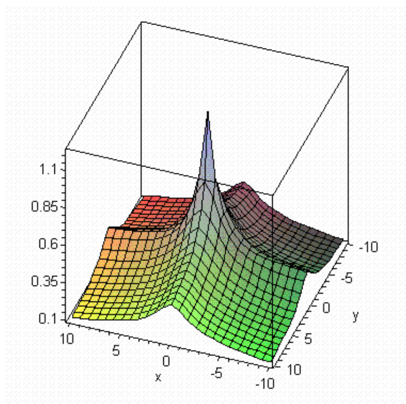

FIGURE 6. Superposition of linear waves. The graphic of (11) in case of $D = 5$; $a = 1$; $z = 1$; $R_1 = R_2 = R_3 = 100$; $R_4 = R_5 = R_6 = 0$.

Using well-known Poisson formula [5,6,8,33], we obtain the non-smooth solution of the Problem 1 for the equation (10)

$$\begin{aligned}\psi_0 = R_1 \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-x)^2 + y^2} + R_2 \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-z)^2 + y^2} \\ + R_3 \frac{|x|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-z)^2 + x^2} + R_4 \frac{|x|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-y)^2 + x^2} \\ + R_5 \frac{|z|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-x)^2 + z^2} + R_6 \frac{|z|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{(t-y)^2 + z^2},\end{aligned} \tag{11}$$

$f(t)$ $(-\infty < t < +\infty)$, is the function vanishing at infinity, having second order continues derivatives and first order continues derivatives except the point $t = 0$, where the following conditions are satisfied

$$(f')^+(0) = -(f')^-(0), \quad (f'')^+(0) = (f'')^-(0), \quad |f(t)| \le e^{-D}, \quad D \ge 5,$$

R_1, R_2, R_3, R_4, R_5, R_6, $|R_1| + |R_2| + |R_3| + |R_4| + |R_5| + |R_6| \neq 0$ are non-negative constants.

The graphics of (11) are given in Figures 5, 6 in the case $f(t) = e^{-a|t|-D}$, $a > 0$.

3) Now, let us consider the equation

$$\Delta\psi + c_1\psi + c_2\psi^2 = 0. \tag{12}$$

The equation (12) represents (1) in case of $a_0 = 1$, $a_1 = a_2 = b_0 = b_1 = b_2 = a_i = b_i = c_i = c_{n+1} = c_{n+2} = 0$, $i = 3, \dots, n$;

The function (4) will be the solution of (12) only in the case $c_2 = 0$ (see Example 2), but by means of the function (4) we can construct approximate solutions of the equation (12) vanishing at infinity for which $c_2 \neq 0$.

Let us introduce the notation

$$\psi = R\sin^2\psi_1, \tag{13}$$

where ψ_1 is a function having negligible fifth degree value, R is some parameter.

Taking into account

$$\sin\psi_1 \approx \psi_1 - \psi_1^3/6; \quad \sin^2\psi_1 \approx \psi_1^2 - \psi_1^4/3; \tag{14}$$

and putting (13) into (12) we obtain the following equation

$$\begin{aligned} 2\left(\psi_1 - \frac{2}{3}\psi_1^3\right)\Delta\psi_1 + 2\left(1 - 2\psi_1^2 + \frac{2}{3}\psi_1^4\right)(\nabla\psi_1)^2& \\ + c_1\left(\psi_1^2 - \frac{1}{3}\psi_1^4\right) + c_2R\psi_1^4 = 0,& \end{aligned} \tag{15}$$

The equation (15) is the approximation of the equation (12) with the accuracy $\frac{8|R|d^2}{3}\psi_1^6$.

If

$$c_1 = -4d^2 = -c_2R \tag{16}$$

the function given by (4) will be the approximate solution of the equation (15) with the accuracy $\frac{8|R|d^2}{3}\exp(-6D)$, i. e. this function is the exact solution of the equation

$$2\left(\psi_1 - \frac{2}{3}\psi_1^3\right)\Delta\psi_1 + 2\left(1 - 2\psi_1^2\right)(\nabla\psi_1)^2 + c_1\left(\psi_1^2 - \frac{1}{3}\psi_1^4\right) + c_2R\psi_1^4 = 0.$$

According to (13), (14), (15), (16) the approximate solution of the equation (12) will be given by

$$\psi = R\sin^2\left\{\exp[-\alpha|x| - \beta|y| - \gamma|z| - D]\right\}, \tag{17}$$

where

$$4(\alpha^2 + \beta^2 + \gamma^2) = c_2R = -c_1, \quad c_1 < 0,$$

and the parameter D is chosen accordingly for the desired accuracy in such a way, that the quantity e^{-5D} is negligible (for example for $D = 3$, $e^{-15} \approx 10^{-7}$).

It is obvious

$$|\psi| \leq R\exp(-2D).$$

The graphics of (17) for some parameters are given in Figures 7, 8 in case of $D = 4$.

Note 4. The equation (12) is connected with the crystal growth [23, 24].

4) Now, let us consider the case $a_0 = 1$; $c_1 = -A_0$; $c_3 = \lambda_0$; $c_2 = a_1 = a_2 = a_3 = b_0 = b_1 = b_2 = b_3 = a_i = b_i = c_i = c_{n+1} = c_{n+2} = 0$, $i = 4, \dots, n$; then the equation (1) takes the form

$$\Delta\psi + \lambda_0\psi^3 - A_0\psi = 0, \tag{18}$$

The function (4) will be the solution of (18) only in the case $\lambda_0 = 0$.

By means of the function (4) we will construct approximate solutions of the equation (18) vanishing at infinity for which $\lambda_0 \neq 0$.

Let us introduce the following notation

$$\psi = R\sin\psi_2, \tag{19}$$

where ψ_2 is a function having negligible fifth degree value, $R > 0$ is some parameter.

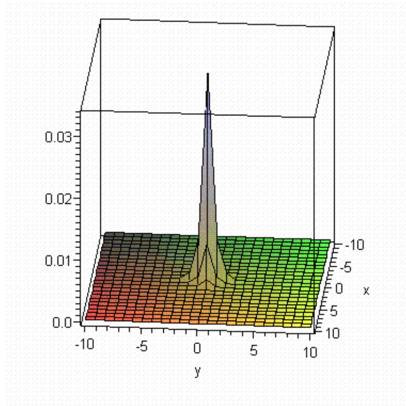

FIGURE 7. The graphic of (17) in case of $R = 100$; $\alpha = \beta = \gamma = 1$; $z = 0$;

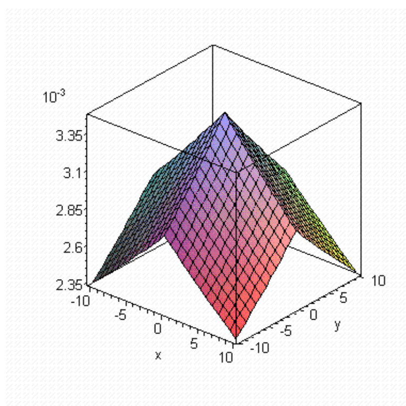

FIGURE 8. The graphic of (17) in case of $R = 1$; $\alpha = \beta = \gamma = 0.01$; $z = 0$;

Putting (19) into the left hand side of (18) and taking into the account (14) one obtains

$$\left(1 - \frac{\psi_2^2}{2} + \frac{\psi_2^4}{24}\right)\Delta\psi_2 - \left(\psi_2 - \frac{\psi_2^3}{6}\right)(\nabla\psi_2)^2$$
$$+ \lambda_0 R^2 \left(\psi_2 - \frac{\psi_2^3}{6}\right)^3 - A_0\left(\psi_2 - \frac{\psi_2^3}{6}\right) = 0. \tag{20}$$

As ψ_2^5 is negligible,the function (4) will be the solution of the equation (20) with the accuracy $A_0 \frac{\exp(-5D)}{2}$ and the exact solution of the equation (2). Hence, the function ψ given by the formula (19) is the solution of the equation (18) with the accuracy $A_0 \frac{\exp(-5D)}{2}$.

According to (4), (5), (19) the approximate solution of (18) will be given by the formula

$$\psi = R \sin\{\exp[-\alpha|x| - \beta|y| - \gamma|z| - D]\}, \tag{21}$$

where

$$\alpha^2 + \beta^2 + \gamma^2 = A_0, \quad \lambda_0 R^2 = 4A_0/3; \quad A_0 > 0, \tag{22}$$

and the constant D is chosen for the desired accuracy in such a way, that e^{-5D} is negligible (for example for $D = 4$, $e^{-20} \approx 2 \times 10^{-9}$).

The equation (18) is the cubic nonlinear Schrödinger type equation (cNLS). By the formulaes (21), (22) the modulus r of some class of solitary waves is given [25, 27]. The different classes of solitary waves are obtained in [1–3, 7, 9–13, 15–21, 26, 28–36, 40, 42–53].

In Figures 9, 10 the graphics of (21) are plotted for different parameters for the case $R = 10$ and $D = 4$ by using "Maple".

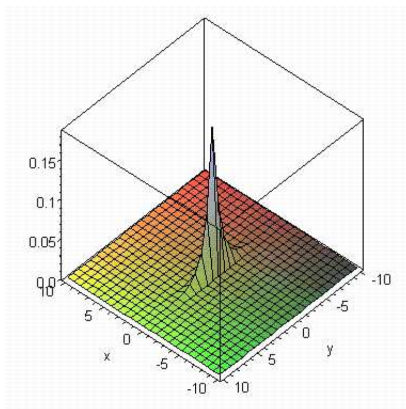

FIGURE 9. The modulus of the solitary wave. The graphic of (21) in case of $\alpha = 10$; $\beta = \gamma = 1$; $z = 0$; $A_0 = 102$; $\lambda_0 = 1.36$.

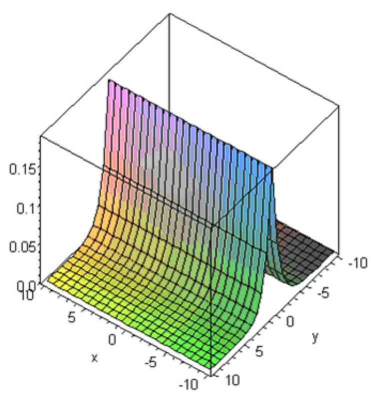

FIGURE 10. The modulus of the solitary wave. The graphic of (21) in case of $\alpha = 1$; $\beta = \gamma = 0$; $A_0 = 1$; $\lambda_0 = 0.013333$.

Note 5. In the works [21, 22] the equation (18) is equivalently reduced to the nonlinear integral equation.

Note 6. The equation (1) was considered in 3D, but the results of the current paper are valid in any dimensions.

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I. Javakhishvili Tbilisi State University, 2 University Str., Tbilisi 0186, Georgia
E-mail address: ninakhatia@gmail.com

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EIGENOSCILLATIONS AND STABILITY OF ORTHOTROPIC SHELLS, CLOSE TO CYLINDRICAL ONES, WITH AN ELASTIC FILLER AND UNDER THE ACTION OF MERIDIONAL FORCES, NORMAL PRESSURE AND TEMPERATURE

S. KUKUDZHANOV

Abstract. Eigenoscillations and stability of closed orthotropic shells of revolution, close by their form to cylindrical ones, with an elastic filler and under the action of meridional forces, external pressure and temperature are investigated. The shells of positive and negative Gaussian curvature are studied. Formulas and universal curves of dependence of the least frequency on orthotropy parameters, meridional loading, external pressure, temperature, rigidity of an elastic filler, as well as on the amplitude of shell deviation from the cylinder, are obtained. Critical values of outer effects are defined.

We study eigenoscillations and stability of closed orthotropic shells of revolution, close by their forms to cylindrical ones, with an elastic filler and under the action of meridional forces uniformly distributed over the end-walls of the shell, external pressure and temperature. We consider a light filler for which the influence of tangential stresses on the contact surface and the inertia forces may be neglected. The shell is considered to be thin and elastic. Temperature in a shell body is uniformly distributed. An elastic filler is modelled by the Winkler's base, its extension by heating is not taken into account. We investigate the shells of middle length whose form of midsurface generatrix is expressed by a parabolic function. We consider the shells of positive and negative Gaussian curvature. The boundary conditions on the end-walls correspond to a free support admitting certain radial displacement in the initial state. Formulas and universal curves of dependance of the least frequency on the orthotropy parameters, meridional loading, external pressure, temperature, rigidity of the elastic filler, as well as on the deviation amplitude of the shell from the cylinder are obtained. It is shown that the elastic orthotropy parameters affect significantly the least frequency and the corresponding form of the waveformation. A degree of influence of orthotropy parameters under separate and joint action of the above-mentioned outer factors on the lower frequencies is revealed. Critical values of outer effects are defined.

We consider the shell whose middle surface is formed by the rotation of a square parabola around the z-axis of the rectangular system of coordinates x, y, z with the origin in the middle of the segment of the axis of rotation. It is assumed that the cross-section radius R of the middle surface is defined by the equality $R = r + \delta_0\big[1 - \xi^2(r/\ell)^2\big]$, where r is the end-wall section radius, δ_0 is the maximal deviation from the cylindrical form (for $\delta_0 > 0$, the shell is convex, and for $\delta_0 < 0$, it is concave), $L = 2\ell$ is the shell length, $\xi = z/r$. We consider the shells of middle length [9], and it is assumed that

$$(\delta_0/r)^2 \ll 1, \quad (\delta_0/\ell)^2 \ll 1. \tag{1}$$

As the basic equations of oscillations we have taken those of the theory of shallow orthotropic shells [8]. For the shells of middle length, the forms of oscillations that correspond to the least frequencies have weak variability in longitudinal direction as compared with the circumferential one, therefore the correlation

$$\frac{\partial^2 f}{\partial \xi^2} \ll \frac{\partial^2 f}{\partial \varphi^2} \quad (f = w, \psi) \tag{2}$$

is valid, where w and ψ are, respectively, the functions of radial displacement and stress. As a result, the system of equations for the shells under consideration is reduced to the following resolving equation

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(due to the adopted assumption, temperature terms are equal to zero [7]):

$$\varepsilon\,\frac{\partial^8 w}{\partial\varphi^8}+\frac{E_1}{E_2}\left(\frac{\partial^4 w}{\partial\xi^4}+4\,\delta\,\frac{\partial^4 w}{\partial\xi^2\,\partial\varphi^2}+4\,\delta^2\,\frac{\partial^4 w}{\partial\varphi^4}-t_1^0\,\frac{\partial^6 w}{\partial\xi^2\,\partial\varphi^4}\right)$$
$$-\,t_2^0\,\frac{\partial^6 w}{\partial\varphi^6}-2\,s^0\,\frac{\partial^6 w}{\partial\xi\,\partial\varphi^5}+\gamma\,\frac{\partial^4 w}{\partial\varphi^4}+\frac{\rho\,r^2}{E_2}\,\frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 w}{\partial\varphi^4}\right)=0, \tag{3}$$

$\varepsilon=h^2/12\,r^2(1-\nu_1\nu_2)$, $\delta=\delta_0 r/\ell^2$, $\tau_i=T_i^0/E_2h$ $(i=1,2)$, $s^0=S^0/E_2h$, $\gamma=\beta r^2/E_2h$, E_1,E_2,ν_1,ν_2 are, respectively, the E_1,E_2,ν_1,ν_2 moduli of elasticity and Poisson coefficients in the axial and circumferential directions $(E_1\nu_2=E_2\nu_1)$; T_1^0, T_2^0 are meridional and circumferential normal forces of the initial state; S^0 is the shearing stress of the initial state; h the shell thickness; ρ is the material density of the shell; β is the "bed" coefficient of the elastic filler (characterizing elastic rigidity); φ is an angular coordinate; t is time.

The initial state is assumed to be momentless. On the basis of the corresponding solution, taking into account the reaction of the filler and also inequalities (1), we obtain the following approximate expressions

$$\begin{aligned}T_1^0&=P_1\Big[1+\frac{\delta_0}{r}\big(\xi^2(r/\ell)^2-1\big)\Big]+q\delta_0\Big[\xi^2(r/\ell)^2-1\Big],\\ T_2^0&=-2\,P_1\delta_0 r/\ell^2-qr+\beta_0 r w_0,\quad S^0=0,\end{aligned} \tag{4}$$

where w_0 and β_0 are, respectively, deflection and a "bed" coefficient of the filler in the initial state; P_1 is meridional stress; q is external pressure.

Taking into account (2), we get

$$\big|\xi^2(r/\ell)^2-1\big|\,\frac{\partial^2 w}{\partial\xi^2}\ll 2(r/\ell)^2\frac{\partial^2 w}{\partial\varphi^2},\qquad \frac{\delta_0}{2}\,\big|\xi^2(r/\ell)^2-1\big|\,\frac{\partial^2 w}{\partial\xi^2}\ll\frac{\partial^2 w}{\partial\varphi^2}.$$

Therefore expressions (4), after substitution into equation (3), can be simplified and written in the following form:

$$T_1^0=P_1,\quad T_2^0=-2\,P_1\delta_0 r/\ell^2-qr+w_0\beta_0 r,\quad T_i^0=\sigma_i^0 h\quad(i=1,2). \tag{4$'$}$$

Taking into account the fact that in the initial state the shell deformation ε_φ^0 in the circumferential direction is defined by the equalities

$$\varepsilon_\varphi^0=\frac{\sigma_2^0-\nu_1\sigma_1^0}{E_2}+\alpha_2 T,\quad \varepsilon_\varphi^0=-\frac{w_0}{r}\,,$$

where α_2 is the coefficient of linear extension in the circumferential direction and T is temperature, we have

$$w_0=\big(-\sigma_2^0+\nu_1\sigma_1^0\big)\frac{r}{E_2}-\alpha_2 T_2. \tag{5}$$

Substituting expression (5) into (4$'$), we obtain

$$\frac{T_2^0}{E_2h}=\frac{\sigma_2^0}{E_2}=-\frac{qr}{E_2h}-2\frac{P_1}{E_2h}\,\delta+\nu_1\frac{\sigma_1^0}{E_2}\,\frac{\beta_0 r^2}{E_2h}-\alpha_2 T\frac{\beta_0 r^2}{E_2h}-\frac{\sigma_2^0}{E_2}\,\frac{\beta_0 r^2}{E_2h}.$$

Introduce the notation

$$E_1=e_1E,\quad E_2=e_2E,$$
$$\frac{qr}{Eh}=\overline{q},\quad \frac{P_1}{Eh}=-p,\quad \frac{\beta_0 r^2}{Eh}=\gamma_0,\quad 1+\gamma_0 e_2^{-1}=g.$$

Then expressions (4$'$) take the form

$$-\frac{\sigma_1^0}{E_2}=-e_2^{-1}p,\quad -\frac{\sigma_2^0}{E_2}=\big(\overline{q}-2p\delta+\nu_1 p\gamma_0+\alpha_2 T\gamma_0\big)e_2^{-1}g^{-1}. \tag{5$'$}$$

Note that since R is close to r, in the expressions for stresses (5$'$) we adopted $R\approx r$.

As a result, equation (3) takes the form

$$\varepsilon\,\frac{\partial^8 w}{\partial\varphi^8}+\frac{e_1}{e_2}\left[\frac{\partial^4 w}{\partial\xi^4}+4\delta\,\frac{\partial^4 w}{\partial\xi^2\,\partial\varphi^2}+4\big(\delta^2+\gamma/4e_1\big)\,\frac{\partial^4 w}{\partial\varphi^4}\right]$$
$$+\big(\overline{q}-2p\delta+\nu_1 p\gamma_0+\alpha_2 T\gamma_0\big)e_2^{-1}g^{-1}\frac{\partial^6 w}{\partial\varphi^6}+p\,\frac{\partial^6 w}{\partial\xi^2\,\partial\varphi^4}e_2^{-1}+\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big)e_2^{-1}=0. \tag{6}$$

We consider the harmonic oscillations. For the given boundary conditions of free support and for equation (6) the solution

$$w=A_{mn}\cos\lambda_m\xi\,\sin n\varphi\,\cos\omega_{mn}t,\quad \lambda_m=m\pi r/2\ell \tag{7}$$
$$(m=2i+1,\quad i=0,1,2,\dots)$$

is satisfied.

Substituting expression (7) into (6), for finding eigenfrequencies, we obtain the following equality (in the sequel, the indices ω_{mn} will be omitted):

$$\omega^2=\frac{E}{\rho r^2}\Big[e_2\varepsilon n^4+e_1\big(\lambda_m^4 n^{-4}+4\,\delta\lambda_m^2 n^{-2}+4\big(\delta^2+\gamma/4e_1\big)$$
$$-\,p\big(\lambda_m^2-2\,\widetilde{\delta}n^2\big)-\big(\overline{q}+d_2T\gamma_0\big)g^{-1}n^2\big)\Big].$$

Introduce the notation

$$\begin{aligned}&\overline{\delta}^{\,2}=\delta^2+\gamma/4e_1\,,\quad \widetilde{\delta}=\Big(\delta-\frac{1}{2}\nu_1\gamma_0\Big)g^{-1},\quad \widetilde{q}=\big(\overline{q}+\alpha T\gamma_0\big)g^{-1},\\&\omega^2=\frac{E}{\rho r^2}\Big[e_2\varepsilon n^4+e_1\big(\lambda_m^4 n^{-4}+4\,\delta\lambda_m^2 n^{-2}+4\,\overline{\delta}^{\,2}-p\big(\lambda_m^2-2\,\widetilde{\delta}n^2\big)-\widetilde{q}n^2\big)\Big].\end{aligned} \tag{8}$$

It is not difficult to see that for $p=0$, $\delta>0$, to the least frequency there corresponds $m=1$. It can also be shown that this condition takes place for $\delta<0$, bearing in mind inequalities (1) and the fact that $\omega^2>0$. Therefore, first of all, we consider the forms of oscillations under which there arises one half-wave ($m=1$) over the whole length of the shell and n waves in the circumferential direction. For the compression $p>0$, and for the tension $p<0$; q is a normal pressure which is assumed to be positive if it is external.

To present expression in a dimensionless form, we introduce the dimensionless values

$$\begin{aligned}&\theta=(e_2/e_1)^{1/4}N,\ \ N=n^2/n_0^2,\ \ \overline{P}=P/\sqrt{e_1e_2},\ \ P=p/p_*,\\&\widetilde{Q}=\widetilde{q}/\overline{q}_{0*},\ \ \widetilde{q}=(\overline{q}+\alpha T\gamma_0)g^{-1},\ \ n_0^2=\lambda_1\varepsilon^{1/4},\ \ p_*=2\varepsilon^{1/2},\\&\overline{q}_{0*}=0,855(1-\nu_1\nu_2)^{-3/4}\Big(\frac{h}{r}\Big)^{3/2}\frac{r}{L}\,,\ \ \delta_*^\nu=(e_1/e_2)\delta_*\,,\\&\delta_*=\delta\varepsilon_*^{-1/2},\ \ \widetilde{\delta}^{\,\nu}=(e_1/e_2)^{1/4}\Big(\delta-\frac{1}{2}\nu_1\gamma_0\Big)\varepsilon_*^{-1/2}g^{-1},\\&\overline{\delta}^{\,\nu^2}=(e_1/e_2)^{1/2}(\delta_*^2+\gamma_*/4e_1)=\overline{\delta}^{\,\nu^2}+e_1e_2)^{-1/2}\frac{\gamma_*}{4}\,,\ \ \gamma_*=\gamma\varepsilon_*^{-1},\\&\omega_*^2=2\lambda_1^2\varepsilon^{1/2}E/3r^2\,,\quad \varepsilon=(1-\nu^2)^{-1/2}\frac{h}{2}\Big(\frac{r}{L}\Big)^2,\end{aligned} \tag{9}$$

where $p_*,\overline{q}_{0*},\omega_*$ are, respectively, critical loading of compression, critical pressure and the least frequency for the cylindrical isotropic shell of middle length [1, 9]. Thus equality (8) can be written in the following dimensionless form:

$$\omega^2(\theta)/\omega_*^2=0,5\sqrt{e_1e_2}\big(\theta^2+\theta^{-2}+2,37\,\delta_*^\nu\theta^{-1}+1,4045\,\overline{\delta}_*^{\,\nu^2}\big)$$
$$-\,1,755\,e_1^{-1/4}e_2^{-3/4}\theta\overline{Q}-2\,\overline{P}\big(1-1,185\,\widetilde{\delta}_*^{\,\nu}\theta\big). \tag{10}$$

The least frequency (for $\omega^2(\theta)>0$) is defined by the condition $[\omega^2(\theta)]'=0$. As a result, we obtain

$$0,8775\,e_1^{-1/4}e_2^{-3/4}\widetilde{Q}-1,185\,\widetilde{\delta}^{\,\nu}\overline{P}=\theta-1,185\,\delta_*^\nu\theta^{-2}-\theta^{-3} \tag{11}$$

or

$$\theta^4 - \big(0,8775\, e_1^{-1/4} e_2^{-3/4} \widetilde{Q} - 1,185\, \widetilde{\delta}^{\,\nu} \overline{P}\big)\theta^3 - 1,185\, \delta_*^{\nu} \theta - 1 = 0. \tag{12}$$

This implies that for $\widetilde{Q} = \overline{P} = 0$, we get

$$\theta^4 - 1,185\, \delta_*^{\nu} \theta - 1 = 0. \tag{12$'$}$$

The above equation for an isotropic shell has been considered in [3]. Investigation of the roots of the above equation, similar to that carried out in [3], leads to

$$\begin{aligned} \theta &= \sqrt{1 - 0,0876\, \delta_*^2 (e_1/e_2)^{1/2}} + 0,2962\, \delta_* (e_1/e_2)^{1/4} \quad (\delta_* > 0), \\ \theta &= \sqrt{1 - 0,0876\, \delta_*^2 (e_1/e_2)^{1/2}} - 0,2962\, \delta_* (e_1/e_2)^{1/4} \quad (\delta_* < 0). \end{aligned} \tag{13}$$

In particular, for $\delta_* = 0$, we get the known formula for the cylindrical orthotropic shell of middle length ($n^2 = (e_1/e_2)^{1/4} \lambda_1 \varepsilon^{-1/4}$) [5].

By θ_0 we denote the value of θ which is defined by virtue of (13).

Defining thus the value of θ_0 (for fixed e_1, e_2, δ_*) and substituting it into expression (10) (for $P = \widetilde{Q} = 0$), we obtain the least frequency of a free shell $\omega(\theta_0)$. For clearness, we will now proceed to considering the value $N = \theta(e_1, e_2)^{1/4}$.

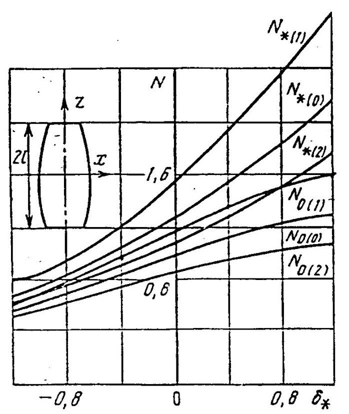

FIGURE 1

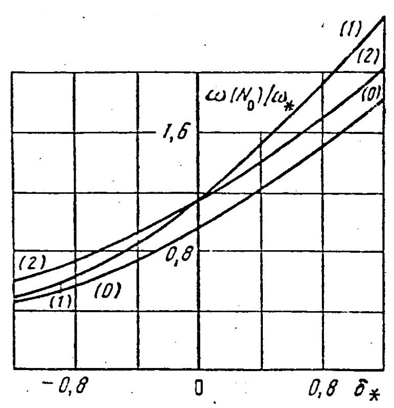

FIGURE 2

In Figures 1 and 2 we can see the graphs of dependencies $N_0 = n^2/n_0^2$ and $\omega(N_0)/\omega_*$ on the parameter δ_* for the cases $e_1 = e_2 = 1(0)$, $e_1 = 1$, $e_2 = 2(1)$; $e_1 = 2$, $e_2 = 1(2)$; the corresponding curves are denoted by $N_{0(i)}$ and (i) $i = 0, 1, 2$. It can be easily seen that for the convex shells ($\delta > 0$) the importance of the elastic parameter is greater in the axial direction than in the circumferential one, whereas for the concave shells ($\delta < 0$), the situation is inverse.

For $\omega = 0$, $P = 0$ from equality (10), we have

$$1,755\, e_1^{-1/4}\, e_2^{-3/4}\, \widetilde{Q} = \theta + \theta^{-3} + 2,37\, \delta_*^{\nu} \theta^{-2} + 1,404\, \overline{\delta}^{\,\nu^2} \theta^{-1}. \tag{14}$$

The least value $\widetilde{Q} > 0$ depending on θ is realized for $\widetilde{Q}'_\theta$. Thus we obtain

$$\theta^4 - 1,404\, \overline{\delta}^{\,\nu^2} \theta^2 - 4,74\, \delta_*^{\nu} \theta - 3 = 0. \tag{15}$$

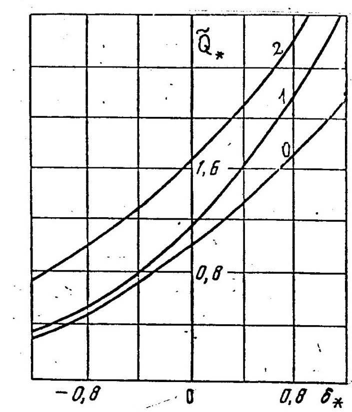

FIGURE 3

The positive root of that equation $\theta = \theta_*$ ($N = N_*$) corresponds to the number of wave in the transverse direction under which is realized the critical loading of stability loss $\widetilde{Q}_*$. This equation for an isotropic shell is considered in [3], where the expression of the positive root is given explicitly. Generalizing this result to the orthotropic case, we present the roots of dependence of N_* on δ_* for the cases $i = 0, 1, 2$ considered above. In Figure 1, these curves are denoted, respectively, by $N_{*(i)}$. The graphs of dependence of $\widetilde{Q}_*$ on δ_* for those cases are given in Figure 3.

Note that expression (14) for finding the critical loading can be simplified on the basis of (15). From this equation implies that

$$2,37\, \delta_*^{\nu} \theta^{-2} + 1,404\, \overline{\delta}_*^{\,\nu^2} \theta^{-1} = -\big(2,37\, \delta_*^{\nu} \theta^{-2} + 3\, \theta^{-3} - \theta\big). \tag{16}$$

Substituting equality (16) into (14), we get

$$\widetilde{Q}_* = 1,15\, e_1^{1/4} e_2^{3/4} \big(\theta_* - \theta_*^{-3} - 1,185\, \delta_*^{\nu} \theta_*^{-2}\big). \tag{17}$$

From the condition of minimality of frequency (11) for $\overline{P} = 0$, we obtain the following dependence between $\widetilde{Q}$ and θ:

$$\widetilde{Q} = 1,15\, e_1^{1/4} e_2^{3/4} \big(\theta - \theta^{-3} - 1,185\, \delta_*^{\nu} \theta_*^{-2}\big). \tag{18}$$

It is not difficult to notice that from the above equality we have also the relation (17). On the basis of equality (18), for $\widetilde{Q} = 0$, we obtain equation (12′), whose root $\theta = \theta_0$ corresponds to the least frequency of the unloaded shell $\omega(\theta_0)$; while for $\widetilde{Q} = \widetilde{Q}_*$, we obtain equation (17), whose root θ_* corresponds to the critical loading, and $\omega = 0$.

Thus, when $\widetilde{Q}$ varies in the interval

$$0 \le \widetilde{Q} \le \widetilde{Q}_* \tag{19}$$

the least frequency $\omega(\theta, \widetilde{Q})$ varies in the interval $\omega(\theta_0, \widetilde{Q}=0) \ge \omega(\theta, \widetilde{Q}) \ge 0$. Relying on the reasoning similar to that cited in [2], we can show that as $\widetilde{Q}$ varies in the interval (19), the value θ, realizing the least frequency $\omega(\theta, \widetilde{Q})$ and connected with $\widetilde{Q}$ by the relation (18), lies in the interval

$$\theta_0 \le \theta \le \theta_* \,. \tag{20}$$

Let us pass now to the value $N = \theta(e_1/e_2)^{1/4}$. In particular, for $\delta = \gamma = 0$, inequalities (19) and (20) take the form

$$0 \le \widetilde{Q} \le e_1^{1/4}\, e_2^{3/4}\,, \quad (e_1/e_2)^{1/4} \le N \le 1,315\, (e_1/e_2)^{1/4}. \tag{21}$$

For an isotropic case, inequalities (21) coincide with those presented in [2], $0 \le \widetilde{Q} \le 1$, $1 \le N \le 1,315$.

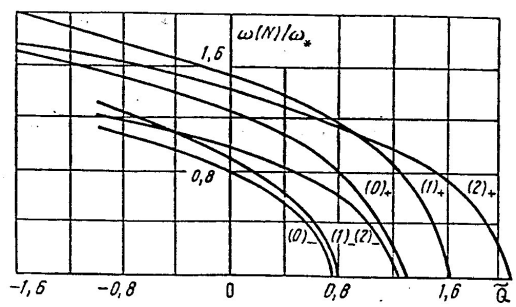

FIGURE 4

By virtue of equality (18) it is not difficult to construct the curves $N(\widetilde{Q})$ realizing the least frequency for different values $e_1, e_2, \delta_*, \gamma_*, T$. Towards this end, we fix these parameters and having the value θ, from the interval (20), we define the corresponding value $\widetilde{Q}$ by formula (18). Substituting these values in formula (10), we obtain (for the case $P = 0$ under considertion) the corresponding value of the least frequency. In Figure 4, we can see the curves of dependence of the least frequency ω/ω_* on $\widetilde{Q}$ (for $\gamma = 0$) for $\delta_* = 0,4$ and $\delta_* = -0,4$ for the cases $i = 0, 1, 2$. The curves are denoted by $(0)_+$, $(1)_+$, $(2)_+$, and $(0)_-$, $(1)_-$, $(2)_-$, respectively.

On the basis of the given curves and the results obtained in [1], it is easy to notice that if for the cylindrical shell in the absence of prestress the influence of orthotropy parameters is practically the same, then for the convex shells this effect occurs only for $\widetilde{Q} \approx 0,9$ and, in addition, on the interval $0 \le \widetilde{Q} \le 0,9$, the leading role belongs to the elastic parameter in the axial direction as compared with the circumferential one, whereas on the interval $0,9 \le \widetilde{Q} \le 1,6$ the situation is inverse.

Consider now the case $\overline{P} \ne 0$, $\widetilde{Q} = 0$ ($q = 0$, $\gamma = 0$) with $\overline{\delta}_*^{\nu^2} = \delta_*^{\nu^2}$, $\widetilde{\delta}_*^{\nu} = \delta_*^{\nu}$. On the basis of (10) and (11), we have

$$\omega^2/\omega_*^2 = 0,5\,\sqrt{e_1 e_2}\,\big[\theta^2 + \theta^{-2} + 2,375\,\delta_*^{\nu}\theta^{-1} + 1,404\,\delta_*^{\nu^2} - 2\,\overline{P}(1 - 1,185\,\delta_*^{\nu}\theta)\big], \tag{22}$$

$$-\,1,185\,\delta_*^{\nu}\overline{P} = Q - 1,185\,\delta_*^{\nu}\theta^{-2} - \theta^{-3} \tag{23}$$

or

$$\theta^4 + 1,185\,\delta_*^{\nu}\overline{P}\theta^3 - 1,185\,\delta_*^{\nu}\theta - 1 = 0. \tag{24}$$

From equation (24), for $\delta_* = 0$, we obtain the equation $\theta^4 - 1 = 0$ whose positive root $\theta = 1$ ($N = (e_1/e_2)^{1/4}$). Consequently, for the orthotropic cylindrical shell of middle length the least frequency is realized for $N = (e_1/e_2)^{1/4}$, independently of $\overline{P}$. For the isotropic case, all the above-said is in a full agreement with [6]. Moreover, from (24), for $\overline{P}=1$, we find that the positive root of that equation does not depend on δ_*^ν.

For $\omega=0$, equation (22) takes the form

$$\overline{P}=\frac{\theta^2+\theta^{-2}+2,37\,\delta_*^\nu\theta^{-1}+1,404\,\delta_*^{\nu^2}}{2(1-1,185\,\delta_*^\nu\theta)}\,. \tag{25}$$

As is known, the least value of $\overline{P}$ is called a critical loading. In particular, for $\delta_*=0$, $\theta=1$, from (25), we get the known formula of the critical contracting force for the cylindrical orthotropic shell $\overline{P}=1$ [9]. The least value $\overline{P}$ ($\overline{P}>0$), depending on θ, realizes for $\overline{P}'_\theta=0$. Thus we get

$$\begin{aligned}&2\big(\theta-\theta^{-3}-1,185\,\delta_*^\nu\theta^{-2}\big)\big(1-1,185\,\delta_*^\nu\theta\big)\\&\quad=-1,185\,\delta_*^\nu\big(\theta^2+\theta^{-2}+2,37\,\delta_*^\nu\theta^{-1}+1,404\,\delta_*^{\nu^2}\big).\end{aligned} \tag{26}$$

In a simpler form, (26) is the fifth degree equation, so it is impossible to define its roots exactly. Therefore we have suggested somewhat different way of finding the positive root of that equation. We denote the positive root of that equation by θ_{*p}. The value $\theta=\theta_{*p}$ corresponds to a number of waves in the transversal direction under which is realized the critical loading of the stability loss $\overline{P}_*$. Substituting equality (26) into (25), we obtain

$$-1,185\,\delta_*^\nu\overline{P}_*=\theta_{*p}-1,185\,\delta_*^\nu\theta_{*p}^{-2}-\theta_{*p}^{-3}. \tag{27}$$

It is not difficult to notice that equality (27) is likewise follows from equality (23) for $\omega=0$.

Consequently, the values $\overline{P},\theta$ satisfying equality (23) for which expression (22) vanishes, are the critical values of $\overline{P}_*$, θ_{*p}.

By virtue of equality (24), for $\overline{P}=0$, we obtain equation (12′) whose positive root is denoted, as above, by θ_0 and corresponds to the least frequency of the unloaded shell, whereas for $\overline{P}=\overline{P}_*$, we obtain equation (27) whose positive root $\theta=\theta_{*p}$ corresponds to $\omega=0$.

Thus, for $\overline{P}$, varying in the interval

$$0\le\overline{P}\le\overline{P}_* \tag{28}$$

the least frequency varies in the interval $[\omega(\theta_0,\overline{P}=0),0]$.

Analogously to the investigation carried out in [2], we can show that when $\overline{P}$ varies in the interval (28) for $\delta_*\ge0$, the value of θ realizing the least frequency $\omega(\theta,\overline{P})$ lies in the interval

$$\theta_0\le\theta\le\theta_*. \tag{29}$$

In particular for $\delta_*=0$ inequalities (28) and (29) take the form $0\le\overline{P}\le1$, $\theta_0=\theta_*=1$ (or $0\le P\le e_1^{1/2}e_2^{1/2}$, $N_0=N_*=(e_1/e_2)^{1/4}$).

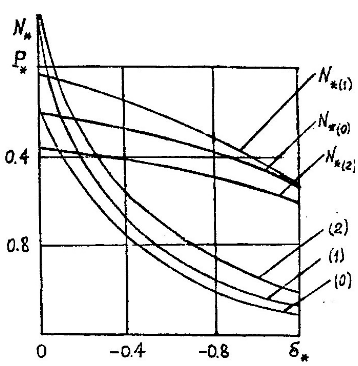

Figure 5

Dependencies $N_* = n_*^2/n_0^2$ and $P = p_*/p_0^2$ on the parameter $\delta \leq 0$ for the cases $i = 0, 1, 2$ are given in Figure 5. The corresponding curves are denoted by $N_{*(i)}$ and (i). It is not difficult to see that for the concave shells of importance is the elastic parameter in the circumferential direction as compared with the axial one.

By virtue of equation (27), we can construct the dependence $N(\overline{P})$ which realizes the least frequency of the prestressed shell for various values of δ_*. To this end, we fix the parameters e_1, e_2, δ_* and having the value of θ from the interval (29), we find $\overline{P}_*$ by formula (27).

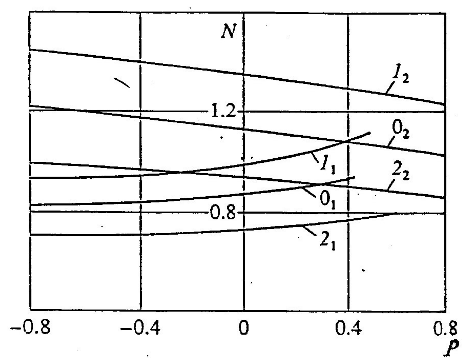

FIGURE 6

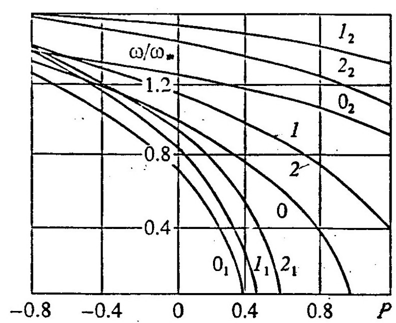

FIGURE 7

In Figure 6, we can see the values $N(P)$ for the cases $i = 0, 1, 2$ (for $\delta_* = 0,4$ and $\delta_* = -0,4$) which are denoted by i_1 and i_2. Figure 7, gives the curves of dependence of dimensionless least frequencies $\omega(N, P)/\omega_*$ on P for the above-mentioned cases which are likewise denoted by i_1 and i_2. Moreover, in Figure 7, we see the graph of dependence of ω/ω_* on P for the cylindrical shell ($\delta_* = 0$) in the cases $i = 0, 1, 2$ denoted, respectively, by $0, 1, 2$. On the basis of these graphs, it is not difficult to notice that if the influence of the ortotropy parameters for the cylindrical shell is practically the same, then for the concave shell, the influence of an elastic parameter in the circumferential direction is much more greater as compared with the axial elastic parameter, whereas the situation is opposite for the convex shells.

In the case of tensile forces $\overline{P} < 0$, equations (22) and (23) take the form

$$\omega^2/\omega_*^2 = 0,5\, e_1^{1/2} e_2^{1/2} \big[\theta^2 + \theta^{-2} + 2,37\, \delta_*^{\nu} \theta^{-1} + 1,404\, \delta_*^{\nu^2} + 2|\overline{P}|(1 - 1,185\, \delta_*^{\nu}\theta)\big], \tag{30}$$

$$1,185\, \delta_*^{\nu} |\overline{P}| = \theta - 1,185\, \delta_*^{\nu} \theta^{-2} - \theta^{-3}. \tag{31}$$

Analogously to the above-said, on the basis of formulas (30) and (31), we can construct the corresponding dependencies. In Figure 7, on the left of the Oy-axis we can see the graphs of dependence of ω/ω_* on $\overline{P}<0$ for the cases $i=0,1,2$ (for $\delta_*=0,4$ and $\delta_*=-0,4$).

Consider now a general case $\overline{P}\neq 0$, $\widetilde{Q}\neq 0$. Just as above, the frequency is defined by equality (10). For $\omega=0$, by virtue of (10), we obtain

$$1,755\,e_1^{-1/4}e_2^{-3/4}\widetilde{Q}=\theta+\theta^{-3}+2,37\,\delta_*^{\nu}\theta^{-2}+1,404\,\overline{\delta}_*^{\nu^2}\theta^{-1}-2\,\overline{P}(\theta^{-1}-1,185\,\overline{\delta}_*^{\nu}). \tag{32}$$

The least value $\widetilde{Q}>0$ depending on θ is realized for $Q'_\theta=0$. Thus we have

$$\begin{gathered}\theta^4+c\theta^2+d\theta+e=0,\quad c=2\overline{P}-1,404\,\overline{\delta}_*^{\nu^2},\\ d=-4,74\,\delta_*^{\nu},\quad e=-3.\end{gathered} \tag{33}$$

The roots of the last equation coincide with those of the two square equations

$$\begin{gathered}\theta^2+\frac{A_{1,2}}{2}\,\theta+\Big(y-\frac{d}{A_{1,2}}\Big)=0,\quad A_{1,2}=\pm\sqrt{8\,\alpha},\\ \theta_{1,2}=-\sqrt{\frac{\alpha}{2}}\pm\sqrt{\frac{d}{\sqrt{8\,\alpha}}-\frac{\alpha_1}{2}},\quad \theta_{3,4}=-\sqrt{\frac{\alpha}{2}}\pm\sqrt{-\frac{d}{\sqrt{8\,\alpha}}-\frac{\alpha_1}{2}},\\ \alpha=y_1-c/2,\quad \alpha_1=y_1+c/2,\end{gathered} \tag{34}$$

where y_1 is any real root of the cubic equation

$$y^3-\frac{c}{2}\,y^2-ey+\Big(\frac{ce}{2}-\frac{d^2}{8}\Big)=0 \tag{35}$$

or

$$z^3+3\,pz+2\,q=0\quad (z=y-c/6), \tag{36}$$

$$\begin{aligned}p&=1-\Big(2\,\overline{P}-1,404\,\overline{\delta}_*^{\nu^2}\Big)^2\Big/36,\\ q&=-\frac{1}{2}\Big(2\,\overline{P}+1,404\,\overline{\delta}_*^{\nu}\Big)^2\left[1-\frac{(2\,\overline{P}-1,404\,\overline{\delta}_*^{\nu^2})^3}{108\,(2\,\overline{P}+1,404\,\overline{\delta}_*^{2})}\right].\end{aligned} \tag{37}$$

If we assume that

$$\Big(2\,\overline{P}-1,404\,\overline{\delta}_*^{\nu^2}\Big)^2\Big/36\ll 1,$$

then expressions (37) take the form $p=1$, $q=-\frac{1}{2}(2\,\overline{P}+1,404\,\overline{\delta}_*^{\nu^2})$. Since the discriminant of equation (36) is $D=q^2+p^3>0$, we have only one real root

$$z=\Big(-q+\sqrt{q^2+p^3}\Big)^{1/3}+\Big(-q-\sqrt{q^2+p^3}\Big)^{1/3} \tag{38}$$

If we assume that

$$\big(2\,\overline{P}+1,404\,\overline{\delta}_*^{\nu^2}\big)\big/36\ll 1 \tag{38$'$}$$

and expand the expressions appearing in (38) in series, omitting all values of the second order of smallness, we arrive at $z=\big[2\,\overline{P}+1,404\,(\overline{\delta}_*^{\nu^2}-\gamma_*/4)\big]/3$. Then on the basis of (34), (36) and (33), we obtain

$$\begin{aligned}\alpha&=z-c/3=2\cdot 1,404\,\delta_*^{\nu^2},\\ \alpha_1&=z+\frac{2}{3}\,c=2\overline{P}-1,404\Big(\delta_*^{\nu^2}+\frac{3}{4}\,\gamma_*\Big)\Big/3.\end{aligned} \tag{39}$$

Taking into account that y_1 is the root of equation (35), we have

$$\frac{d^2}{8(y_1-c/2)}=y_1^2-e,$$

whence we get

$$\frac{|d|}{\sqrt{8\,\alpha}}=\sqrt{y_1^2-e}>y_1=\frac{y_1}{2}+\frac{y_1}{2}+\frac{c}{4}-\frac{c}{4}=\frac{1}{2}\Big(y_1-\frac{c}{2}\Big)+\frac{1}{2}\Big(y_1+\frac{c}{2}\Big).$$

Consequently,

$$\frac{|d|}{\sqrt{8\,\alpha}}-\frac{\alpha_1}{2}>\frac{\alpha}{2}\,. \tag{40}$$

Since $N^2=n^2/n_0^2$, of our interest are only positive roots of equation (33). Taking into account inequality (40), we find that for $\delta_*<0$ ($d>0$), positive is only the root θ_1, and for $\delta_*>0$ ($d<0$), positive is the root θ_3. Substituting the values d,α,α_1, according to equalities (33) and (39), into (34), we obtain

$$\theta_{1,2}=\sqrt{\sqrt{3}+0,234\Big(\delta_*^{\nu^2}+\frac{3}{4\,e_1}\,\gamma_*^{\nu}\Big)-\overline{P}}\pm 0,684\,|\delta_*^{\nu}|, \tag{41}$$

where the indices "1" and "2" correspond to $\delta_*>0$ and $\delta_*<0$, respectively. It should be noted that the above formula is, according to inequality ($38'$), valid for comparatively not large values of rigidity of the elastic filler γ_*^{ν}. Taking into account that θ in an expanded form is $\theta=(e_1/e_2)^{1/4}n^2/\lambda_1\varepsilon^{-1}4$, we have

$$\begin{aligned} n_{1,2}^2=(e_1/e_2)^{1/4}\bigg\{\bigg(&\sqrt{3}+0,270(e_1/e_2)^{1/2}\varepsilon^{-1/2}\Big[\Big(\frac{\delta_0}{\ell}\Big)^2\\ &+\frac{3}{4}\,\frac{\gamma}{e_1}\Big(\frac{\ell}{r}\Big)^2\Big]-\overline{P}\bigg)^{1/2}\pm 0,735\,\Big(\frac{e_1}{e_2}\Big)^{1/4}\varphi^{-1/4}\,\frac{|\delta_0|}{\ell}\bigg\}\lambda_1\varepsilon^{-1/4}. \end{aligned} \tag{42}$$

In particular, for $\delta_0=\gamma=p=0$, we obtain the well-known formula for a critical number of waves of the cylindrical shell of middle length: $n^2=(e_1/e_2)^{1/4}\sqrt{3}\,\lambda_1\varepsilon^{-1/4}$ [5].

From formula (42), it is not difficult to notice that under the action of contracting forces a number of critical circumferential waves decreases, while under the action of tensile forces this number increases.

Formula (39), as it has been mentioned above, takes place if condition ($38'$) is fulfilled. In the case if this condition is not fulfilled we have to proceed from full expressions (37). Defining thus the values θ_* (for fixed $\delta_*^{\nu},\gamma_*^{\nu},\overline{P},e_1,e_2$) and substituting into (32), we obtain the corresponding critical value of $\widetilde{Q}_*$. In an expanded form, formula (32) for a critical pressure has the form

$$\begin{aligned} \overline{q}_{kp}=0,570\,e_1^{1/4}e_2^{3/4}g\Big[&\theta_*+\theta_*^{-3}+2,37\,\delta_*^{\nu}\theta_*^{-2}\\ &+1,404\big(\delta_*^{\nu^2}+\gamma_*^{\nu}/4e_1\big)\theta_*^{-1}-2\,\overline{P}\big(\theta_*^{-1}-1,185\,\widetilde{\delta}_*^{\nu}\big)\Big]\overline{q}_{0*}-\gamma_0\alpha_2T. \end{aligned}$$

Note that the obtained value of $\widetilde{Q}_*$ on the basis of formula (32) for the isotropic cylindrical shell coincides for ($\delta_*=0$, $\gamma_*=0$) $P>0$ practically with the results obtained in [4].

Consider now equation (12) and write it in the form

$$\begin{gathered} \theta^4+b\theta^3+d\theta+e=0,\quad b=1,185\,\delta_*^{\nu}\overline{P}-0,8775\,\widetilde{Q}^{\,\nu},\\ \widetilde{Q}^{\,\nu}=e_1^{-1/4}e_2^{3/4}\widetilde{Q},\quad d=-1,185\,\delta_*^{\nu},\quad e=-1. \end{gathered} \tag{43}$$

The roots of this equation coincide with those of the following two equations

$$\theta^2+(b+B_{1,2})\frac{\theta}{2}+\Big(y_1+\frac{by_1-d}{B_{1,2}}\Big)=0,\quad B_{1,2}=\pm\sqrt{8\big(y_1-b^2/8\big)}. \tag{44}$$

Introduce the notation

$$\gamma_1=y_1+b^2/8,\quad \gamma_2=y_1-b^2/4. \tag{45}$$

Then the roots of these equations will take the form

$$\theta_{1,2} = -\frac{\sqrt{8\,\gamma_1 + b}}{4} \pm \sqrt{-\frac{by_1 - d}{\sqrt{8\,\gamma_1}} + \frac{b\sqrt{8\,\gamma_1} - 4\,\gamma_2}{8}}, \tag{46}$$

$$\theta_{3,4} = \frac{\sqrt{8\,\gamma_1 + b}}{4} \pm \sqrt{\frac{by_1 - d}{\sqrt{8\,\gamma_1}} - \frac{b\sqrt{8\,\gamma_1} + 4\,\gamma_2}{8}}, \tag{47}$$

where y_1 is any real root of the cubic equation

$$y^3 + 3\,py + 2q = 0, \quad 3\,p = 1 - \frac{1,185^2\widetilde{\delta}_*^{\nu^2}\overline{P}M}{4}\,,$$

$$2\,q = -\frac{1,185^2\widetilde{\delta}_*^{\nu^2}(1 - \overline{P}^2M^2)}{8}\,, \quad M = 1 - 0,7405\,\widetilde{Q}/\widetilde{\delta}_*^{\nu}\overline{P}$$

for

$$\begin{gathered}\frac{1,185^2\widetilde{\delta}^{\nu^2}|\overline{P}M|}{4} \ll 1 \quad \big(\widetilde{\delta}_*^{\nu} \le 0,5, \;\; |\overline{P}M| \le 0,5\big), \\ p = \frac{1}{3}, \quad q = -1,185^2\,\widetilde{\delta}_*^{\nu^2}\big(1 - \overline{P}^{\,\nu}M^2\big)/16.\end{gathered} \tag{48}$$

Since the discriminant of this equation $D > 0$, we have one real root

$$y_1 = \big(-q + \sqrt{D}\big)^{1/3} + \big(-q - \sqrt{D}\big)^{1/3},$$

$$\sqrt{D} = \sqrt{1 + 0,208\,\widetilde{\delta}_*^{\nu^4}\big(1 - \overline{P}^2M^2\big)/3^{3/2}}.$$

If we assume

$$0,208\,\widetilde{\delta}_*^{\nu^4}\big(1 - \overline{P}^2M^2\big) \ll 1 \tag{49}$$

then in a full analogy with the above-said, we obtain $y_1 = 0,1755\,\widetilde{\delta}_*^{\nu^2}\big(1 - \overline{P}^2M^2\big)$. Under the restrictions (48), inequality (49) is all the more fulfilled. Substituting the values y_1, b, d, e_1, e_2 into expressions (46) and (47) and also taking into account inequality (48), we find that for $d > 0$ ($\delta_*^{\nu} < 0$), positive is only the root θ_1, whereas for $d < 0$ ($\delta_*^{\nu} > 0$), positive is the root θ_3. As a result, we have

$$\begin{aligned}\theta_1 = \Big[1 + 0,1755\,\widetilde{\delta}_*^{\nu^2}\overline{P}M_1\big(1 - \overline{P}^2M_1^2\big) - 0,0877\,\widetilde{\delta}_*^{\nu^2}\big(1 + 2\,\overline{P}M_1 \\ - 2\,\overline{P}^2M_1^2\big)\Big]^{1/2} + 0,2962\,\widetilde{\delta}_*^{\nu}\big(1 - \overline{P}M_1\big) \;\; (\delta_*^{\nu} > 0),\end{aligned} \tag{50}$$

$$\begin{aligned}\theta_2 = \Big[1 + 0,1755\,\widetilde{\delta}_*^{\nu^2}PM_2\big(1 - \overline{P}^2M_2^2\big) - 0,0877\,\widetilde{\delta}_*^{\nu^2}\big(1 + 2\,\overline{P}M_2 \\ - 2\,\overline{P}M_2\big)\Big]^{1/2} - 0,2962\,\widetilde{\delta}_*^{\nu}\big(1 - \overline{P}M_1\big) \;\; (\delta_*^{\nu} < 0)\end{aligned} \tag{51}$$

$$M_1 = 1 - 0,7405\,\widetilde{Q}^{\nu}/\delta^{\nu}\overline{P}, \quad M_1 = 1 + 0,7405\,\widetilde{Q}^{\nu}/|\delta^{\nu}|\overline{P}.$$

For $\widetilde{\delta}_*^{\nu} > 0$, $\overline{P}/\widetilde{Q} > 0$ the value $M_1 = 0$, if $\delta_*^{\nu} = 0,7405\,\overline{P}/\widetilde{Q}^{\nu}$; for $\delta_*^{\nu} < 0$, $\overline{P}/\widetilde{Q}^{\,\nu} < 0$, the value $M_2 = 0$, if $|\delta_*^{\nu}| = -0,7405\,\overline{P}/\widetilde{Q}^{\,\nu}$, and formulas (50), (51) take the form

$$\theta = \sqrt{1 - 0,0877\,\widetilde{\delta}_*^{\nu^2}} + 0,2962\,\widetilde{\delta}_*^{\nu} \quad (\delta_*^{\nu} > 0),$$

$$\theta = \sqrt{1 - 0,0877\,\widetilde{\delta}_*^{\nu^2}} - 0,2962\,\big|\widetilde{\delta}_*^{\nu}\big| \quad (\delta_*^{\nu} < 0).$$

Note that this case of the certain values $\widetilde{\delta}_*^{\nu}$ corresponds to the cases for which the normal circumferential stresses due to meridional loading, external pressure and also temperature effect neutralise mutually each other.

For $\gamma_0 = 0$, $e_1 = e_2 = 1$ we have $\widetilde{\delta}_*^{\nu} = \delta_*$, $\widetilde{q} = \overline{q}$ and for θ, we obtain the formula given in [3].

Substituting the obtained expression for θ (for fixed δ_*^{ν}, $\overline{P}$, $\widetilde{Q}$, γ^{ν}) into formula (10), we obtain the corresponding least value of the dimensionless frequency ω/ω_*.

The above obtained formulas and graphs show how much substantially vary critical loading, the least frequency and the forms of wave formation depending on the orthotropy parameters, shell shape and external effects.

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A. Razmadze Mathematical Institute I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

E-mail address: kotic13@mail.ru

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ON MEIR–KEELER CONTRACTION IN BRANCIARI b-METRIC SPACES

Z. MITROVIĆ[1] AND S. RADENOVIĆ[2]

Abstract. In this paper we consider Meir–Keeler type results in the context of Branciari b-metric spaces. Our results generalize, improve and complement several ones in the existing literature.

1. Introduction and Preliminaries

In the paper [14] the authors introduced the concept of $b_v(s)$-metric space as follows.

Definition 1.1 ([14]). Let X be a set, let d be a function from $X \times X$ into $[0, \infty)$ and let $v \in \mathbf{N}$. Then (X, d) is said to be a $b_v(s)$-metric space if for all $x, y \in X$ and for all distinct points $u_1, u_2, \dots, u_v \in X$, each of them different from x and y the following hold:

$(B1)$ $d(x, y) = 0$ if and only if $x = y$;

$(B2)$ $d(x, y) = d(y, x)$;

$(B_v3(s))$ there exists a real number $s \geq 1$ such that

$$d(x, y) \leq s[d(x, u_1) + d(u_1, u_2) + \cdots + d(u_v, y)].$$

Note that:

- $b_1(1)$-metric space is usual metric space,
- $b_1(s)$-metric space is b-metric space with coefficient s of Czerwik [3, 4],
- $b_2(1)$-metric space is rectangular metric space or Branciari metric space [2],
- $b_2(s)$-metric space is rectangular b-metric space with coefficient s of George et al [8] or Branciari b-metric space [9],
- $b_v(1)$-metric space is v-generalized metric space of Branciari [2],
- Let (X, d_K) be a N-polygonal K-metric space over an ordered Banach space $(V, ||\cdot||, K)$ (see [7]) such that K is a closed normal cone with normal constant λ and the function $D : X \times X \to [0, \infty)$ defined by $D(x, y) = ||d_K(x, y)||$. Then (X, D) is $b_N(\lambda)$-metric space.

Definition 1.2 ([14]). Let (X, d) be a $b_v(s)$-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent in (X, d) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$.

(c) (X, d) is said to be a complete $b_v(s)$-metric space if every Cauchy sequence in X converges to some $x \in X$.

Definition 1.3 ([11]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called Meir–Keeler contraction if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \Rightarrow d(Tx, Ty) < \epsilon \text{ for all } x, y \in X.$$

Definition 1.4 ([16]). A mapping $T : X \to X$ is called α-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1,$$

where $\alpha : X \times X \to [0, \infty)$ is a given function. A function α is transitive if, given $x, y, z \in X$,

$$\alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1.$$

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Lemma 1.1 ([1])**.** *Let $T : X \to X$ be an α-admissible mapping and let $\{x_n\}$ be a Picard sequence of T based on a point $x_0 \in X$. If x_0 satisfies $\alpha(x_0, Tx_0) \geq 1$, then $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbf{N}$. Additionally, if α is transitive, then $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbf{N}$ such that $n < m$.*

One generalization on Meir–Keeler mappings was given by Gülyaz et al in the paper [9].

Definition 1.5 ([9])**.** Let (X, d) be a Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-admissible mapping. If for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq M(x, y) < \epsilon + \delta \text{ implies } \alpha(x, y)d(Tx, Ty) < \frac{\epsilon}{s}, \tag{1}$$

where

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Ty, y)\}$$

for all $x, y \in X$, then T is called generalized α-Meir–Keeler contraction.

Definition 1.6 ([9])**.** A Branciari b-metric space (X, d) is called α-regular if for any sequence $\{x_n\}$ such that $\lim d(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbf{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbf{N}$.

We note that Gülyaz et al in the paper [9] define Brancari b-metric spaces, but this class of space has already been defined by George et al in the paper [8] and others called them rectangular b-metric spaces. Also in the paper [9] Gülyaz et al prove Lemma 2. 5 (see [9, p. 5449]).

Lemma 1.2 (Lemma 2. 5. in [9])**.** *Let (X, d) be a Branciari b-metric space with a constant $s \geq 1$. Let $\{x_n\}$ be a sequence in X satisfying*

1. *$x_m \neq x_n$ for all $m \neq n, m, n \in \mathbf{N}$,*
2. *$d(x_n, x_{n+1}) \leq \frac{1}{s} d(x_{n-1}, x_n)$, for all $n \in \mathbf{N}$,*
3. *$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. Then $\{x_n\}$ is a Cauchy sequence in (X, d).*

Unfortunately, the Lemma 1.2 is not correct, as shown in the following example.

Example 1.1. Put $X = \mathbf{R}$, $d(x, y) = |x - y|, x, y \in X$ and $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then (X, d) is Branciari b-metric space with coefficient $s = 1$ and sequence $\{x_n\}$ fulfills the conditions of Lemma 1.2 but not the Cauchy sequence.

Of course, then main result in the [9] is not correct, because its proof is needed by Lemma 2. 5. Here we prove the new version of Lemma 2. 5. in [9], also we show that continuity of function T is not necessary. Also, note that condition (1) follows the following condition

$$\alpha(x, y)d(Tx, Ty) \leq \lambda M(x, y),$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{s})$. In addition, the authors in [9] use that is the next result.

Proposition 1.1 (Proposition 1.6. in [9])**.** *Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X, d) such that $\lim d(x_n, x) = 0$, where $x \in X$. Then $\lim d(x_n, y) = d(x, y)$, for all $y \in X$. In particular, the sequence $\{x_n\}$ does not converge to y if $y \neq x$.*

For proof of the main result in [9] (Theorem 2.6) authors used that the Proposition 1.1 is valid if replace Branciari metric space by a Branciari b-metric space.

Unfortunately, Proposition 1.1 is not true in Branciari b-metric space (see Example 1.7. in [8]).

2. Main Results

Lemma 2.1. *Let (X, d) be a complete $b_2(s)$-metric space and let $\{x_n\}$ be a sequence in X such that x_n $(n \geq 0)$ are all different. Suppose that exists $\lambda \in [0, \frac{1}{\sqrt{s}})$ such that*

(1) *$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$,*
(2) *$d(x_n, x_{n+2}) \leq \lambda d(x_{n-1}, x_{n+1})$,*

for all $n \geq 1$. Then $\{x_n\}$ is a convergent sequence in (X, d). Additionally, if d is continuous, then for x^* for which $x^* = \lim x_n$ the next estimate holds

$$d(x_n, x^*) \leq \frac{2s\lambda^n}{1 - s\lambda^2} d(x_0, x_1) + 3\lambda^n[d(x_0, x_1) + d(x_0, x_2)]. \tag{2}$$

Proof. First, we note that from conditions 1 and 2 we follow

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1), \tag{3}$$

and

$$d(x_n, x_{n+2}) \leq \lambda^n d(x_0, x_2), \tag{4}$$

for all $n \geq 1$.

Let $n, m \in \mathbf{N}$ and $m > n$.

1. Case: $m - n = 2k$ for any $k \in \mathbf{N}$.

From condition $(B_2 3(s))$ we have

$$\begin{array}{rcl} d(x_n, x_m) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ & + & s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ & + & s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] \\ & \vdots & \\ & + & s^{k-2}[d(x_{n+2k-6}, x_{n+2k-5}) + d(x_{n+2k-5}, x_{n+2k-4})] \\ & + & s^{k-1}[d(x_{n+2k-4}, x_{n+2k-3}) + d(x_{n+2k-3}, x_{n+2k-2})] \\ & + & s^{k-1} d(x_{n+2k-2}, x_{n+2k}) \end{array}$$

From conditions (3) and (4) we obtain

$$\begin{array}{rcl} d(x_n, x_m) & \leq & s\lambda^n(1+\lambda) d(x_0, x_1) \\ & + & s^2\lambda^{n+2}(1+\lambda) d(x_0, x_1) \\ & + & s^3\lambda^{n+4}(1+\lambda) d(x_0, x_1) \\ & \vdots & \\ & + & s^k\lambda^{n+2k-2}(1+\lambda) d(x_0, x_1) \\ & + & s^k\lambda^{n+2k-2} d(x_0, x_2). \end{array}$$

So,

$$\begin{array}{rcl} d(x_n, x_m) & \leq & s\lambda^n(1+\lambda) d(x_0, x_1)[1 + s\lambda^2 + \cdots + (s\lambda^2)^{k-1}] \\ & + & (s\lambda^2)^{k-1}\lambda^n d(x_0, x_2). \end{array}$$

How is it $0 \leq s\lambda^2 < 1$, we obtain

$$d(x_n, x_m) \leq \frac{s\lambda^n(1+\lambda) d(x_0, x_1)}{1 - s\lambda^2} + \lambda^n d(x_0, x_2). \tag{5}$$

Now from (5), we conclude that $\{x_n\}$ is Cauchy.

2. Case: $m - n = 2k + 1$ for any $k \in \mathbf{N}$. Similar to the previous case from condition $B_2 3(s)$ we have

$$\begin{array}{rcl} d(x_n, x_m) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ & + & s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ & + & s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] \\ & \vdots & \\ & + & s^{k-2}[d(x_{n+2k-6}, x_{n+2k-5}) + d(x_{n+2k-5}, x_{n+2k-4})] \\ & + & s^{k-1}[d(x_{n+2k-4}, x_{n+2k-3}) + d(x_{n+2k-3}, x_{n+2k-2})] \\ & + & s^k[d(x_{n+2k-2}, x_{n+2k-1}) + d(x_{n+2k-1}, x_{n+2k}) \\ & + & d(x_{n+2k}, x_{n+2k+1})], \end{array}$$

and from here again using the inequalities (3) and (4) we get

$$\begin{aligned} d(x_n, x_m) &\leq s\lambda^n(1+\lambda)d(x_0, x_1) \\ &+ s^2\lambda^{n+2}(1+\lambda)d(x_0, x_1) \\ &+ s^3\lambda^{n+4}(1+\lambda)d(x_0, x_1) \\ &\vdots \\ &+ s^k\lambda^{n+2k-2}(1+\lambda)d(x_0, x_1) \\ &+ s^k\lambda^{n+2k-2}(1+\lambda+\lambda^2)d(x_0, x_1). \end{aligned}$$

So we have

$$\begin{aligned} d(x_n, x_m) &\leq \frac{s\lambda^n(1+\lambda)d(x_0, x_1)}{1-s\lambda^2} \\ &+ s\lambda^n(s\lambda^2)^{k-1}(1+\lambda+\lambda^2)d(x_0, x_1) \end{aligned}$$

$$d(x_n, x_m) \leq \frac{s\lambda^n(1+\lambda)d(x_0, x_1)}{1-s\lambda^2} + \lambda^n(1+\lambda+\lambda^2)d(x_0, x_1). \tag{6}$$

So, $\{x_n\}$ is Cauchy. The estimate (2) follows from (5) and (6) when we let us m run infinitely. □

Lemma 2.2. *Let $T : X \to X$ be an α-admissible mapping and let $\{x_n\}$ be a Picard sequence of T based on a point $x_0 \in X$. If α is transitive, x_0 satisfies $\alpha(x_0, Tx_0) \geq 1$ and*

$$\alpha(x, y)d(Tx, Ty) \leq \lambda d(x, y), \tag{7}$$

for all $x, y \in X$, where $\lambda \in (0, 1)$, then it is

$$d(x_{m+k}, x_{n+k}) \leq \lambda^k d(x_m, x_n),$$

for all $m, n, k \in \mathbf{N}, n < m$.

Proof. Using Lemma 1.1 we get

$$\alpha(x_m, x_n) \geq 1 \text{ for all } n < m.$$

From condition (7) follows

$$\begin{aligned} d(x_{m+k}, x_{n+k}) &\leq \frac{\lambda}{\alpha(x_{m+k-1}, x_{n+k-1})} d(x_{m+k-1}, x_{n+k-1}) \\ &\leq \lambda d(x_{m+k-1}, x_{n+k-1}) \\ &\vdots \\ &\leq \lambda^k d(x_m, x_n). \end{aligned}$$

□

Lemma 2.3. *Let (X, d) be a $b_v(s)$-metric space, $T : X \to X$ be a mapping and let $\{x_n\}$ be a sequence in X such that $x_0 \in X$ and $x_{n+1} = Tx_n$. If there exists $\lambda \in [0, 1)$ and such that*

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \textit{ for all } n \geq 1, \tag{8}$$

then T has a fixed point or $x_n \neq x_m$ for all $n \neq m$.

Proof. If $x_n = x_{n+1}$ then x_n is fixed point of T and proof is hold. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Then $x_n \neq x_{n+k}$ for all $n \geq 0, k \geq 1$. Namely, if $x_n = x_{n+k}$ for some $n \geq 0$ and $k \geq 1$ we have that $Tx_n = Tx_{n+k}$ and $x_{n+1} = x_{n+k+1}$. Then (8) implies that

$$d(x_{n+1}, x_n) = d(x_{n+k+1}, x_{n+k}) \leq \lambda^k d(x_{n+1}, x_n) < d(x_{n+1}, x_n)$$

is a contradiction. Thus we assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbf{N}$. □

Theorem 2.1. *Let (X,d) be a complete α-regular $b_2(s)$-metric space and $T : X \to X$ be a α-admissible such that T satisfies the conditions*

$$\alpha(x,y)d(Tx,Ty) \le \lambda d(x,y),$$

for all $x,y \in X$, where $\lambda \in (0,1)$. If $\alpha(x_0,Tx_0) \ge 1$ for some $x_0 \in X$ and α transitive then T has a fixed point in X.

Proof. Let $\lambda \in [0,1)$. Since $\lim_{n\to\infty} \lambda^n = 0$, there exists a natural number N such that

$$0 < \lambda^k \cdot s < 1, \tag{9}$$

for all $k \ge N$.

Let $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$. From Lemma 1.1 we have that

$$\alpha(x_n,x_{n+1}) \ge 1 \text{ for all } n \in \mathbf{N}.$$

Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \ge 0$. If $x_n = x_{n+1}$ then x_n is fixed point of T and proof is hold. So, suppose that $x_n \ne x_{n+1}$ for all $n \ge 0$. Then $x_n \ne x_m$ for all $n < m$. Since, (X,d) is $b_2(s)$-metric space, from condition $(B_2(s))$ we have

$$d(x_m,x_n) \le s[d(x_m,x_{m+k}) + d(x_{m+k},x_{n+k}) + d(x_{n+k},x_n)].$$

Using Lemma 2.2 we get

$$\begin{aligned} d(x_m,x_n) &\le s[\lambda^m d(x_0,x_k) + \lambda^k d(x_m,x_n) + \lambda^n d(x_0,x_k)] \\ (1-s\lambda^k)d(x_m,x_n) &\le s(\lambda^m + \lambda^n)d(x_0,x_k). \end{aligned}$$

From this, together with (9), we obtain

$$d(x_m,x_n) \le \frac{s(\lambda^m+\lambda^n)}{1-s\lambda^k} d(x_0,x_k).$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X,d) there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Now we obtain that x^* is a fixed point of T. Namely, for any $n \in \mathbf{N}$ we have

$$\begin{aligned} d(x^*,Tx^*) &\le s[d(x^*,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},Tx^*)] \\ &= s[d(x^*,x_n) + d(x_n,x_{n+1}) + d(Tx_n,Tx^*)] \\ &\le s\Big[d(x^*,x_n) + d(x_n,x_{n+1}) + \frac{\lambda d(x_n,x^*)}{\alpha(x_n,x^*)}\Big] \\ &\le s[d(x^*,x_n) + d(x_n,x_{n+1}) + \lambda d(x_n,x^*)]. \end{aligned}$$

Since, $\lim_{n\to\infty} d(x^*,x_n) = 0$ and $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, we have $d(x^*,Tx^*) = 0$ i. e., $Tx^* = x^*$. □

Remark 2.1. We note that the previous Theorem is an improvement in the results in [13] (Theorem 2.1).

In the next Theorem we do not assume that the function α is transitive.

Theorem 2.2. *Let (X,d) be a complete α-regular $b_2(s)$-metric space and $T : X \to X$ be a α-admissible such that T satisfies the conditions*

$$\alpha(x,y)d(Tx,Ty) \le \lambda M(x,y), \tag{10}$$

for all $x,y \in X$, where $\lambda \in (0,\frac{1}{s})$. If $\min\{\alpha(x_0,Tx_0),\alpha(x_0,T^2x_0)\} \ge 1$ for some $x_0 \in X$, then T has a fixed point in X.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$ and $x_{n+1} = Tx_n, n = 1, 2, \ldots$. Since T is a α-admissible, from Lemma 1.1 we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbf{N}. \tag{11}$$

Similarly, from $\alpha(x_0, T^2x_0) \geq 1$ follows

$$\alpha(x_n, x_{n+2}) \geq 1 \text{ for all } n \in \mathbf{N}. \tag{12}$$

From conditions (10) and (11) we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) \\ &\leq \lambda M(x_n, x_{n+1}), \end{aligned}$$

since

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$$

and

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_{n+1}, x_{n+2})$$

not possible, we conclude that it is

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}), \tag{13}$$

so, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \lambda^n d(x_1, x_0).$$

Similarly, from conditions (10) and (12) we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha(x_{n-1}, x_{n+1})d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \lambda M(x_{n-1}, x_{n+1}), \end{aligned}$$

since

$$M(x_{n-1}, x_{n+1}) = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}$$

and

$$d(x_{n+1}, x_{n+2}) \leq \lambda^2 d(x_{n-1}, x_n),$$

we conclude that it is

$$d(x_n, x_{n+2}) \leq \lambda \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}. \tag{14}$$

From conditions (13) and (14) we obtain

$$d(x_n, x_{n+2}) \leq \lambda^n \max\{d(x_1, x_0), d(x_0, x_2)\}. \tag{15}$$

From (13) and (15) and Lemma 2.1 we conclude that $\{x_n\}$ is Cauchy, so it converges to a limit $x^* \in X$. How is (X, d) α-regular $b_2(s)$-metric space, from (11) we get that $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbf{N}$. From Lemma 2.3 we conclude that $x_n \neq x_m$ for all $n \neq m$. Now we obtain that x^* is the fixed point of T. Namely, for any $n \in \mathbf{N}$ we have

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &= s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tx^*)] \\ &\leq s\Big[d(x^*, x_n) + d(x_n, x_{n+1}) + \frac{\lambda M(x_n, x^*)}{\alpha(x_n, x^*)}\Big] \\ &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) \\ &+ \lambda \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*)\}]. \end{aligned}$$

Since, $\{x_n\}$ converges to x^* and $\lambda < \frac{1}{s}$, we have $Tx^* = x^*$. □

Remark 2.2. We note that in the previous Theorem 2.2, for the proof of the convergence of the sequence $\{x_n\}$, a sufficient condition is that it is $\lambda \in (0, \frac{1}{\sqrt{s}})$. Also, if $M(x,y) = d(x,y)$, we get that

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &= s[d(x^*, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tx^*)] \\ &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + \frac{\lambda d(x_n, x^*)}{\alpha(x_n, x^*)}] \\ &\leq s[d(x^*, x_n) + d(x_n, x_{n+1}) + \lambda d(x_n, x^*)]. \end{aligned}$$

So, $Tx^* = x^*$.

Thus, the following result follows from the Theorem 2.2 and Remark 2.2.

Theorem 2.3. *Let (X, d) be a complete α-regular $b_2(s)$-metric space and $T : X \to X$ be a α-admissible such that T satisfies the conditions*

$$\alpha(x, y)d(Tx, Ty) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{\sqrt{s}})$. If $\min\{\alpha(x_0, Tx_0), \alpha(x_0, T^2x_0)\} \geq 1$ for some $x_0 \in X$, then T has a fixed point in X.

Remark 2.3. If $\alpha(x, y) = 1$, for all $x, y \in X$ then T has unique fixed point. Let y^* be another fixed point of T. Then it follows from (8) that $d(x^*, y^*) = d(Tx^*, Ty^*) \leq \lambda d(x^*, y^*) < d(x^*, y^*)$, is a contradiction. Therefore, we must have $d(x^*, y^*) = 0$, i.e., $x^* = y^*$.

We note that from Theorem 2.3 we obtain the following result (Theorem 2.1. in [8]).

Theorem 2.4 ([8]). *Let (X, d) be a complete rectangular b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s}]$. Then T has a unique fixed point.

Remark 2.4. As $\frac{1}{s} < \frac{1}{\sqrt{s}}, (s > 1)$, using the Lemma 2.1, the following results can be improved Theorem 2.1. in [6], Theorem 2. 1. in [5], Theorem 1. in [15], Theorem 2.1. in [18].

The following result is known for $b_1(s)$-metric space (see R. Miculescu and A. Mihail [12, Lemma 2.2] and T. Suzuki [17, Lemma 6]).

Lemma 2.4 ([12, 17]). *Every sequence $(x_n)_{n\in\mathbf{N}}$ of elements from a b-metric space (X, d, s), having the property that there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}),$$

for every $n \in \mathbf{N}$, is Cauchy.

It is therefore natural to ask the following question.

Question. Does the conclusion of Lemma 2.1 hold if $\frac{1}{\sqrt{s}}$ is replaced by 1?

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[1]University of Banja Luka, Faculty of Electrical Engineering, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina
E-mail address: zoran.mitrovic@etf.unibl.org

[2]Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia
E-mail address: radens@beotel.rs

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CHARACTERIZATION OF SETS OF SINGULAR ROTATIONS FOR A CLASS OF DIFFERENTIATION BASES

G. ONIANI AND K. CHUBINIDZE

Abstract. We study the dependence of differential properties of an indefinite integral on rotations of the coordinate system. Namely, the following problem is studied: For a summable function f of what kind can be the set of rotations γ for which $\int f$ is not differentiable with respect to the γ-rotation of a given basis B? The result obtained in the paper implies a solution of the problem for any homothecy invariant differentiation basis B of two-dimensional intervals which has symmetric structure.

1. Definitions and Notation

A collection B of open bounded and non-empty subsets of $\mathbb{R}^n$ is called a *differentiation basis* (briefly: *basis*) if for every $x \in \mathbb{R}^n$ there exists a sequence (R_k) of sets from B such that $x \in R_k$ $(k \in \mathbb{N})$ and $\lim\limits_{k\to\infty} \operatorname{diam} R_k = 0$.

For a basis B by $B(x)$ $(x \in \mathbb{R}^n)$ it will be denoted the collection of all sets from B containing the point x.

Let B be a basis. For $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the upper and lower limits of the integral means $\frac{1}{|R|}\int_R f$, where R is an arbitrary set from $B(x)$ and $\operatorname{diam} R \to 0$, are called the *upper and the lower derivatives with respect to B of the integral of f at the point x*, and are denoted by $\overline{D}_B(\int f, x)$ and $\underline{D}_B(\int f, x)$, respectively. If the upper and the lower derivatives coincide, then their common value is called the *derivative of $\int f$ at the point x* and denoted by $D_B(\int f, x)$. We say that B *differentiates* $\int f$ (or $\int f$ *is differentiable with respect to* B) if $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each f in the class of functions $F \subset L(\mathbb{R}^n)$ we say that B *differentiates* F. By F_B denote the class of all functions $f \in L(\mathbb{R}^n)$ the integrals of which are differentiable with respect to B. The *maximal operator M_B corresponding to B* is defined as follows: $M_B(f)(x) = \sup_{R\in B(x)} \frac{1}{|R|}\int_R |f|$, where $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

A basis B is called *translation invariant* (*homothecy invariant*) if for any set R from B and any translation (homothecy) $M : \mathbb{R}^n \to \mathbb{R}^n$ the set $M(R)$ also belongs to B. It is easy to check that each homothecy invariant basis is translation invariant also. Let us call a basis B convex if each set $R \in B$ is convex.

Denote by $\mathbf{I} = \mathbf{I}(\mathbb{R}^n)$ the basis consisting of all n-dimensional intervals. Differentiation with respect to $\mathbf{I}$ is called *strong differentiation*.

Let us call a basis B *non-standard* if there exists a function $f \in L(\mathbb{R}^n)$ the integral of which is not differentiable with respect to B (i.e. if B does not differentiate $L(\mathbb{R}^n)$).

The basis $\mathbf{I}$ is non-standard (see, e.g., [3, Ch. IV, §1]). Note that (see, [3, Appendix III]) a homothecy invariant basis B of multi-dimensional intervals is non-standard if and only if $\sup\{I \in B : l^I/l_I\} = \infty$, where l^I and l_I are the lengthes of the biggest and of the smallest edges of an interval I, respectively. Moreover, a clear geometrical criterion for the non-standartness it is known also for translation invariant bases of multi-dimensional intervals (see [14, 16]).

By $\Gamma(\mathbb{R}^n)$ denote the collection of all rotations in $\mathbb{R}^n$.

Let B be a basis in $\mathbb{R}^n$ and $\gamma \in \Gamma(\mathbb{R}^n)$. The *$\gamma$-rotated basis B* is defined as follows: $B(\gamma) = \{\gamma(R) : R \in B\}$.

Denote by ρ_k $(k = 0, 1, 2, 3)$ the rotation of the plane by the angle $\pi k/2$.

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Let us call a set $E \subset \Gamma(\mathbb{R}^2)$ *symmetric* if for any $\gamma \in E$ the rotations $\rho_1 \circ \gamma, \rho_2 \circ \gamma$ and $\rho_3 \circ \gamma$ also belong to the set E.

Let us call a translation invariant basis B of two-dimensional intervals *symmetric* if the bases $B(\rho_1), B(\rho_2)$ and $B(\rho_3)$ are equal to B. Obviously, the basis $\mathbf{I}(\mathbb{R}^2)$ is symmetric.

The set of two-dimensional rotations $\Gamma(\mathbb{R}^2)$ can be identified with the circumference $\mathbb{T} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, if to a rotation γ we put into correspondence the point $\gamma((1,0))$. The distance $d(\gamma, \sigma)$ between rotations $\gamma, \sigma \in \Gamma(\mathbb{R}^2)$ is assumed to be equal to the length of the smallest arch of the circumference $\mathbb{T}$ connecting the points $\gamma((1,0))$ and $\sigma((1,0))$.

A class of functions F is called *invariant with respect to a class of transformations of a variable* Λ if $(f \in F, \lambda \in \Lambda) \Rightarrow f \circ \lambda \in F$.

2. Introduction

The dependence of properties of functions of several variables on rotations of the system of coordinates (that is, on a transformation of the variables that is a rotation) has been studied by various authors.

Zygmund posed the following problem (see, [3, Ch. IV, §2]): Is it possible to improve an arbitrary function $f \in L(\mathbb{R}^2)$ by means of a rotation of the coordinate system to achieve strong differentiability of the integral of f? In [7] Marstrand gave a negative answer to this problem by constructing a non-negative function $f \in L(\mathbb{R}^2)$ such that $\overline{D}_{\mathbf{I}}\left(\int f \circ \gamma, x\right) = \infty$ a.e. for every $\gamma \in \Gamma(\mathbb{R}^2)$. In the works [6, 10, 13] and [11] the result of Marstrand was extended to bases of quite general type.

As established by Lepsveridze [5], Oniani [8] and Stokolos [15], the property of strong differentiability (that is, the class $F_{\mathbf{I}}$) is not invariant with respect to linear changes of variables and, in particular, to rotations. A similar result was proved by Dragoshanskii [2] for the class of continuous functions of two variables whose Fourier series (Fourier integral) is Pringsheim convergent almost everywhere.

In [11] non-invariance of a class F_B with respect to rotations was proved for any non-standard translation invariant basis B of multi-dimensional intervals.

Suppose B is a translation invariant basis. Then it is easy to verify that the differentiation of the integral of a "rotated" function $f \circ \gamma$ with respect to B at a point x is equivalent to the differentiation of the integral of f with respect to the "rotated" basis $B(\gamma^{-1})$ at the point $\gamma^{-1}(x)$. Consequently, we can reduce the study of the behavior of functions $f \circ \gamma$ $(\gamma \in \Gamma(\mathbb{R}^n))$ with respect to the basis B to the study of the behavior of f with respect to the rotated bases $B(\gamma)$ $(\gamma \in \Gamma(\mathbb{R}^n))$. Below we will use this approach.

If for a translation invariant basis B the class F_B is not invariant with respect to the rotations then there exists a function $f \in L(\mathbb{R}^n)$ having non-homogeneous behaviour with respect to rotated bases $B(\gamma)$ $(\gamma \in \Gamma(\mathbb{R}^n))$, more exactly, $\int f$ is not differentiable with respect to $B(\gamma)$ for some rotations and $\int f$ is differentiable with respect $B(\gamma)$ for some other rotations. Thus, for f some rotations γ are "singular" and some other rotations γ are "regular". In this connection naturally arises the problem: *Of what kind can be the sets of singular and of regular rotations for a fixed function*? Note that by duality argument we can restrict ourselves by studying sets of singular rotations.

In connection to the posed problem let us formulate rigor definition of a set of singular rotations: Suppose B is a translation invariant basis in $\mathbb{R}^n$ and $E \subset \Gamma(\mathbb{R}^n)$. Let us call E a *W_B-set* if there exists a function $f \in L(\mathbb{R}^n)$ with the following two properties: 1) $f \notin F_{B(\gamma)}$ for every $\gamma \in E$; 2) $f \in F_{B(\gamma)}$ for every $\gamma \notin E$.

Let us formulate also the definition of a set of "strongly" singular rotations: Suppose B is a translation invariant basis in $\mathbb{R}^n$ and $E \subset \Gamma(\mathbb{R}^n)$. Let us call E an *R_B-set* if there exists a function $f \in L(\mathbb{R}^n)$ with the following two properties: 1) $\overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty$ a.e. for every $\gamma \in E$; 2) $f \in F_{B(\gamma)}$ for every $\gamma \notin E$.

Now the problem can be formulated as follows: *For a given translation invariant basis B what kind of sets are W_B-sets(R_B-sets)*?

Note that for a standard basis B, i.e. for a basis B differentiating $L(\mathbb{R}^n)$, the problem is trivial. Here note also that if a translation invariant basis B of two-dimensional intervals is symmetric then every W_B-set and every R_B-set is symmetric.

In [1] for an arbitrary translation invariant basis B in $\mathbb{R}^2$ it was established the following three structural properties of sets of singular rotations: 1) Each W_B-set is of type $G_{\delta\sigma}$; 2) Each R_B-set is of type G_δ; 3) At most countable union of R_B-sets is a W_B-set.

Sets of singular rotations for the case of strong differentiability process on the plane (i.e., for the case $B = \mathbf{I}(\mathbb{R}^2)$) was characterized by G. Karagulyan [4] proving that: 1) a set $E \subset \Gamma(\mathbb{R}^2)$ is a $W_{\mathbf{I}(\mathbb{R}^2)}$-set if and only if E is symmetric and of type $G_{\delta\sigma}$; 2) a set $E \subset \Gamma(\mathbb{R}^2)$ is an $R_{\mathbf{I}(\mathbb{R}^2)}$-set if and only if E is symmetric and of type G_δ.

Our purpose is to show that the idea in Karagulyan's construction works for bases of two-dimensional intervals of quite general type.

3. Result

For a translation invariant convex basis B let us define the following function $\sigma_B(\lambda) = \overline{\lim\limits_{\varepsilon\to 0}} |\{M_B(\chi_{V_\varepsilon}) > \lambda\}|/|V_\varepsilon|$ $(0 < \lambda < 1)$, where V_ε is the ball with the centre at the origin and with the radius ε. Here and below everywhere χ_E denotes the characteristic function of a set E. We call σ_B a *spherical halo function of* B. It is easy to check that if B is homothecy invariant, then $\sigma_B(\lambda) = |\{M_B(\chi_V) > \lambda\}|$, where V is the unit ball.

We say that a translation invariant convex basis B has the *non-regular spherical halo function* if $\overline{\lim\limits_{\lambda\to 0}} \lambda\sigma_B(\lambda) = \infty$.

Theorem 1. *Let B be a non-standard translation invariant basis of two-dimensional intervals which is symmetric and has the non-regular spherical halo function. Then*:

- *a set $E \subset \Gamma(\mathbb{R}^2)$ is a W_B-set if and only if E is symmetric and of type $G_{\delta\sigma}$;*
- *a set $E \subset \Gamma(\mathbb{R}^2)$ is an R_B-set if and only if E is symmetric and of type G_δ.*

In [11] (see Lemma 2.4) it was shown that every non-standard homothecy invariant convex basis B has the non-regular spherical halo function. Taking into account this fact, we obtain from Theorem 1 the following corollary.

Corollary 1. *Let B be a non-standard homothecy invariant basis of two-dimensional intervals which is symmetric. Then for W_B-sets and R_B-sets characterizations analogous to the ones given in Theorem* 1 *are true.*

4. Auxiliary Propositions

By $\mathfrak{B}_{\mathrm{TI}}$ and $\mathfrak{B}_{\mathrm{HI}}$ we will denote the classes of all translation invariant and homothecy invariant bases in $\mathbb{R}^2$, respectively. By $\mathfrak{B}_{\mathrm{I}}$ it will be denoted the class of all bases consisting of two-dimensional intervals. The lower left vertex of an interval $I \subset \mathbb{R}^2$ denote by $a(I)$. For a set $A \subset \mathbb{R}^n$ with the centre of symmetry at a point x and for a number $\alpha > 0$ we denote by αA the dilation of A with the coefficient α, i.e. the set $\alpha A = \{x + \alpha(y - x) : y \in A\}$.

Let $B \in \mathfrak{B}_{\mathrm{I}}$. For a square interval Q and $\lambda \in (0, 1)$ by $\Omega_B(Q, \lambda)$ denote the collection of all intervals $I \in B$ with the properties: $a(I) = a(Q), I \supset Q$ and $|Q|/|I| > \lambda$. The set $E_B(Q, \lambda)$ will be defined as the union of all intervals from the collection $\Omega_B(Q, \lambda)$. Obviously, $\frac{1}{|I|}\int_I \chi_Q > \lambda$ for each $I \in \Omega_B(Q, \lambda)$ and $E_B(Q, \lambda) \subset \{M_B(\chi_Q) > \lambda\}$.

Lemma 1. *Let $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{I}}$, Q be a square interval and $0 < \lambda < 1$. Then $|E_B(Q, \lambda)| \geq c(|\{M_B(\chi_Q) > \lambda\}| - 18|Q|/\lambda)$, where c is a positive absolute constant.*

Proof. Without loss of generality let us assume that Q is a square interval of the type $(-\varepsilon, \varepsilon)^2$. Let Θ be the collection of all intervals $I \in B$ such that $\frac{1}{|I|}\int_I \chi_Q > \lambda$. Obviously, $\{M_B(\chi_Q) > \lambda\} = \bigcup_{I\in\Theta} I$.

Denote by Θ_0 the collection of all intervals $I \in \Theta$ having at least one side with the length smaller than 2ε. It is easy to check that every $I \in \Theta_0$ is contained in the union of the intervals $(-3\varepsilon, 3\varepsilon) \times (-\varepsilon - 2\varepsilon/\lambda, \varepsilon + 2\varepsilon/\lambda)$ and $(-\varepsilon - 2\varepsilon/\lambda, \varepsilon + 2\varepsilon/\lambda) \times (-3\varepsilon, 3\varepsilon)$. Consequently, $|\bigcup_{I\in\Theta_0} I| < 18|Q|/\lambda$.

Let $\mathbb{R}^2_k$ $(k \in \overline{1,4})$ be the k-th coordinate quarter. Denote by Θ_k $(k \in \overline{1,4})$ the collection of all intervals $I \in \Theta \setminus \Theta_0$ for which $|I \cap \mathbb{R}^2_k| = \max\{|I \cap \mathbb{R}^2_m| : m \in \overline{1,4}\}$. Obviously, $\Theta = \bigcup_{k=0}^4 \Theta_k$. The unions $\bigcup_{I\in\Theta_k} I$ and $\bigcup_{I\in\Theta_m} I$ are symmetric with respect to Ox_2 if $k = 1, m = 2$ or $k = 3, m = 4$ and are symmetric with respect to Ox_1 if $k = 2, m = 3$ or $k = 4, m = 1$. Hence, the sets $\bigcup_{I\in\Theta_k} I$ $(k \in \overline{1,4})$ have one and the same measure. Consequently,

$$\Big|\bigcup_{I\in\Theta_1} I\Big| \geq \frac{1}{4}\Big(\Big|\bigcup_{I\in\Theta} I\Big| - \Big|\bigcup_{I\in\Theta_0} I\Big|\Big) \geq \frac{1}{4}\bigg(|\{M_B(\chi_Q) > \lambda\}| - \frac{18|Q|}{\lambda}\bigg). \tag{1}$$

For arbitrary $I \in \Theta_1$ let us consider the translation T for which $T(I) \in \Omega_B(Q,\lambda)$. It is clear that $I \subset 2T(I)$. Consequently, $\bigcup_{I\in\Theta_1} I \subset \bigcup_{I\in\Omega_B(Q,\lambda)} 2I$. Therefore, by (1): $|\bigcup_{I\in\Omega_{B(Q,\lambda)}} 2I| \geq \frac{1}{4}(|\{M_B(\chi_Q) > \lambda\}| - 18|Q|/\lambda)$. On the other hand, by virtue of the inclusion $\bigcup_{I\in\Omega_B(Q,\lambda)} 2I \subset \{M_{\mathbf{I}(\mathbb{R}^2)}(\chi_A) \geq 1/4\}$, where $A = \bigcup_{I\in\Omega_B(Q,\lambda)} I$, and the strong maximal inequality (see, e.g., [3, Ch. II, §3]), we have: $|\bigcup_{I\in\Omega_B(Q,\lambda)} 2I| \leq C|\bigcup_{I\in\Omega_B(Q,\lambda)} I|$, where C is a positive absolute constant. From the last two estimations it follows the validity of the lemma. $\square$

Lemma 2. *Let $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{I}}$ and $0 < \lambda < 1$. If $\sigma_B(\lambda) > 144/\lambda$, then for every $\varepsilon > 0$ there is a square interval Q such that* $\operatorname{diam} Q < \varepsilon$ *and $|E_B(Q,\lambda)| \geq c\sigma_B(\lambda)|Q|/8$, where c is the constant from Lemma* 1.

Proof. Taking into account the definition of the spherical halo function σ_B, we can find a ball $V_\delta = \{x \in \mathbb{R}^2 : \operatorname{dist}(x,O) < \delta\}$ such that $\delta < \varepsilon/4$ and $|\{M_B(\chi_{V_\delta}) > \lambda\}|/|V_\delta| > \sigma_B(\lambda)/2$. Let us consider the square interval Q superscribed around V_δ, i.e. $Q = (-\delta,\delta)^2$. Then $\operatorname{diam} Q < \varepsilon$ and $|\{M_B(\chi_Q) > \lambda\}| \geq |\{M_B(\chi_{V_\delta}) > \lambda\}| > \sigma_B(\lambda)|V_\delta|/2 > \sigma_B(\lambda)|Q|/4$. Now, taking into account the estimation $\sigma_B(\lambda) > 144/\lambda$, by virtue of Lemma 1, we write: $|E_B(Q,\lambda)| \geq c(\sigma_B(\lambda)|Q|/4 - 18|Q|/\lambda) \geq c\sigma_B(\lambda)|Q|/8$. This proves the lemma. $\square$

Suppose, $S = (0,\varepsilon) \times (0,\varepsilon)$, $0 < \alpha \leq \pi/4$ and $n \in \mathbb{N}$. For each $k \in \overline{1,n}$ let us define the points $P_k^+(S,\alpha)$, $P_k^-(S,\alpha)$ and the balls $V_k^+(S,\alpha,n)$, $V_k^-(S,\alpha,n)$ as follows:

$$P_k^+(S,\alpha) = \Big(\frac{\varepsilon}{2^k}, \frac{\varepsilon}{2^k}\tan(\alpha)\Big), \quad P_k^-(S,\alpha) = \Big(\frac{\varepsilon}{2^k}, -\frac{\varepsilon}{2^k}\tan(\alpha)\Big),$$
$$V_k^+(S,\alpha,n) = \Big\{x \in \mathbb{R}^2 : \operatorname{dist}(x, P_k^+(S,\alpha)) < \frac{\varepsilon}{4^n}\tan(\alpha)\Big\},$$
$$V_k^-(S,\alpha,n) = \Big\{x \in \mathbb{R}^2 : \operatorname{dist}(x, P_k^-(S,\alpha)) < \frac{\varepsilon}{4^n}\tan(\alpha)\Big\}.$$

Suppose, Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1, 0 < \alpha \leq \pi/4$ and $n \in \mathbb{N}$. Let $\xi = \xi_{Q,h,S,\alpha,n}$ be the function which is proportional to the function $\sum_{k=1}^n \chi_{V_k^+(S,\alpha,n)} - \sum_{k=1}^n \chi_{V_k^-(S,\alpha,n)}$, and satisfies the following conditions: $\{\xi > 0\} = \bigcup_{k=1}^n V_k^+(S,\alpha,n)$, $\{\xi < 0\} = \bigcup_{k=1}^n V_k^-(S,\alpha,n)$ and $\|\xi\|_L = 2\|h\chi_Q\|_L$. The function $\xi_{Q,h,S,\alpha,n}$ we will call (S,α,n)-*oscillator corresponding to the function* $h\chi_Q$. It is easy to see that:

1) the balls $V_k^+(S,\alpha,n)$ are disjoint and contained in the square S;
2) the balls $V_k^-(S,\alpha,n)$ are disjoint and contained in the square $S^- = (0,\varepsilon) \times (-\varepsilon,0)$;
3) $\int_{V_k^+(S,\alpha,n)} \xi = -\int_{V_k^-(S,\alpha,n)} \xi = h|Q|/n$ for each $k \in \overline{1,n}$.

For $\gamma \in \Gamma(\mathbb{R}^2)$ and $\varepsilon > 0$ denote $V[\gamma,\varepsilon] = \{\rho \in \Gamma(\mathbb{R}^2) : \operatorname{dist}(\rho,\gamma) \leq \varepsilon\}$.

For a basis B by $\overline{M}_B$ denote the following type maximal operator: $\overline{M}_B(f)(x) = \sup_{R\in B(x)} \frac{1}{|R|}\int_R f$ $(f \in L(\mathbb{R}^n),\ x \in \mathbb{R}^n)$.

Lemma 3. *Let $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{I}}$. Suppose Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1, 0 < \alpha \leq \pi/4$ and $n \in \mathbb{N}$. Then for the oscillator $\xi = \xi_{Q,h,S,\alpha,n}$ it is valid the following estimation*: $\frac{1}{|\gamma(I)|}\int_{\gamma(I)} \xi > 1$ *for every $I \in \Omega_B(Q,1/h)$ and $\gamma \in V[\rho_0,\alpha/2]$; consequently, $\{\overline{M}_{B(\gamma)}(\xi) > 1\} \supset \gamma(E_B(Q,1/h))$ for every $\gamma \in V[\rho_0,\alpha/2]$.*

Proof. Let $I \in \Omega_B(Q,1/h)$ and $\gamma \in V[\rho_0,\alpha/2]$. Using simple geometry it is easy to see that $\gamma(I) \supset \{\xi > 0\}$ and $\gamma(I) \cap \{\xi < 0\} = \emptyset$. Consequently, taking into account the properties of the oscillator ξ, we write: $\frac{1}{|\gamma(I)|}\int_{\gamma(I)} \xi = \frac{1}{|I|}\int_{\{\xi>0\}} \xi = \|h\chi_Q\|_L/|I| = h|Q|/|I| > 1$. The lemma is proved. $\square$

Remark 1. On the basis of Lemmas 1 and 3 the oscillator $\xi = \xi_{Q,h,S,\alpha,n}$ may be interpreted as the transformation of the function $h\chi_Q$ that conserves values of integral means with respect to the bases $B(\gamma)$ for rotations γ belonging to the neighbourhood $V[\rho_0, \alpha/2]$. In particular, if it is known that the set $\{M_B(h\chi_Q) > 1\}$ has a big measure, then the sets $\{\overline{M}_{B(\gamma)}(\xi) > 1\}$ have big measures of the same order for every $\gamma \in V[\rho_0, \alpha/2]$.

The following Lemma was shown in [4] (see Lemma 2) and plays an essential role in achieving differentiation effect for desired rotations.

Lemma A. *Let S be a square interval, $0 < \alpha < \pi/12$ and $n \in \mathbb{N}$. Then for arbitrary rectangle R the sides of which compose with the line Ox_1 angles greater than 3α it is valid the estimation $|\nu_+ - \nu_-| \le 2$, where ν_+ is a number of all points $P_k^+(S,\alpha)$ $(k \in \overline{1,n})$ belonging to R and ν_- is a number of all points $P_k^-(S,\alpha)$ $(k \in \overline{1,n})$ belonging to R.*

For a square $S = (0,\varepsilon)^2$ by $\Delta(S)$ denote the union of the strips $(-7\varepsilon, 7\varepsilon) \times \mathbb{R}$ and $\mathbb{R} \times (-7\varepsilon, 7\varepsilon)$.

For a basis B let $\widehat{M}_B$ be the following type maximal operator: $\widehat{M}_B(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|}\left|\int_R f\right|$ $(f \in L(\mathbb{R}^n), x \in \mathbb{R}^n)$.

For a non-empty set $E \subset \Gamma(\mathbb{R}^2)$ and a number $\varepsilon > 0$ denote $V[E,\varepsilon] = \{\gamma \in \Gamma(\mathbb{R}^2) : \operatorname{dist}(\gamma, E) \le \varepsilon\}$.

Below the set of the rotations ρ_0, ρ_1, ρ_2 and ρ_3 will be denoted by Π.

Lemma 4. *Let Q be a square interval with $a(Q) = (0,0)$, $h > 1$ and $0 < \alpha < \pi/12$. Then for every square interval $S \subset Q$ with $a(S) = (0,0)$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for the oscillator $\xi = \xi_{Q,h,S,\alpha,n}$ it is valid the following inclusion: $\{\widehat{M}_{\mathbf{I}(\gamma)}(\xi) \ge \varepsilon\} \subset \gamma(\Delta(S))$ for every $\gamma \notin V[\Pi, 3\alpha]$.*

Proof. Suppose $x \notin \gamma(\Delta(S)), \gamma \notin V[\Pi, 3\alpha]$, $R \in \mathbf{I}(\gamma)(x)$ and $R \cap \operatorname{supp}\xi \ne \emptyset$. For $n \in \mathbb{N}$ denote by $N_+, N_-, N_+^*, N_-^*, N_+^{**}$ and N_-^{**} the sets of indexes $k \in \overline{1,n}$ satisfying conditions $V_k^+(S,\alpha,n) \cap R \ne \emptyset$, $V_k^-(S,\alpha,n) \cap R \ne \emptyset$, $P_k^+(S,\alpha) \in R$, $P_k^-(S,\alpha) \in R$, $V_k^+(S,\alpha,n) \subset R$ and $V_k^-(S,\alpha,n) \subset R$, respectively.

It is easy to see that if n is big enough, then every line l composing an angle with the axis Ox_1 greater than 3α may intersect at most one among balls $V_k^+(S,\alpha,n)(V_k^-(S,\alpha,n))$. Below we will assume that n has the just mentioned property. Consequently, the boundary of the rectangle R may intersect at most 4 among balls $V_k^+(S,\alpha,n)(V_k^-(S,\alpha,n))$. Thus, there are true the following estimations: $\operatorname{card}(N_+ \setminus N_+^*) + \operatorname{card}(N_+^* \setminus N_+^{**}) \le 4$ and $\operatorname{card}(N_- \setminus N_-^*) + \operatorname{card}(N_-^* \setminus N_-^{**}) \le 4$. Herewith, by virtue of Lemma A: $|\operatorname{card} N_+^* - \operatorname{card} N_-^*| \le 2$.

Let us estimate $|\int_R \xi|$. We have

$$\begin{aligned}
\Big|\int_R \xi\Big| &= \Big|\sum_{k \in N_+} \int_{V_k^+(S,\alpha,n)\cap R} \xi + \sum_{k \in N_-} \int_{V_k^-(S,\alpha,n)\cap R} \xi\Big| \\
&\le \Big|\sum_{k \in N_+^*} \int_{V_k^+(S,\alpha,n)\cap R} \xi + \sum_{k \in N_-^*} \int_{V_k^-(S,\alpha,n)\cap R} \xi\Big| \\
&\quad + \Big|\sum_{k \in N_+} \int_{V_k^+(S,\alpha,n)\cap R} \xi - \sum_{k \in N_+^*} \int_{V_k^+(S,\alpha,n)\cap R} \xi\Big| \\
&\quad + \Big|\sum_{k \in N_-} \int_{V_k^+(S,\alpha,n)\cap R} \xi - \sum_{k \in N_-^*} \int_{V_k^+(S,\alpha,n)\cap R} \xi\Big| = a_1 + a_2 + a_3.
\end{aligned}$$

The term a_1 can be estimated as follows

$$\begin{aligned}
a_1 &\le \Big|\sum_{k \in N_+^*} \int_{V_k^+(S,\alpha,n)} \xi + \sum_{k \in N_-^*} \int_{V_k^-(S,\alpha,n)} \xi\Big| \\
&\le \sum_{k \in N_+^* \setminus N_+^{**}} \int_{V_k^+(S,\alpha,n)} \xi + \sum_{k \in N_-^* \setminus N_-^{**}} \int_{V_k^-(S,\alpha,n)} |\xi| = a_{1,1} + a_{1,2} + a_{1,3}.
\end{aligned}$$

By virtue of equalities $\int_{V_k^+(S,\alpha,n)} \xi = -\int_{V_k^-(S,\alpha,n)} \xi = h|Q|/n$ ($k \in \overline{1,n}$), we write:

$$a_{1,1} = |\operatorname{card} N_+^* - \operatorname{card} N_-^*| \frac{h|Q|}{n},$$

$$a_{1,2} \le \sum_{k \in N_+^* \setminus N_+^{**}} \int\limits_{V_k^+(S,\alpha,n)} \xi = \operatorname{card}(N_+^* \setminus N_+^{**}) \frac{h|Q|}{n},$$

$$a_{1,3} \le \sum_{k \in N_-^* \setminus N_-^{**}} \int\limits_{V_k^-(S,\alpha,n)} |\xi| = \operatorname{card}(N_-^* \setminus N_-^{**}) \frac{h|Q|}{n},$$

$$a_2 \le \sum_{k \in N_+ \setminus N_+^*} \int\limits_{V_k^+(S,\alpha,n)} \xi = \operatorname{card}(N_+ \setminus N_+^*) \frac{h|Q|}{n},$$

$$a_3 \le \sum_{k \in N_- \setminus N_-^*} \int\limits_{V_k^-(S,\alpha,n)} |\xi| = \operatorname{card}(N_- \setminus N_-^*) \frac{h|Q|}{n}.$$

Consequently,

$$\Big| \int\limits_R \xi \Big| \le a_{1,1} + a_{1,2} + a_{1,3} + a_2 + a_3 \le \frac{10h|Q|}{n}.$$

Since $x \notin \gamma(\Delta(S)), R \in \mathbf{I}(\gamma)(x)$ and $R \cap \operatorname{supp} \xi \ne \emptyset$, it is easy to check that the side lengths of R are not less than the length of the sides of S. Therefore, $|R| \ge |S|$. Hence,

$$\frac{1}{|R|} \Big| \int\limits_R \xi \Big| \le \frac{10h|Q|}{n|S|}.$$

The last estimation implies that if n is big enough, then for every $\gamma \notin V[\Pi, 3\alpha]$ it is valid the needed inclusion: $\{\widehat{M}_{\mathbf{I}(\gamma)}(\xi) \ge \varepsilon\} \subset \gamma(\Delta(S))$. The lemma is proved. $\square$

Remark 2. On the basis of Lemma 4 the oscillator $\xi = \xi_{Q,h,S,\alpha,n}$ may be considered as the transformation of the function $h\chi_Q$ that decreases values of integral means with respect to the bases $\mathbf{I}(\gamma)$ for rotations γ not belonging to the neighbourhood $V[\Pi, 3\alpha]$.

Let us define an oscillator for more general parameters. Suppose, Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1, 0 < \alpha \le \pi/4, n \in \mathbb{N}, \gamma \in \Gamma(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. Denote by T the translation: $T(y) = y - x$. The *oscillator* $\xi_{Q,h,S,\alpha,n,\gamma,x}$ define as the function $(\xi_{Q,h,S,\alpha,n} \circ \gamma^{-1}) \circ T$.

For $\gamma \in \Gamma(\mathbb{R}^2)$ the set of the rotations $\gamma, \rho_1 \circ \gamma, \rho_2 \circ \gamma$ and $\rho_3 \circ \gamma$ will be denoted by Π_γ.

From Lemmas 3 and 4 we can easily obtain the following two assertions.

Lemma 5. *Let $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{I}}$. Suppose, Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1, 0 < \alpha \le \pi/4$, $n \in \mathbb{N}$, $\gamma \in \Gamma(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. Then for the oscillator $\xi = \xi_{Q,h,S,\alpha,n,\gamma,x}$ it is valid the following condition: $\frac{1}{|\gamma^*(I)+x|} \int_{\gamma^*(I)+x} \xi > 1$ for every $I \in \Omega_B(Q, 1/h)$ and $\gamma^* \in V[\gamma, \alpha/2]$; consequently, $\{\overline{M}_{B(\gamma^*)}(\xi) > 1\} \supset \gamma^*(E_B(Q, 1/h)) + x$ for every $\gamma^* \in V[\gamma, \alpha/2]$.*

Lemma 6. *Let Q be a square interval with $a(Q) = (0,0)$, $h > 1$ and $0 < \alpha < \pi/12$. Then for every square interval $S \subset Q$ with $a(S) = (0,0)$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for every $\gamma \in \Gamma(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ the oscillator $\xi = \xi_{Q,h,S,\alpha,n,\gamma,x}$ satisfies the following inclusion:*

$$\{\widehat{M}_{\mathbf{I}(\gamma^*)}(\xi) \ge \varepsilon\} \subset \gamma^*(\Delta(S)) + x \quad \textit{for every} \quad \gamma^* \notin V[\Pi_\gamma, 3\alpha].$$

Recall that a one-dimensional interval I is called *dyadic* if it has the form $(k/2^m, (k+1)/2^m)$, where $k, m \in \mathbb{Z}$. A square interval Q is called *dyadic* if it is a product of two dyadic intervals.

The length of the sides of a square Q denote by $d(Q)$. If $d(Q) = 1/2^m$, then let us call the number m an *order of a dyadic square Q*.

Suppose Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1, 6hd(Q) \leq 1, 0 < \alpha < \pi/12$, $n \in \mathbb{N}$ and $\gamma \in \Gamma(\mathbb{R}^2)$. For this parameters we will define the function $f_{Q,h,S,\alpha,n,\gamma}$ below.

Let $W(Q,h)$ be the smallest square interval concentric with Q containing the square $6hQ$ and having $d(W)$ of the type $1/2^j$ $(j \in \mathbb{Z})$. Note that by virtue of the condition $6hd(Q) \leq 1$, we have: $d(W) \leq 1$. Let us decompose the unit square $(0,1)^2$ into pair-wise non-overlapping square intervals congruent to $W(Q,h)$ and the obtained squares denote by $W_1, \ldots, W_k$. By $x_1, \ldots, x_k$ denote the centres of $W_1, \ldots, W_k$, respectively. The order of the dyadic squares $W_1, \ldots, W_k$ denote by $m(Q,h)$.

The function $f_{Q,h,S,\alpha,n,\gamma}$ define as follows: $f_{Q,h,S,\alpha,n,\gamma} = \sum_{j=1}^{k} \xi_{Q,h,S,\alpha,n,\gamma,x_j}$. It is clear that $\operatorname{supp} f_{Q,h,S,\alpha,n,\gamma} \subset (0,1)^2$.

Let Θ be a some collection of rectangles and Δ be a subinterval of $(0,\infty)$. Then by Θ_Δ denote the collection of all rectangles $R \in \Theta$ the side lengths of which belong to the interval Δ.

Let B be a some basis consisting of rectangles and Δ be a subinterval of $(0,\infty)$. Then by M_B^Δ and $\overline{M}_B^\Delta$ denote the following type operators: $M_B^\Delta(f)(x) = \sup_{R \in B(x)_\Delta} \frac{1}{|R|}\int_R |f|$ and $\overline{M}_B^\Delta(f)(x) = \sup_{R \in B(x)_\Delta} \frac{1}{|R|}\int_R f$, where $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Let $B \in \mathfrak{B}_{\mathrm{I}} \cap \mathfrak{B}_{\mathrm{TI}}$ and Q be a square interval. By $\sigma_{B,Q}$ denote the function defined as follows: $\sigma_{B,Q}(\lambda) = |E_B(Q,\lambda)|/|Q|$ $(0 < \lambda < 1)$.

By $\mathbf{P}$ it will be denoted the basis of all two-dimensional rectangles.

Lemma 7. *Let $B \in \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{I}}$. Suppose, Q and S are square intervals with $Q \supset S$ and $a(Q) = a(S) = (0,0)$, $h > 1$, $6hd(Q) \leq 1$, $0 < \alpha < \pi/12$, $n \in \mathbb{N}$, $\gamma \in \Gamma(\mathbb{R}^2)$, $W = W(Q,h)$ and $m = m(Q,h)$. Then the function $f = f_{Q,h,S,\alpha,n,\gamma}$ has the following properties*:

1) $\|f\|_L < 1/h$;

2) *for every $\gamma^* \in V[\gamma, \alpha/2]$ there is a set $A(\gamma^*)$ such that*:

(a) $A(\gamma^*) \subset \{\overline{M}_{B(\gamma^*)}^{[d(Q),d(W)]}(f) > 1\}$;

(b) $|A(\gamma^*)| \geq \sigma_{B,Q}(1/h)/(300h^2)$;

(c) $A(\gamma^*)$ *is uniformly distributed in the dyadic squares of order m contained in $(0,1)^2$, i.e. if $W_1, \ldots, W_k$ are all dyadic squares of order m contained in $(0,1)^2$, then the sets $A(\gamma^*) \cap W_k$ are congruent*;

(d) $A(\gamma^*)$ *is a union of dyadic squares of the fixed order, moreover, the order is one and the same for every $\gamma^* \in V[\gamma, \alpha/2]$*;

3) $|\{M_{\mathbf{P}}^{(0,d(Q))}(f) > 0\}| < 1/h^2$;

4) $M_{\mathbf{P}}^{(d(W),\infty)}(f)(x) < 2/h$ *for every $x \in \mathbb{R}^2$.*

Proof. Let W_j, x_j and $\xi_{Q,h,S,\alpha,n,\gamma,x_j}$ $(j \in \overline{1,k})$ be parameters from the definition of the function $f_{Q,h,S,\alpha,n,\gamma}$. Denote $\xi_j = \xi_{Q,h,S,\alpha,n,\gamma,x_j}$ $(j \in \overline{1,k})$.

Using the inclusion $6hQ \subset W$ it is easy to see that $\|f\|_L = \sum_{j=1}^{k}\|\xi_j\|_L = \sum_{j=1}^{k} 2h|Q| = 2h|Q|k = 2h\frac{|Q|}{|W|}k|W| \leq 2h \cdot \frac{1}{36h^2} \cdot 1 < 1/h$.

Let $I \in \Omega_B(Q,1/h), j \in \overline{1,k}$ and $\gamma^* \in V[\gamma, \alpha/2]$. It is easy to check that the side lengthes of I belong to the interval $[d(Q), hd(Q)]$. Consequently, taking into account the inclusion $6hQ \subset W$, we have: $\gamma^*(I) + x_j \subset W_j$. Thus, the rectangle $\gamma^*(I) + x_j$ does not intersect supports of functions ξ_ν with $\nu \neq j$. Therefore, by virtue of Lemma 5, $\frac{1}{|\gamma^*(I)+x_j|}\int_{\gamma^*(I)+x_j} f = \frac{1}{|\gamma^*(I)+x_j|}\int_{\gamma^*(I)+x_j} \xi_j > 1$. Now taking into account estimation $6hd(Q) \leq d(W)$, we conclude that for every $\gamma^* \in V[\gamma, \alpha/2]$,

$$\bigcup_{j=1}^{k} \bigcup_{I \in \Omega_B(Q,1/h)} (\gamma^*(I) + x_j) \subset \{\overline{M}_{B(\gamma^*)}^{[d(Q),d(W)]}(f) > 1\}. \tag{2}$$

For a set $E \subset \mathbb{R}^2$ by $E(\nu)$ $(\nu \in \mathbb{Z})$ let us denote the union of all dyadic squares of order ν contained in E.

Since the set $E_B(Q,1/h)$ is open and the sets $\gamma^*(E_B(Q,1/h))$ ($\gamma^* \in V[\gamma,\alpha/2]$) are congruent, then it is possible to find a number $\nu > m$ (see, e.g., [10, Lemma 7] for details) for which

$$|\gamma^*(E_B(Q,1/h))(\nu)| \geq |\gamma^*(E_B(Q,1/h))|/2 = |E_B(Q,1/h)|/2 \tag{3}$$

for every $\gamma^* \in V[\gamma,\alpha/2]$.

Let us define the set $A(\gamma^*)$ ($\gamma^* \in V[\gamma,\alpha/2]$) as the union of the translations: $\gamma^*(E_B(Q,1/h))(\nu)+x_j$ ($j \in \overline{1,k}$). By virtue of the inclusions $\gamma^*(I)+x_j \subset W_j$ we obtain:

$$\gamma^*(E_B(Q,1/h))(\nu)+x_j \subset \gamma^*(E_B(Q,1/h))+x_j \subset W_j, \tag{4}$$

for every $\gamma^* \in V[\gamma,\alpha/2]$ and $j \in \overline{1,k}$.

From (3), (4) and the obvious inclusion $W \subset 12hQ$, for arbitrary $\gamma^* \in V[\gamma,\alpha/2]$ we write

$$\begin{aligned}|A(\gamma^*)| &= \sum_{j=1}^{k} |\gamma^*(E_B(Q,1/h))(\nu)+x_j| \geq k\frac{|E_B(Q,1/h)|}{2} \\ &= k|I|\frac{|Q|}{|I|}\frac{|E_B(Q,1/h)|}{2|Q|} \geq 1\cdot\frac{1}{144h^2}\cdot\frac{\sigma_{B,Q}(1/h)}{2} \geq \frac{\sigma_{B,Q}(1/h)}{300h^2}.\end{aligned}$$

This proves the property (b) of the sets $A(\gamma^*)$. The properties (a), (c) and (d) directly follow from the definition of the sets $\gamma^*(E_B(Q,1/h))(\nu)$ and the relations (2) and (4).

Let $x \notin \bigcup_{j=1}^{k} 5(\gamma(Q)+x_j)$. Then it is easy to see that $\operatorname{dist}(x,\operatorname{supp} f) \geq 2d(Q)$. Therefore, for every $R \in \mathbf{P}(x)_{(0,d(Q))}$ we have: $\int_R f = 0$, and consequently, $M_{\mathbf{P}}^{(0,d(Q))}(f)(x) = 0$. Thus, $\{M_{\mathbf{P}}^{(0,d(Q))}(f) > 0\} \subset \bigcup_{j=1}^{k} 5(\gamma(Q)+x_j)$. By virtue of the last inclusion,

$$|\{M_{\mathbf{P}}^{(0,d(Q))}(f) > 0\}| \leq 25k|Q| = 25k|W|\frac{|Q|}{|W|} < 25\cdot 1\cdot\frac{1}{36h^2} < \frac{1}{h^2}.$$

Let $x \in \mathbb{R}^2$ and $R \in \mathbf{P}(x)_{(d(W),\infty)}$. By N denote the set of all numbers $j \in \overline{1,k}$ for which $W_j \cap R \neq \emptyset$. It is easy to check that $\bigcup_{j\in N} W_j \subset 5R$. This inclusion implies that $(\operatorname{card} N)|I| = \sum_{j\in N}|I_j| \leq 25|R|$. Thus, $\operatorname{card} N \leq 25|R|/|I|$. Now we can write,

$$\begin{aligned}\int_R |f| &\leq \sum_{j\in N}\int_{W_j}|f| = \sum_{j\in N}\int_{W_j}|\xi_j| = \sum_{j\in N} 2h|Q| \\ &= (\operatorname{card} N)2h|Q| \leq 50h\frac{|R||Q|}{|W|} = 50h|R|\frac{1}{36h^2} < \frac{3}{2h}|R|.\end{aligned}$$

The obtained estimation implies that $M_{\mathbf{P}}^{(d(W),\infty)}(f)(x) < 2/h$ for every $x \in \mathbb{R}^2$. The lemma is proved. □

Lemma 8. *Let Q be a square interval with $a(Q) = (0,0)$, $h > 1$ and $0 < \alpha < \pi/12$. Then for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there are a square interval $S \subset Q$ with $a(S) = (0,0)$ and a number $n \in \mathbb{N}$ such that for every $\gamma \in \Gamma(\mathbb{R}^2)$ and $x_1,\dots,x_k \in \mathbb{R}^2$ the functions $\xi_j = \xi_{Q,h,S,\alpha,n,\gamma,x_j}$ ($j \in \overline{1,k}$) satisfy the following estimation*:

$$\Big|\Big\{\widehat{M}_{\mathbf{I}(\gamma^*)}\Big(\sum_{j=1}^{k}\xi_j\Big) \geq \varepsilon\Big\} \cap (0,1)^2\Big| < \varepsilon \quad \textit{for every} \quad \gamma^* \notin V[\Pi_\gamma, 3\alpha].$$

Proof. Let us choose a square interval $S \subset Q$ with $a(S) = (0,0)$ so that $28\sqrt{2}\operatorname{diam} S < \varepsilon/k$, and using Lemma 6 let us choose a number $n \in \mathbb{N}$ so that for every $\gamma \in \Gamma(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ the oscillator $\xi = \xi_{Q,h,S,\alpha,n,\gamma,x}$ satisfies the following condition: $\{\widehat{M}_{\mathbf{I}(\gamma^*)}(\xi) \geq \varepsilon/k\} \subset \gamma^*(\Delta(S)) + x$ for every $\gamma^* \notin V[\Pi_\gamma, 3\alpha]$.

Suppose, $\gamma \in \Gamma(\mathbb{R}^2)$, $x_1,\dots,x_k \in \mathbb{R}^2$ and $\xi_j = \xi_{Q,h,S,\alpha,n,\gamma,x_j}$ ($j \in \overline{1,k}$). Let us consider an arbitrary $\gamma^* \notin V[\Pi_\gamma, 3\alpha]$. Then taking into account the estimation $\widehat{M}_{\mathbf{I}(\gamma^*)}\Big(\sum_{j=1}^{k}\xi_j\Big) \leq \sum_{j=1}^{k}\widehat{M}_{\mathbf{I}(\gamma^*)}(\xi_j)$, we have

$$\Big\{\widehat{M}_{\mathbf{I}(\gamma^*)}\Big(\sum_{j=1}^{k}\xi_j\Big) \geq \varepsilon\Big\} \subset \bigcup_{j=1}^{k}\{\widehat{M}_{\mathbf{I}(\gamma^*)}(\xi_j) \geq \varepsilon/k\} \subset \bigcup_{j=1}^{k}(\gamma^*(\Delta(S)) + x_j). \tag{5}$$

Note that: 1) For any strip Δ it is true the estimation: $|\Delta \cap (0,1)^2| \leq \sqrt{2}$ (width of Δ); 2) $\gamma^*(\Delta(S)) + x_j$ ($j \in \overline{1,k}$) is a union of two strips with the widthes less than $14 \operatorname{diam} S$. Consequently, on the basis of choosing of S, for each j we write: $|(\gamma^*(\Delta(S))+x_j)\cap(0,1)^2| \leq 2(\sqrt{2}\, 14 \operatorname{diam} S) < \varepsilon/k$. Hence, using (5) we obtain that $|\{\widehat{M}_{\mathbf{I}(\gamma^*)}(\sum_{j=1}^{k}\xi_j) \geq \varepsilon\} \cap (0,1)^2| < \varepsilon$. The lemma is proved. □

Lemma 9. *Let $I \subset \mathbb{R}$ be an open interval. For every $s > 1$ and $\varepsilon \in (0,1)$ there are pairwise non-overlapping closed intervals $I_k \subset I$ ($k \in \mathbb{N}$) such that $I = \bigcup_{k=1}^{\infty} I_k$, $|I_k| < \varepsilon|I|$ ($k \in \mathbb{N}$), $sI_k \subset I$ ($k \in \mathbb{N}$) and $\sum_{k=1}^{\infty}\chi_{sI_k}(x) \leq c(s)$ ($x \in I$), where $c(s)$ is a constant depending only on the parameter s.*

Proof. Let x_0 be a midpoint of I and for a number $t \in (0,1)$ let us consider the points $x_m = \sup I - t^m|I|/2$, $x_{-m} = \inf I + t^m|I|/2$ ($m \in \mathbb{N}$). It is easy to check that if t is quite close to 1 then the intervals $[x_m, x_{m+1}]$ ($m \in \mathbb{Z}$) generate the needed decomposition of I. □

Lemma 10. *For an arbitrary non-empty symmetric set $E \subset \Gamma(\mathbb{R}^2)$ of type G_δ there are sequences of rotations (γ_k) and numbers (α_k) from the interval $(0,\pi/12)$ such that $\overline{\lim\limits_{k\to\infty}}\, V[\gamma_k, \alpha_k/2] = \overline{\lim\limits_{k\to\infty}}\, V[\Pi_{\gamma_k}, 3\alpha_k] = E$.*

Proof. For an interval $I \subset [0,2\pi)$ denote $I_{\mathbb{T}} = \{(\cos(t), \sin(t)) : t \in I\}$ and $\Gamma_I = \{\gamma \in \Gamma(\mathbb{R}^2) : \gamma((1,0)) \in I_{\mathbb{T}}\}$.

First let us prove the following statement: For an arbitrary non-empty set $W \subset \Gamma_{[0,\pi/2)}$ of G_δ type there are sequences of rotations (σ_m) and numbers (β_m) from the interval $(0,\pi/12)$ such that $\overline{\lim\limits_{m\to\infty}}\, V[\sigma_m, \beta_m/2] = \overline{\lim\limits_{m\to\infty}}\, V[\sigma_m, 3\beta_m] = W$.

Without loss of generality we can assume that $\rho_0 \notin W$, i.e. $W \subset \Gamma_{(0,\pi/2)}$. Using identification of $\Gamma_{(0,\pi/2)}$ with the interval $(0,\pi/2)$ by the mapping $\Gamma_{(0,\pi/2)} \ni \gamma \mapsto \operatorname{dist}(\gamma, \rho_0) \in (0,\pi/2)$ we can formulate our statement in the following equivalent way: For an arbitrary non-empty set $V \subset (0,\pi/2)$ of G_δ type there exists a sequence of closed intervals $I_m \subset (0,\pi/2)$ such that $|I_m| < \pi/12$ and $\overline{\lim\limits_{m\to\infty}}\, I_m = \overline{\lim\limits_{m\to\infty}}\,(6I_m) = V$.

Consider a sequence of open sets $G_n \subset (0,\pi/2)$ with $G_1 \supset G_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} G_n = V$. Let $\{I_p^{(n)}\}$ be the collection of open intervals decomposing G_n. For each n and p let us consider a sequence of closed intervals $(I_{p,q}^{(n)})_{q\in\mathbb{N}}$ corresponding to the parameters $s = 6, \varepsilon = 1/12$ and $I = I_p^{(n)}$ according to Lemma 9. If we enumerate the intervals $I_{p,q}^{(n)}$ by one index $m \in \mathbb{N}$, then it is easy to see that the obtained sequence of intervals (I_m) will satisfy the needed conditions. This proves the statement.

Now let us consider an arbitrary non-empty symmetric set $E \subset \Gamma(\mathbb{R}^2)$ of G_δ type. Let (σ_m) and (β_m) be sequences corresponding to the set $E \cap \Gamma_{[0,\pi/2)}$ according to the above proved statement. By $\lceil x \rceil$ ($x \in \mathbb{R}$) denote the number $\min\{n \in \mathbb{Z} : x \leq n\}$. Then it is easy to check that the sequences: $\gamma_k = \rho_{(k-1)(\mathrm{mod}\ 4)} \circ \sigma_{\lceil k/4 \rceil}$, $\alpha_k = \beta_{\lceil k/4\rceil}$ ($k \in \mathbb{N}$), will satisfy the needed conditions. □

5. Proof of Theorem 1

Let B be a basis satisfying the conditions of the theorem. In the introduction it was mentioned that the following three statements are true: 1) Each W_B-set is of type $G_{\delta\sigma}$ and each R_B-set is of type G_δ; 2) Every W_B-set and every R_B-set is symmetric; 3) Not more than countable union of R_B-sets is a W_B-set.

Taking into account three statements above it suffices to prove that an arbitrary symmetric set $E \subset \Gamma(\mathbb{R}^2)$ of type G_δ is an R_B-set. If E is empty, then the statement is trivial. Thus let us consider the case of a non-empty set E.

By virtue of Lemma 10 there are sequences $\gamma_k \in \Gamma(\mathbb{R}^2)$ and $\alpha_k \in (0,\pi/12)$ such that $\overline{\lim\limits_{k\to\infty}}\, V[\gamma_k, \alpha_k/2] = \overline{\lim\limits_{k\to\infty}}\, V[\Pi_{\gamma_k}, 3\alpha_k] = E$.

Taking into account non-regularity of the spherical halo function σ_B and the estimation $\sigma_B(1/h) \le Ch\ln h$ $(h \ge 2)$ (which is valid by virtue of strong maximal inequality (see, e.g., [3, Ch. II, §3]) it is not difficult to choose sequences (h_j) and (η_j) with the properties: $h_j \ge 2$, $0 < \eta_j < h_j$, $\lim\limits_{j\to\infty} h_j = \infty$, $\lim\limits_{j\to\infty} \eta_j = \infty$, $\sigma_B(1/h_j) > 144h_j$, $\sigma_B(1/h_j)/h_j^2 < 1$, $\sum_{j=1}^\infty \sigma_B(1/h_j)/h_j^2 = \infty$ and $\sum_{j=1}^\infty \eta_j/h_j < \infty$.

On the basis of divergence of the series $\sum_j \sigma_B(1/h_j)/h_j^2$ we can choose numbers $1 = j_0 < j_1 < j_2 < \cdots$ so that $\prod_{j=j_{k-1}}^{j_k-1}\Big(1 - \frac{c}{2400}\frac{\sigma_B(1/h_j)}{h_j^2}\Big) < \frac{1}{2^k}$ for every $k \in \mathbb{N}$. Here c is the constant from Lemma 1.

Denote $J_k = \{j \in \mathbb{N} : j_{k-1} \le j \le j_k - 1\}$ $(k \in \mathbb{N})$.

Using Lemmas 7 and 8 we can find sequences of square intervals (Q_j) and (S_j) with $a(Q_j) = a(S_j) = (0,0)$ and a sequence of natural numbers (n_j) for which the functions $f_j = f_{Q_j,h_j,S_j,\alpha_j,n_j,\gamma_j}$, $g_j = \eta_j f_j$ $(j \in \mathbb{N})$ satisfy the following conditions:

1) $\|g_j\| = \eta_j\|f_j\| < \eta_j/h_j$;

2) $d(W_1) > d(Q_1) > d(W_2) > d(Q_2) > \cdots$. Here $W_j = W(Q_j, h_j)$ is a square interval from the definition of the function $f_{Q,h,S,\alpha,n,\gamma}$;

3) there are sets $A_j(\gamma)$ $(k \in \mathbb{N}, \gamma \in V[\gamma_k, \alpha_k/2], j \in J_k)$ such that:

(a) $A_j(\gamma) \subset \{\overline{M}_{B(\gamma)}^{[d(Q_j),d(W_j)]}(f_j) > 1\} = \{\overline{M}_{B(\gamma)}^{[d(Q_j),d(W_j)]}(g_j) > \eta_j\}$;

(b) $|A_j(\gamma)| \ge c\sigma_B(1/h_j)/(2400h_j^2)$;

(c) $A_j(\gamma)$ is uniformly distributed in the dyadic squares of order $m_j = m(Q_j, h_j)$ contained in $(0,1)^2$, i.e. if $W_1, \ldots, W_\nu$ are all dyadic squares of order m_j contained in $(0,1)^2$, then the sets $A_j(\gamma) \cap W_i$ $(i \in \overline{1,\nu})$ are congruent. Here $m(Q_j, h_j)$ is the number from the definition of the function $f_{Q,h,S,\alpha,n,\gamma}$;

(d) $A_j(\gamma)$ is an union of dyadic squares of the order $m_j^* > m_j$, where m_j^* does not depend on $\gamma \in V[\gamma_k, \alpha_k/2]$;

4) the numbers m_j and m_j^* from the conditions 3)–(c) and 3)–(d) satisfy inequalities: $m_1 < m_1^* < m_2 < m_2^* < \cdots$;

5) $|\{M_{\mathbf{P}}^{(0,d(Q_j))}(g_j) > 0\}| = |\{M_{\mathbf{P}}^{(0,d(Q_j))}(f_j) > 0\}| < 1/h_j^2$ for every $j \in \mathbb{N}$;

6) $M_{\mathbf{P}}^{(d(W_j),\infty)}(g_j)(x) = \eta_j M_{\mathbf{P}}^{(d(W_j),\infty)}(f_j)(x) < 2\eta_j/h_j$ for every $j \in \mathbb{N}$ and $x \in \mathbb{R}^2$;

7) $|\{\widehat{M}_{\mathbf{I}(\gamma)}(f_j) \ge 1/(\eta_j 2^j)\} \cap (0,1)^2| < 1/(\eta_j 2^j)$ for every $k \in \mathbb{N}, \gamma \notin V[\Pi_{\gamma_k}, 3\alpha_k]$ and $j \in J_k$. Consequently, $|\{\widehat{M}_{\mathbf{I}(\gamma)}(g_j) \ge 1/2^j\} \cap (0,1)^2| < 1/2^j$ for every $k \in \mathbb{N}, \gamma \notin V[\Pi_{\gamma_k}, 3\alpha_k]$ and $j \in J_k$.

Set $g = \sum_{j=1}^\infty g_j$. First note that $\|g\|_L \le \sum_{j=1}^\infty \|g_j\|_L < \sum_{j=1}^\infty \eta_j/h_j < \infty$. Thus, g is a summable function. Suppose $\gamma \notin E$. Let us prove that $\mathbf{I}(\gamma)$ differentiates $\int g$. Since $\operatorname{supp} g \subset (0,1)^2$, then $\mathbf{I}(\gamma)$ differentiates $\int g$ at every point $x \notin [0,1]^2$. Further, denote

$$T_j = \{\widehat{M}_{\mathbf{I}(\gamma)}(g_j) \ge 1/2^j\} \cap (0,1)^2, \quad T = \varlimsup_{j\to\infty} T_j.$$

We have that $\gamma \notin \varlimsup\limits_{k\to\infty} V[\Pi_{\gamma_k}, 3\alpha_k]$. Consequently, there is $k_0 \in \mathbb{N}$ for which $\gamma \notin V[\Pi_{\gamma_k}, 3\alpha_k]$ for every $k \ge k_0$. The last condition on the basis of the estimation 7) implies: $|T_j| < 1/2^j$ for every $j \ge j_{k_0}$. Now taking into account that $|T_j| \le 1$ $(j \in \mathbb{N})$ we have: $\sum_{j=1}^\infty |T_j| < \infty$. Consequently, $|T| = 0$. Thus, for arbitrary given point $x \in (0,1)^2 \setminus T$ there is $j^* \in \mathbb{N}$ for which $\widehat{M}_{\mathbf{I}(\gamma)}(g_j)(x) < 1/2^j$ for every $j > j^*$. Now taking into account boundedness of the functions g_j we write: $\widehat{M}_{\mathbf{I}(\gamma)}(g)(x) \le \sum_{j=1}^\infty \widehat{M}_{\mathbf{I}(\gamma)}(g_j)(x) \le \sum_{j=1}^{j^*} \widehat{M}_{\mathbf{I}(\gamma)}(g_j)(x) + \sum_{j=j^*+1}^\infty 1/2^j < \infty$. Thus, $(0,1)^2 \setminus T \subset \{\widehat{M}_{\mathbf{I}(\gamma)}(g) < \infty\}$. Note that by virtue of the result of Besicovitch (see, e.g., [3, Ch. IV, §3]) the sets $\{g < \overline{D}_B(\int g, \cdot) < \infty\}$ and $\{-\infty < \underline{D}_B(\int g, \cdot) < g\}$ have zero measure. Therefore, taking into account the last inclusion, we conclude that $\mathbf{I}(\gamma)$ differentiates $\int g$.

Suppose $\gamma \in E$. Then $\gamma \in \varlimsup\limits_{k\to\infty} V[\gamma_k, \alpha_k/2]$. Thus, the set $N = \{k \in \mathbb{N} : \gamma \in V[\gamma_k, \alpha_k/2]\}$ is infinite. Let $k \in N$. Taking into account the properties 3)–(c), 3)–(d) and 4) it is easy to see that the sets $A_j(\gamma)$ $(j \in J_k)$ are probabilistically independent. Therefore,

$$\Big|\bigcup_{j\in J_k} A_j(\gamma)\Big| = 1 - \Big|\bigcap_{j\in J_k}((0,1)^2 \setminus A_j(\gamma))\Big| = 1 - \prod_{j\in J_k}(1 - |A_j(\gamma)|).$$

Now using 3)–(b) and taking into account the choice of the numbers j_k, we obtain: $|\bigcup_{j\in J_k} A_j(\gamma)| > 1 - 1/2^k$. From this estimation we conclude: if A denotes the upper limit of the sequence of the sets $\bigcup_{j\in J_k} A_j(\gamma)$ $(k \in N)$, then A is of full measure in $(0,1)^2$, i.e. $|(0,1)^2 \setminus A| = 0$.

Let F be the upper limit of the sequence of the sets $\{M_{\mathbf{P}}^{(0,d(Q_j))}(g_j) > 0\}$ $(j \in \mathbb{N})$. By virtue of the property 5), $\sum_{j=1}^{\infty} |\{M_{\mathbf{P}}^{(0,d(Q_j))}(g_j) > 0\}| < \infty$. Therefore the set F is of zero measure.

For any $x \in A \setminus F$ let us prove the equality $\overline{D}_{B(\gamma)}(\int g, x) = +\infty$. It will imply that the equality is valid for almost every point from $(0,1)^2$.

We can find an infinite set $N^* \subset N$, a sequence $j(k) \in J_k$ $(k \in N^*)$ and a number $j(0) \in \mathbb{N}$ with the properties: i) $x \in A_{j(k)}(\gamma)$ for every $k \in N^*$; ii) $x \notin \{M_{\mathbf{P}}^{(0,d(Q_j))}(g_j) > 0\}$ for every $j > j(0)$. We can assume that $j(k) > j(0)$ $(k \in N^*)$.

For every $k \in N^*$ we can find a rectangle $R_k \in B(\gamma)(x)_{[d(Q_{j(k)}),d(W_{j(k)})]}$ for which $\frac{1}{|R_k|}\int_{R_k} g_{j(k)} > \eta_{j(k)}$. Let us estimate the integral means on R_k of the functions g_j with $j \neq j(k)$. Taking into account the property 2), we have: $\frac{1}{|R_k|}\int_{R_k} g_j = 0$ if $j(0) < j < j(k)$ and $\frac{1}{|R_k|}\int_{R_k} g_j < \eta_j/h_j$ if $j > j(k)$. Consequently,

$$\begin{aligned}\frac{1}{|R_k|}\int\limits_{R_k} g &= \frac{1}{|R_k|}\int\limits_{R_k} g_{j(k)} - \sum_{j=1}^{j(0)} \frac{1}{|R_k|}\int\limits_{R_k} g_j - \sum_{j=j(0)+1}^{j(k)-1} \frac{1}{|R_k|}\int\limits_{R_k} g_j \\ &\quad - \sum_{j=j(k)+1}^{\infty} \frac{1}{|R_k|}\int\limits_{R_k} g_j > \eta_{j(k)} - \sum_{j=1}^{j(0)} \|g_j\|_{L^\infty} - \sum_{j=j(k)+1}^{\infty} \frac{\eta_j}{h_j}.\end{aligned}$$

Thus, the rectangles R_k $(k \in N^*)$ satisfy conditions: $R_k \in B(\gamma)(x)$ $(k \in N^*)$, $\operatorname{diam} R_k \to 0$ $(N^* \ni k \to \infty)$ and $\frac{1}{|R_k|}\int_{R_k} g \to +\infty$ $(N^* \ni k \to \infty)$. Therefore, $\overline{D}_{B(\gamma)}\big(\int g, x\big) = +\infty$.

Summarizing above established properties of the function g we have: i) $g \in L(\mathbb{R}^2)$ and $\operatorname{supp} g \subset (0,1)^2$; ii) $\overline{D}_{B(\gamma)}\big(\int g, x\big) = +\infty$ a.e. on $(0,1)^2$ for every $\gamma \in E$; iii) $\mathbf{I}(\gamma)$ differentiates $\int g$ for every $\gamma \notin E$.

Set $f(x_1, x_2) = \sum_{i,j\in\mathbb{Z}} g(x_1 + i, x_2 + j)/2^{i+j}$ $((x_1, x_2) \in \mathbb{R}^2)$. Then we can easily check that f satisfies the conditions providing E to be an R_B-set. The theorem is proved.

Remark 3. The function f constructed in the proof of Theorem 1 for any rotation $\gamma \notin E$ satisfies stronger condition than it is required. Namely, $\int f$ is differentiable with respect to the basis $\mathbf{I}(\gamma)$ which is broader than the basis $B(\gamma)$.

Remark 4. The function f constructed in the proof of Theorem 1 takes values of both signs. For non-negative summable functions the problem of characterization of singular rotation's sets is open even for the case of the basis $\mathbf{I}(\mathbb{R}^2)$. Some partial results in this direction are obtained in [8] and [12].

Remark 5. For the multidimensional case the problem of characterization of $W_{\mathbf{I}(\mathbb{R}^n)}$-sets and $R_{\mathbf{I}(\mathbb{R}^n)}$-sets is open. Note that a class of $R_{\mathbf{I}(\mathbb{R}^n)}$-sets is found in [9].

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Department of Mathematics, Akaki Tsereteli State University, 59, Tamar Mepe Str., Kutaisi 4600, Georgia

E-mail address: oniani@atsu.edu.ge

E-mail address: kaxachubi@gmail.com

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ABOUT ONE CONTACT PROBLEM FOR A VISCOELASTIC HALFPLATE

N. SHAVLAKADZE, G. KAPANADZE, AND L. GOGOLAURI

Abstract. Exact solutions of two-dimensional singular integro-differential equations related to the problems of interaction of an elastic thin infinite homogeneous inclusion with a plate for the Kelvin-Voigt linear model are considered. The Kolosov-Muskhelishvili's type formulas are obtained, and the problem is reduced to the Volterra type integral equations. Using the method of integral transformation, the boundary value problem of the theory of analytic functions is obtained. The solution of the problem is represented explicitly and asymptotic analysis is carried out.

Introduction

The theories of viscoelasticity including the Maxwell model, the Kelvin-Voigt model and the standard linear solid model were used to predict response of a material under the action of different loading conditions. Viscoelastic materials play an important role in many branches of civil and geotechnical engineering, technology and biomechanics.

Significant development of hereditary Bolzano–Volterra mechanics is determined by many technical applications in the theory of polymers, metals, plastics, concrete and in the mining engineering. The fundamentals of the theory of viscoelasticity, the methods for solving linear and nonlinear problems of the creep theory, the problems of mechanics of inhomogeneous ageing viscoelastic materials, some boundary value problems of the theory of growing solids, the contact and mixed problems of the theory of viscoelasticity for composite inhomogeneously ageing and nonlinearly ageing bodies are considered in [1,5,7,11,12,14,15,20].

A complete study of various possible forms of viscoelastic relations and some aspects of the general theory of viscoelasicity are studied in [8,9,13,19]. The research dealing with the material creep can be found in [2–4,17].

In [6,21], we have considered integro-differential equations with a variable coefficient related to the interaction of an elastic thin inclusion and a plate, when the inclusion and plate materials possess a creep property. Using the investigation of different boundary value problems of the theory of analytic functions, we have got solutions of those integro-differential equations and established asymptotics of unknown contact stresses.

1. Kolosov-Muskhelishvili's Type Formulas for One Model of the Plane Theory of Viscoelasticity

For viscoelastic bodies, following the Kelvin–Voigt model [20], the Hook's law has the form

$$\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{xx}, \\ Y_y &= \lambda\theta + 2\mu e_{yy} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{yy}, \\ X_y &= \mu\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) + \mu^*\Big(\frac{\partial \dot{v}}{\partial x} + \frac{\partial \dot{u}}{\partial y}\Big), \end{aligned} \tag{1.1}$$

where X_x, Y_y, X_y, $u, v, \theta = e_{xx} + e_{yy}, e_{xx}, e_{yy}, e_{xy}$ are the functions of variables x, y, t. The points in the expressions $\dot{\theta}$, $\dot{e}_{xx}$, $\dot{e}_{yy}$, $\dfrac{\partial \dot{v}}{\partial x}$, $\dfrac{\partial \dot{u}}{\partial y}$ denote derivatives with respect to the time t; λ, μ are the elastic and λ^*, μ^* are viscoelastic constants.

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The components of stresses X_x, Y_y, X_y defined by the relations (1.1), when body forces are absent, just as in the classical case, must satisfy the equilibrium equations and the compatibility condition in the plane theory of elasticity [18].

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0, \qquad \frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}=0, \qquad \Delta\left(X_x+Y_y\right)=0.$$

Moreover, X_x, Y_y, X_y should be single-valued and continuous functions up to the boundary, together with their second derivatives with respect to the variables x, y. If these conditions are fulfilled, there exists a function $U\left(x,y,t\right)$ satisfying the biharmonic equation with respect to the variables x, y,

$$\Delta\Delta U=0, \tag{1.2}$$

through which the stresses are expressed as follows:

$$X_x=\frac{\partial^2 U}{\partial y^2}, \qquad Y_y=\frac{\partial^2 U}{\partial x^2}, \qquad X_y=-\frac{\partial^2 U}{\partial x\partial y}. \tag{1.3}$$

In view of formula (1.3), we write the first two equations of the relations (1.1) in the form

$$\begin{aligned}\lambda\theta+2\mu e_{xx}+\lambda^*\dot{\theta}+2\mu^*\dot{e}_{xx}&=\frac{\partial^2 U}{\partial y^2},\\ \lambda\theta+2\mu e_{yy}+\lambda^*\dot{\theta}+2\mu^*\dot{e}_{yy}&=\frac{\partial^2 U}{\partial x^2}.\end{aligned} \tag{1.4}$$

Summing up the above equations, we obtain the following equality

$$2\left(\lambda+\mu\right)\theta+2\left(\lambda^*+\mu^*\right)\dot{\theta}=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\equiv\Delta U,$$

which we write as

$$\dot{\theta}+k\theta=\frac{\Delta U}{2\left(\lambda^*+\mu^*\right)}, \tag{1.5}$$

where $k=\dfrac{\lambda+\mu}{\lambda^*+\mu^*}$.

Assuming that at the moment of time t_0 (i.e., at the moment when the body is under the action of loading) $e_{xx}\left(x,y,t_0\right)=0$ and $e_{yy}\left(x,y,t_0\right)=0$, a solution of the linear first order differential equation (1.5) takes the form

$$\theta\left(x,y,t\right)=e^{-k(t-t_0)}\int\limits_{t_0}^{t}\frac{\Delta U\left(x,y,\tau\right)}{2\left(\lambda^*+\mu^*\right)}e^{k(\tau-t_0)}d\tau. \tag{1.6}$$

Equations (1.4) are given the form

$$\begin{aligned}\dot{e}_{xx}+me_{xx}&=\frac{1}{2\mu^*}\left[\frac{\partial^2 U}{\partial y^2}-\lambda\theta-\lambda^*\dot{\theta}\right]\equiv\Psi_1,\\ \dot{e}_{yy}+me_{yy}&=\frac{1}{2\mu^*}\left[\frac{\partial^2 U}{\partial x^2}-\lambda\theta-\lambda^*\dot{\theta}\right]\equiv\Psi_2,\end{aligned} \tag{1.7}$$

where $m=\dfrac{\mu}{\mu^*}$.

It follows from (1.7) that

$$\begin{aligned}\frac{\partial u}{\partial x}&=e^{-m(t-t_0)}\int\limits_{t_0}^{t}\Psi_1\left(x,y,\tau\right)e^{m(\tau-t_0)}d\tau,\\ \frac{\partial v}{\partial y}&=e^{-m(t-t_0)}\int\limits_{t_0}^{t}\Psi_2\left(x,y,\tau\right)e^{m(\tau-t_0)}d\tau.\end{aligned} \tag{1.8}$$

The functions Ψ_1, Ψ_2 introduced by the relations (1.7), in view of equation (1.5) and formula (1.6), after some transformations are reprsented as follows:

$$\Psi_1(x,y,t) = \frac{1}{2\mu^*}\left[-\frac{\partial^2 U}{\partial x^2} + \Delta U + \frac{n_2 e^{-k(t-t_0)}}{4}\int\limits_{t_0}^{t} e^{k(\tau-t_0)}\Delta U d\tau - \frac{\lambda^*\Delta U}{2(\lambda^*+\mu^*)}\right],$$
$$\Psi_2(x,y,t) = \frac{1}{2\mu^*}\left[-\frac{\partial^2 U}{\partial y^2} + \Delta U + \frac{n_2 e^{-k(t-t_0)}}{4}\int\limits_{t_0}^{t} e^{k(\tau-t_0)}\Delta U d\tau - \frac{\lambda^*\Delta U}{2(\lambda^*+\mu^*)}\right] \tag{1.9}$$

where $n_2 = \dfrac{2(\mu\lambda^* - \lambda\mu^*)}{(\lambda^*+\mu^*)^2}$.

Following [18], we introduce the notation $\Delta U = P$, where P is a harmonic function of variables x, y, according to equation (1.2). Let Q be a conjugate to it function. Introduce the function $\varphi(z,t)$ in such a way that

$$\varphi(z,t) = p + iq = \frac{1}{4}\int (P+iQ)\,dz, \tag{1.10}$$

from which it follows that

$$P = 4\frac{\partial p}{\partial x} = 4\frac{\partial q}{\partial y}. \tag{1.11}$$

Substituting into equations (1.8) the values of the functions Ψ_1, Ψ_2 from (1.9) and taking into account equalities (1.11), we can represent the expressions $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial v}{\partial y}$ as

$$2\mu^*\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left\{-\int\limits_{t_0}^{t} e^{m(\tau-t_0)}\left(\frac{\partial U}{\partial x} - n_1 p\right)d\tau + n_2\int\limits_{t_0}^{t} e^{n(\tau-t_0)}\left(\int\limits_{t_0}^{\tau} e^{k(s-t_0)}p ds\right)d\tau\right\}e^{-m(t-t_0)}$$
$$2\mu^*\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left\{-\int\limits_{t_0}^{t} e^{m(\tau-t_0)}\left(\frac{\partial U}{\partial y} - n_1 q\right)d\tau + n_2\int\limits_{t_0}^{t} e^{n(\tau-t_0)}\left(\int\limits_{t_0}^{\tau} e^{k(s-t_0)}q ds\right)d\tau\right\}e^{-m(t-t_0)},$$

where $n_1 = \dfrac{2(\lambda^*+2\mu^*)}{(\lambda^*+\mu^*)}$, $n = m - k = \dfrac{\mu\lambda^* - \lambda\mu^*}{\mu^*(\lambda^*+\mu^*)}$.

Integrating the last expressions, we obtain

$$2\mu^* u = \left\{-\int\limits_{t_0}^{t} e^{m(\tau-t_0)}\left(\frac{\partial U}{\partial x} - n_1 p\right)d\tau\right.$$
$$\left.+n_2\int\limits_{t_0}^{t} e^{n(\tau-t_0)}\left(\int\limits_{t_0}^{\tau} e^{k(s-t_0)}p ds\right)d\tau\right\}e^{-m(t-t_0)} + f_1(y,t), \tag{1.12}$$

$$2\mu^* v = \left\{-\int\limits_{t_0}^{t} e^{m(\tau-t_0)}\left(\frac{\partial U}{\partial y} - n_1 q\right)d\tau\right.$$
$$\left.+n_2\int\limits_{t_0}^{t} e^{n(\tau-t_0)}\left(\int\limits_{t_0}^{\tau} e^{k(s-t_0)}q ds\right)d\tau\right\}e^{-m(t-t_0)} + f_2(x,t). \tag{1.13}$$

Omitting the functions $f_1(y,t)$, $f_2(x,t)$ due to the fact that they provide only rigid displacement at any fixed moment of time t and taking into account (1.10), it follows from equalities (1.12) and (1.13) that

$$2\mu^*(u+iv) = \left\{-\int\limits_{t_0}^{t} e^{m(\tau-t_0)}\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} - n_1\varphi(z,\tau)\right)d\tau\right.$$

$$+n_2\int_{t_0}^{t} e^{n(\tau-t_0)}\Big(\int_{t_0}^{\tau} e^{k(s-t_0)}\varphi(z,s)ds\Big)d\tau\Big\}e^{-m(t-t_0)}.$$

Taking into account that [18]

$$\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}=\varphi\left(z,t\right)+z\overline{\varphi'\left(z,t\right)}+\overline{\psi\left(z,t\right)}, \tag{1.14}$$

from (1.14), after some transformations, we obtain

$$2\mu^*(u+iv)=\int_{t_0}^{t} e^{m(\tau-t)}\left(\varphi(z,\tau)-z\overline{\varphi'(z,\tau)}-\overline{\psi(z,\tau)}\right)d\tau+\omega\int_{t_0}^{t} e^{k(\tau-t)}\varphi(z,\tau)d\tau, \tag{1.15}$$

where $\omega=\dfrac{2\mu_*}{\lambda^*+\mu^*}$ (here "prime" means the derivative with respect to a variable z).

Thus formulas (1.14) and (1.15) are generalize Kolosov-Muskhelishvili's formulas for a viscoelastic material.

2. Statement of the Problem and Reduction to the Integral Equation

Consider now the contact problem of interaction of a semi-infinite stringer of constant rigidity (constant cross-section) and an infinite viscoelastic plate occupying the lower half-plane, when the stringer is under the action of tangential stresses $T_0(x)H(t-t_0)$ ($H(t-t_0)$ is the Heaviside function) and free from normal stresses. We have to find tangential contact stresses $T(x,t)$ along the contact line.

From equation (1.15), we get

$$2\mu^*\frac{\partial u\left(x,y,t\right)}{\partial x}=\operatorname{Re}\Big\{\int_{t_0}^{t} e^{m(\tau-t)}\Big[\Phi\left(z,\tau\right)-\overline{\Phi\left(z,\tau\right)}-z\overline{\Phi'\left(z,\tau\right)}-\overline{\Psi\left(z,\tau\right)}\Big]d\tau\Big\}$$
$$+\omega\operatorname{Re}\int_{t_0}^{t} e^{k(\tau-t)}\Phi(z,\tau)d\tau. \tag{2.1}$$

Since normal stresses in our case are absent, the complex potentials Φ, Ψ will have the form [18]:

$$\Phi\left(z,t\right)=\frac{1}{2\pi}\int_0^{\infty}\frac{T\left(\sigma,t\right)}{\sigma-z}d\sigma, \tag{2.2}$$

$$\Psi\left(z,t\right)=-\frac{1}{2\pi}\int_0^{\infty}\frac{T\left(\sigma,t\right)}{\sigma-z}d\sigma-\frac{1}{2\pi}\int_0^{\infty}\frac{T\left(\sigma,t\right)}{\left(\sigma-z\right)^2}\sigma d\sigma. \tag{2.3}$$

If we substitute into equation (2.1) the values of the functions Φ,Ψ from formulas (2.2) and (2.3) and pass to the real values, then after some transformations we get

$$2\mu^*\frac{\partial u(x,0,t)}{\partial x}=\int_{t_0}^{t} e^{m(\tau-t)}\Phi(x,\tau)d\tau+\omega\int_{t_0}^{t} e^{k(\tau-t)}\Phi(x,\tau)d\tau. \tag{2.4}$$

Introducing a time operator

$$L\Phi(x,t)=\int_{t_0}^{t} e^{m\tau}\Phi(x,\tau)d\tau+\omega\int_{t_0}^{t} e^{nt+k\tau}\Phi(x,\tau)d\tau, \tag{2.5}$$

(2.4) will have the form

$$2\mu^*\frac{\partial u(x,0,t)}{\partial x}=e^{-mt}L\Phi(x,t). \tag{2.6}$$

On the other hand, for deformations of the stringer points we have

$$\frac{\partial u_0(x,t)}{\partial x} = \frac{1}{E}\int_0^x [T(y,t) - T_0(y)H(t-t_0)]dy, \tag{2.7}$$

and the condition for the principal vector to be equal to zero yields

$$\int_0^\infty [T(y,t) - T_0(y)H(t-t_0)]dy = 0.$$

From the condition of the stringer and plate contact

$$\frac{\partial u(x,0,t)}{\partial x} = \frac{\partial u_0(x,t)}{\partial x},$$

in view of the relations (2.6), (2.7) and (2.2), we obtain

$$e^{-mt} L\left(\int_0^\infty \frac{T(\sigma,t)d\sigma}{\sigma - x}\right) = \frac{4\pi\mu^*}{E}\int_0^x [T(y,t) - T_0(y)H(t-t_0)]\, dy. \tag{2.8}$$

Introducing the notation

$$\int_0^x [T(y,t) - T_0(y)H(t-t_0)]\, dy \equiv K(x,t), \qquad K(0,t) = K(\infty,t) = 0,$$

we write the relation (2.8) in the form

$$L\int_0^\infty \frac{K'(\sigma,t)d\sigma}{\sigma - x} = \alpha e^{mt} K(x,t) - F(x)B(t), \tag{2.9}$$

where

$$B(t) = LH(t-t_0) = \frac{1}{m}(e^{mt} - e^{mt_0}) + \frac{\omega}{k}e^{nt}(e^{kt} - e^{kt_0}), \quad \alpha = \frac{4\pi\mu^*}{E}, \quad F(x) = \int_0^\infty \frac{T_0(\sigma)d\sigma}{\sigma - x}.$$

After transformation of variables

$$\sigma = e^\zeta, \qquad x = e^\xi,$$

the integral differential equation (2.9) takes the form

$$-L\int_{-\infty}^\infty \frac{K'_0(\zeta,t)d\zeta}{1 - e^{-(\xi-\zeta)}} = \alpha e^\xi e^{mt} K_0(\xi,t) - B(t)e^\xi F_0(\xi), \quad |\xi| < \infty \tag{2.10}$$

$$K_0(-\infty,t) = K_0(\infty,t) = 0,$$

where

$$K_0(\zeta,t) = K(e^\zeta,t), F_0(\xi) = F(e^\xi).$$

Performing the generalized Fourier transformation [10] of both parts of equation (2.10), we obtain

$$\pi\, s\, cth \pi s\, L\hat{K}_0(s,t) = -\alpha e^{mt}\hat{K}_0(s-i,t) + B(t)\hat{F}(s), \tag{2.11}$$

where

$$\hat{F}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F_0(\xi)e^\xi e^{i\xi s}d\xi, \quad \hat{K}_0(s,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty K_0(\xi,t)e^\xi e^{i\xi s}d\xi.$$

The problem can be formulated as follows: Find the function $\hat{K}_0(z,t)$, analytic in the strip $-1 < \operatorname{Im} z < 1$, (with the exception of a finite number of points lying in the strip $0 < \operatorname{Im} z < 1$ in which it may have poles), vanishing at infinity and satisfying condition (2.11). obviously, if we will be able to find the function $\hat{K}_0^-(z,t)$, holomorphic in the strip $-1 < \operatorname{Im} z < 0$, vanishing at infinity, continuously extendable on the strip boundary, by the boundary condition (2.11), then the solution of the above-formulated problem is

$$\hat{K}_0(z,t)=\begin{cases} L\hat{K}_0^-(z,t), & -1<\operatorname{Im} z<0\\ \dfrac{-\alpha e^{mt}\hat{K}_0^-(z-i,t)+B(t)\hat{F}(t)}{\pi z cth\pi z}, & 0<\operatorname{Im} z<1\end{cases} \tag{2.11.1}$$

To factorize the coefficient of problem (2.11), we represent the function $M(s)=s\,cth\pi s$ as follows:

$$M(s)=iscth\pi s\cdot th\frac{\pi}{2}s\cdot\frac{sh\frac{\pi}{2}(s-i)}{sh\frac{\pi}{2}s}.$$

Owing to the fact that the index of the function $M_0(s)=\ cth\pi s\, th\frac{\pi}{2}s$ equals zero and $\ln\left[cth\pi s\, th\frac{\pi}{2}s\right]$ is integrable on the real axis, we can represent the function $M_0(s)$ in the form

$$M_0(s)=\frac{\mathrm{X}_0(s-i)}{\mathrm{X}_0(s)},$$

where $\mathrm{X}_0(z)=\exp\left(-\frac{1}{2i}\int\limits_{-\infty}^{\infty}\ln\left[cth\pi s\, th\frac{\pi}{2}s\right]cth\pi(t-z)dz\right)$.

Then introducing the notation

$$\tilde{K}(s,t)=\frac{is\hat{K}_0(s,t)}{sh\frac{\pi}{2}s\cdot\mathrm{X}_0(s)},$$

the relation (2.11) can rewritten as

$$\frac{1+is}{\alpha_0}L\,\tilde{K}(s,t)+\tilde{K}(s-i,t)e^{mt}=G(s)B(t), \tag{2.12}$$

where

$$G(s)=\frac{1}{\pi}\frac{(1+is)\hat{F}_0(s)}{\alpha_0 sh\frac{\pi}{2}(s-i)\cdot\mathrm{X}_0(s-i)}.$$

Using the known representation [16]

$$\frac{1+is}{\alpha_0}=\frac{\mathrm{X}_1(s-i)}{\mathrm{X}_1(s)},\quad \mathrm{X}_1(z)=\exp(-iz\ln\alpha_0)\Gamma(1+iz)$$

the condition (2.12) takes the form

$$L\,A(s,t)+A(s-i,t)e^{mt}=G_1(s)B(t), \tag{2.13}$$

where

$$A(s,t)=\frac{\tilde{K}(s,t)}{\mathrm{X}_1(s)}\qquad G_1(s)=\frac{G(s)}{\mathrm{X}_1(s-i)}.$$

Performing the generalized Fourier transformation of both sides of equation (2.13), we obtain the Volterra second kind integral equation

$$L\hat{A}(u,t)+e^{-u}\hat{A}(u,t)e^{mt}=\hat{G}_1(u)B(t). \tag{2.14}$$

Taking into account the form of the operator L, from the last equation, we have

$$\hat{A}(u,t_0)=0. \tag{2.15}$$

Having differentiated the relation (2.14) by using the notation (2.5), and get

$$(\omega+1)e^{mt}\hat{A}(u,t)+\omega n\int\limits_{t_0}^{t}e^{nt+k\tau}\hat{A}(u,\tau)d\tau+e^{-u}\dot{\hat{A}}(u,t)e^{mt}$$
$$+me^{-u}\hat{A}(u,t)e^{mt}=\hat{G}_1(u)\dot{B}(t). \tag{2.16}$$

At the point $t=t_0$ the previous expression yields

$$\dot{\hat{A}}(u,t_0)=(1+\omega)\hat{G}_1(u)e^{u}. \tag{2.17}$$

Multiplying both parts of equation (2.16) by e^{-mt} and differentiating with respect to the variable t, after some transformations, we obtain

$$\ddot{\hat{A}}(u,t) + a(u)\dot{\hat{A}}(u,t) + b(u)\hat{A}(u,t) = \gamma \hat{G}_1(u)e^u, \tag{2.18}$$

where

$$a(u) = (\omega+1)e^u + m + k, \qquad b(u) = \gamma e^u + mk, \qquad \gamma = \frac{\lambda + 3\mu}{\lambda^* + \mu^*}.$$

The discriminant of the corresponding characteristic equation will have the form

$$D = [(\omega+1)e^u + n]^2 - 4n\omega e^u.$$

It is not difficult to show that the above discriminant is always positive, and a generalized solution of the inhomogeneous differential equation (2.18) has the form

$$\hat{A}(u,t) = \bar{\hat{A}}(u,t) + \tilde{\hat{A}}(u,t), \tag{2.19}$$

where

$$\bar{\hat{A}}(u,t) = c_1(u)e^{-p_1(u)t} + c_2(u)e^{-p_2(u)t}, \qquad p_1(u), p_2(u) > 0$$

is a general solution of the homogeneous equation corresponding to equation (2.18), and $\tilde{\hat{A}}(u,t) = \dfrac{\gamma \hat{G}_1(u)e^u}{\gamma e^u + mk}$ is a particular solution of that equation.

From (2.19), using initial conditions (2.15) and (2.17) and defining coefficients $c_1(u)$ and $c_2(u)$, we obtain the solution of the differential equation (2.18) in the form

$$\hat{A}(u,t) = \hat{G}_1(u)G_2(u,t) + \frac{\hat{G}_1(u)}{1+\delta e^{-u}},$$

where

$$G_2(u,t) = \left[\frac{(\gamma p_2(u) - (1+\omega)b(u))e^{-p_1(u)(t-t_0)} + ((1+\omega)b(u) - \gamma p_1(u))e^{-p_2(u)(t-t_0)}}{(p_1(u)-p_2(u))b(u)}\right]e^u, \tag{2.20}$$

$$\delta = \frac{mk}{\gamma} = \frac{\mu(\lambda+\mu)}{\gamma\mu^*(\lambda^*+\mu^*)}.$$

Using Parceval's generalized formula, the inverse Fourier transformation [10] yields

$$A(s,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G_1(y)\hat{G}_2(y-s,t)dy + \frac{i}{2}\int_{-\infty}^{\infty}\frac{G_1(y)e^{i(y-s)\ln\delta}dy}{sh\pi(y-s)},$$

where $A(s,t)$ is the boundary value on the real axis of the function $A(z,t)$, $(z = s+iy)$ holomorphic on the strip $-1 < \operatorname{Im} z < 1$, with the exception of the point $z = \dfrac{i}{2}$, at which it has the first order pole continuously extendable in the strip boundary and vanishing at infinity (see formula (2.11.1)).

Respectively,

$$\hat{K}_0(z,t) = A(z,t)\mathrm{X}(z),$$

where $\mathrm{X}(z) = \dfrac{\mathrm{X}_0(z)\mathrm{X}_1(z)}{iz}sh\dfrac{\pi}{2}z.$

Performing again the inverse Fourier generalized transformation and getting back to our variables, we obtain

$$K'(x,t) = T(x,t) - T_0(x)H(t-t_0) = \frac{x^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(s,t)\mathrm{X}(s)e^{-is\ln x}ds.$$

Using Cauchy's formula and residue theorem for the holomorphic in the strip $-1 < \operatorname{Im} z < 1$ function $A(z,t)$, we obtain the following asymptotic estimates for the unknown tangential contact stress

$$K'(x,t) = O(x^{-1/2}), \qquad x \to 0-$$
$$K'(x,t) = O(x^{-2}), \qquad x \to \infty.$$

As for the behaviour of the tangential contact stress and other mechanical values concerning time $t \geq t_0$, it is clearly seen from the expression (2.20).

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A. Razmadze Mathematical Institute I. Javakhishvili Tbilisi State University 6, Tamarashvili Str., Tbilisi 0177, Georgia

E-mail address: nusha@rmi.ge

E-mail address: kapanadze.49@mail.ru

E-mail address: lida@rmi.ge

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ROUGH STATISTICAL CONVERGENCE ON TRIPLE SEQUENCE OF RANDOM VARIABLES IN PROBABILITY

N. SUBRAMANIAN[1] AND A. ESI[2]

Abstract. This paper aims a further improvement from the works of Phu [9], Aytar [1] and Ghosal [7]. We propose a new apporach to extend the application area of rough statistical convergence usually used in triple sequence of real numbers to the theory of probability distributions. The introduction of this concept in probability of rough statistical convergence, rough strong Cesàro summable, rough lacunary statistical convergence, rough $N_\theta-$ convergence, rouugh $\lambda-$ statistical convergence and rough strong $(V,\lambda)-$ summable generalize the convergence analysis to accommodate any form of distribution of random variables. Among these six concepts in probability only three convergences are distinct rough statistical convergence, rough lacunary statistical convergence and rough $\lambda-$ statistical convergence where rough strong Cesàro summable is equivalent to rough statistical convergence, rough $N_\theta-$ convergence is equivalent to rough lacunary statistical convergence, rough strong $(V,\lambda)-$ summable is equivalent to rough $\lambda-$ statistical convergence. Basic properties and interrelations of above mentioned three distinct convergences are investigated and some observations are made in these classes and in this way we show that rough statistical convergence in probability is the more generalized concept compared to the usual rough statistical convergence.

1. Introduction

In probability theory, a new type of convergence called statistical convergence in probability was introduced in Ghosal [7]. Let $(X_{mnk})_{m,n,k\in\mathbb{N}}$ be a triple sequence of random variables where each X_{mnk} is defined on the same sample spaces W (for each (m,n,k)) with respect to a given class of events Δ and a given probability function $P:\Delta\to\mathbb{R}^3$. Then the triple sequence (X_{mnk}) is said to be statistical convergent in probability to a random variable $X:W\to\mathbb{R}^3$ if for any $\epsilon,\delta>0$

$$\lim_{uvw\to\infty}\frac{1}{uvw}\Big|\Big\{m\le u, n\le v,\, k\le w: P\big(|X_{mnk}-l\ge\delta\big)\Big\}\Big|=0.$$

In this case we write $X_{mnk}\to^{S^P} l$. The class of all triple sequences of random variables which are statistical convergence in probability is denoted by S^P.

In this paper we introduce new notions namely rough statistical convergence in probability, rough strong Cesàro summable in probability rough lacunary statistical convergence in probability, rough $N_\theta-$ convergence in probability rough strong $(V,\lambda)-$ summable in probability and rough $\lambda-$ statistical convergence in probability. Among these six concepts in probability only three convergences are distinct-rough statistical convergence in probability, rough lacunary statistical convergence in probability and rough $\lambda-$ statistical convergence in probability, rough $N_\theta-$ convergence in probability is equivalent to rough lacunary statistical convergence in probability, rough strong $(V,\lambda)-$ summable in probability is equivalent to rough $\lambda-$ statistical convergence in probability. Basic properties and interrelations of above mentioned three distinct convergences are investigated and make some observations about these classes.

The idea of statistical convergence was introduced by H. Steinhaus and also independently by H. Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence. Later on the notion was investigated by Tripathy ([13], [15]), Tripathy and Sen [14], Tripathy and Baruah [16], Tripathy and Goswami ([17–20]) and others.

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Key words and phrases. Rough statistical convergence; Rough strong Cesàro summable; Rough lacunary statistical convergence; Rough $N_\theta-$ convergence; Rough $\lambda-$ statistical convergence; Rough strong $(V,\lambda)-$ summable.

Let K be a subset of the set $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, and let us denote the set $\{(m,n,k)\in K : m \leq u,\ n \leq v,\ k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = \lim_{uvw\to\infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, every finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$, where $K^c = \mathbb{N} - K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$, $m,n,k \in \mathbb{N}$.

A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st-\lim\ x = 0$, provided that the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\in > 0$, infinitely many terms of the sequence may remain outside the $\in -$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m,n,k except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (m,n,k) and we abbreviate this by a.a. (m,n,k).

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set $K = \{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i,j,\ell) \in \mathbb{N}^3\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non-thin subsequence of a triple sequence x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\}$$

is different from zero for every $\in > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0.$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [9], who introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [8] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. Different types of notions of triple sequence were introduced and investigated at the initial by Sahiner et al. [10, 11], Esi et al. [2–4], Dutta et al. [5], Subramanian et al. [12], Debnath et al. [6] and many others.

Throughout the paper let r be a nonnegative real number.

2. Triple Rough Statistical Convergence in Probability

Definition 2.1. Let r be a non-negative real number. A triple sequence (x_{mnk}) is said to be rough convergent to l with respect to the roughness degree r (or shortly: $r-$ convergent to x) if for every $\in > 0$, there exist some numbers u, v and w such that

$$\left|x_{mnk} - l\right| < r + \in \quad \text{for all} \ \ m \geq u,\ n \geq v,\ k \geq w$$

and denoted by $x_{mnk} \to^{r} l$. if we take $r = 0$, then we obtain the ordinary convergence.

Definition 2.2. Let r be a non-negative real number. A triple sequence (x_{mnk}) is said to be rough statistically convergent to l with respect to the roughness of degree r (or shortly: $r-$ statistically convergent to l) if for every $\epsilon > 0$, the set

$$\left\{ (m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq r + \in \right\}$$

has asymptotic density zero or equivalently, if the condition $S - \lim_{mnk\to\infty} \sup |x_{mnk} - l| \leq r$ is satisfied and we denote by $x_{mnk} \to_r^{S^P} l$.

If we take $r = 0$, then we obtain the ordinary statistical convergence.

Definition 2.3. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough statistically convergent in probability to a random variable $X : W \to \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r-$ statistically convergent in probability to l) if for each $\in, \delta > 0$,

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \left| \left\{ m \leq u,\, n \leq v,\, k \leq w : P\big(|X_{mnk} - l| \geq r + \in \big) \geq \delta \right\} \right| = 0,$$

or, equivalently,

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \left| \left\{ m \leq u,\, n \leq v,\, k \leq w : 1 - P\big(|X_{mnk} - l| < r + \in \big) \geq \delta \right\} \right| = 0,$$

and we write $X_{mnk} \to_r^{S^P} l$. The class of all $r-$ statistically convergent triple squences of random variables in probability will be simply denoted by rS^P.

Theorem 2.4. *If $X_{mnk} \to_r^{S^P} l_1$ and $Y_{mnk} \to_r^{S^P} l_2$ then $P\{|l_1 - l_2| \geq r\} = 0$.*

Proof. Let ϵ, δ be any two positive real numbers and let

$$(u,v,w) \in \left\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l_1| \geq r + \frac{\epsilon}{2} \Big) < \frac{\delta}{2} \right\}$$
$$\bigcap \left\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l_2| \geq r + \frac{\epsilon}{2} \Big) < \frac{\delta}{2} \right\} \text{(existence of } (u,v,w)$$

is guaranteed since asymptotic density of both the sets is equal to 1. Then

$$P\Big(\big| l_1 - l_2 \big| \geq r + \epsilon \Big) \leq P\left(\big| X_{mnk} - l_1 \big| \geq r + \frac{\epsilon}{2} \right) + P\left(\big| X_{mnk} - l_2 \big| \geq r + \frac{\epsilon}{2} \right) < \delta.$$

This implies $P(|l_1 - l_2| \geq r) = 0$. ∎

Remark 2.5.

(i) If $X_{mnk} \to_r^{S^P} l_1$ and $X_{mnk} \to_r^{S^P} l_2$, then $P\{l_1 = l_2\} = 1$ (here $r = 0$).

(ii) If $X_{mnk} \to_r^{S^P} l_1$ and $X_{mnk} \to_r^{S^P} l_2$, then $\{P\{l_1 - l_2\} < r\} = 1$.

Definition 2.6. A discrete random variable X is said to be one-point distribution at the point c if the spectrum consists of a single point c and $P(X = c) = 1$. Here c is a parameter of the one point distribution.

Theorem 2.7. *If a triple sequence of constants $x_{mnk} \to_r^{S} l$ then $x_{mnk} \to_r^{S^P} l$.*

Proof. Here for every $(u,v,w), x_{mnk}$ can be regarded as a random variable with one element X_{mnk} in the corresponding spectrum. Let $\in$ be a positive real number. Since $x_{mnk} \to_r^{S} l$ then

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} \left| \left\{ m \leq u, n \leq v, k \leq w : \big| X_{mnk} - l \big| \geq r + \in \right\} \right| = 0,$$

$$\Longrightarrow \lim_{uvw\to\infty} \frac{1}{(uvw)} \left| \left\{ m \leq u, n \leq v, k \leq w : \big| X_{mnk} - l \big| < r + \in \right\} \right| = 1.$$

Now the event $\{w : w \in W \text{ } and \text{ } |X_{mnk}(w) - l(w)| < r+ \in\}$ is the same as the event $|x_{mnk} - l| < r+ \in$ which is here the certain event W for all

$$(u,v,w) \in \Big\{ (m,n,k) \in \mathbb{N}^3 : \big|x_{mnk} - l\big| < r+ \in \Big\}.$$

So $P\big(\{w : w \in W \text{ and } |X_{mnk}(w) - l(w)| < r+ \in \}\big) = P(|x_{mnk} - l| < r+ \in) = P(W) = 1$ for all $(u,v,w) \in \{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| < r+ \in\}$. Thus for any $\delta > 0$,

$$\Big\{(u,v,w) \in \mathbb{N}^3 : 1 - P\big(|x_{uvw} - l| < r+ \in \big)\Big\} \subset \mathbb{N}^3 \backslash \Big\{m \le u,\, n \le v,\, k \le w : |x_{mnk} - l| < r+ \in \Big\}$$

$$= \Big\{m \le u,\, n \le v,\, k \le w : \big|x_{mnk} - l\big| \ge r+ \in \Big\}.$$

In general converse is not true, i.e., if a triple sequence of random variables (x_{mnk}) is a rough statistical convergence in probability to a real number l then each of X_{mnk} may not have one point distribution so each X_{mnk} can not be treated as a constant which is rough statistical convergence to l i.e., rough statistical convergence in probability is the more generalized concept than usual rough statistical convergence. □

Example. Let a triple sequence of random variables (X_{mnk}) be defined by,

$$X_{mnk} \in \begin{cases} \{-10, 10\} & \text{with probability } P(X_{mnk} = -10) = P(X_{mnk} = 10), \\ & \text{if } (m,n,k) = (u,v,w)^2 \text{ for some } (u,v,w) \in \mathbb{N}^3 \\ \{0, 1\} & \text{with probability } P(X_{mnk} = 0) = P(X_{mnk} = 1), \\ & \text{if } (m,n,k) \ne (u,v,w)^2 \text{ for any } (u,v,w) \in \mathbb{N}^3. \end{cases}$$

Let $0 < \in < 1$ be given. Then

$$P\Big(\big|X_{mnk} - 1\big| \ge 2+ \in \Big) = \begin{cases} 1 & \text{if } (m,n,k) = (u,v,w)^2 \text{ for some } (u,v,w) \in \mathbb{N}^3 \\ 0 & \text{if } (m,n,k) \ne (u,v,w)^2 \text{ for any } (u,v,w) \in \mathbb{N}^3. \end{cases}$$

This implies $X_{mnk} \to_2^{S^P} 1$. But it is not ordinary rough statistical convergence of a triple sequence of numbers to 1.

Theorem 2.8.

(i) $X_{mnk} \to_r^{S^P} l \Longleftrightarrow X_{mnk} - l \to_r^{S^P} 0$,
(ii) $X_{mnk} \to_r^{S^P} l \Longrightarrow cX_{mnk} \to_{|c|r}^{S^P} c$ *where* $c \in \mathbb{R}$,
(iii) $X_{mnk} \to_r^{S^P} l_1$ *and* $Y_{mnk} \to_r^{S^P} l_2 \Longrightarrow X_{mnk} + Y_{mnk} \to_r^{S^P} l_1 + l_2$,
(iv) $X_{mnk} \to_r^{S^P} l_1$ *and* $Y_{mnk} \to_r^{S^P} l_2 \Longrightarrow X_{mnk} - Y_{mnk} \to_r^{S^P} l_1 - l_2$,
(v) $X_{mnk} \to_r^{S^P} 0 \Longrightarrow X^2_{mnk} \to_{r^2}^{S^P} 0$,
(vi) $X_{mnk} \to_r^{S^P} l \Longrightarrow X^2_{mnk} \to_{r^2+2|x|r}^{S^P} l^2$,
(vii) $X_{mnk} \to_r^{S^P} l_1$ *and* $Y_{mnk} \to_r^{S^P} l_2 \Longrightarrow X_{mnk} \cdot Y_{mnk} \to_{\frac{r}{2} + \frac{r(|l_1+l_2|+|l_1-l_2|)}{2}}^{S^P} l_1 \cdot l_2$,
(viii) *If* $0 \le X_{mnk} \le Y_{mnk}$ *and* $Y_{mnk} \to_r^{S^P} 0 \Longrightarrow X_{mnk} \to_r^{S^P} 0$,
(ix) $X_{mnk} \to_r^{S^P} l$, *then for each* $\in > 0$ *there exists* $(uvw) \in \mathbb{N}^3$ *such that for any* $\delta > 0$

$$\lim_{uvw \to \infty} \frac{1}{(uvw)} \left| \Big\{ m \le u,\ n \le v,\ k \le w : P\big(|X_{mnk} - l_{uvw}| \ge 2r+ \in \big) \ge \delta \Big\} \right| = 0,$$

which is called rough statistical Cauchy condition in probability.

Proof. Let $\in, \delta$ be any positive real numbers. Then for

(i) proof is straight forward hence omitted.
(ii) If $c = 0$ then the claim is obvious. So suppose assuming $c \ne 0$, then

$$\Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|c\, X_{mnk} - c\, l| \ge |c|\, r + \epsilon \Big) \ge \delta \Big\}$$
$$= \Big\{ (m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l| \ge r + \frac{\epsilon}{|c|} \Big) \ge \delta \Big\}.$$

Hence, $cX_{mnk} \to^{S^P}_{|c|r} cX$.

(iii) $P\Big(\Big|(X_{mnk}+Y_{mnk})-(l_1+l_2)\Big| \geq r+\epsilon\Big) = P\Big(\Big|(X_{mnk}-l_1)+(Y_{mnk}-l_2)\Big| \geq r+\epsilon\Big) \leq P\Big(\big|X_{mnk}-l_1\big| \geq r+\frac{\epsilon}{2}\Big) + P\Big(|Y_{mnk}-l_2| \geq r+\frac{\epsilon}{2}\Big)$.

This implies

$$\Big\{(m,n,k)\in\mathbb{N}^3 : P\Big(\big|(X_{mnk}+Y_{mnk})-(l_1+l_2)\big| \geq r+\epsilon\Big) \geq \delta\Big\} \subseteq$$
$$\Big\{(m,n,k)\in\mathbb{N}^3 : P\Big(|(X_{mnk}-l_1)| \geq r+\frac{\epsilon}{2}\Big) \geq \frac{\delta}{2}\Big\} \bigcup$$
$$\Big\{(m,n,k)\in\mathbb{N}^3 : P\Big(|(Y_{mnk}-l_2)| \geq r+\frac{\epsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}.$$

Hence $X_{mnk}+Y_{mnk} \to^{S^P}_{r} l_1+l_2$.

(iv) Similar to the proof of (iii) and therefore omitted.

(v) $\Big\{(m,n,k)\in\mathbb{N}^3 : P\Big(\big|X^2_{mnk}\big| \geq r^2+\delta\Big)\Big\} = \Big\{(m,n,k)\in\mathbb{N}^3 : P\Big(\big|X^2_{mnk}\big| \geq r^2+2r\eta+\eta^2\Big)\Big\}$ (where $\eta = -r+\sqrt{r^2+\delta} > 0$) $= \Big\{(m,n,k)\in\mathbb{N}^3 : P\Big(\big|X^2_{mnk}\big| \geq r+\eta\Big)\Big\}$. Hence, $X^2_{mnk} \to^{S^P}_{r^2} 0$.

(vi) $X^2_{mnk} = (X_{mnk}-l)^2 + 2l(X_{mnk}-l) + l^2$, so $X^2_{mnk} \to^{S^P}_{2r^2+2|l|r} l^2$.

(vii) $(X_{mnk}+Y_{mnk})^2 \to^{S^P}_{r^2+2r|l_1+l_2|} (l_1+l_2)^2$ and $(X_{mnk}-Y_{mnk})^2 \to^{S^P}_{r^2+2r|l_1-l_2|} (l_1-l_2)^2$ $\Longrightarrow X_{mnk}\cdot Y_{mnk} = \frac{1}{4}\Big\{\Big\{(X_{mnk}+Y_{mnk})^2 - (X_{mnk}-Y_{mnk})^2\Big\}\Big\} \to^{S^P}{}_{\frac{r^2}{2}+\frac{r(|l_1+l_2|+|l_1-l_2|)}{2}}$ $\frac{1}{4}\Big\{(l_1+l_2)^2-(l_1-l_2)^2\Big\} = l_1\cdot l_2$.

(viii) Proof is straight forward hence omitted.

(ix) Choose $(u,v,w)\in\mathbb{N}^3$ be such that $P\left(|X_{uvw}-X| \geq r+\frac{\in}{2}\right) < \frac{\delta}{2}$. Then the claim is obvious from the inequality

$$P\Big(|X_{mnk}-X_{uvw}| \geq 2r+\in\Big) \leq P\Big(|X_{mnk}-X| \geq r+\frac{\in}{2}\Big) + P\Big(|X_{uvw}-X| \geq r+\frac{\in}{2}\Big)$$
$$\leq \frac{\delta}{2} + P\Big(|X_{mnk}-X| \geq r+\frac{\in}{2}\Big).$$
□

Theorem 2.9. *Let (X_{mnk}) be a triple sequence of random variables then there exists a triple sequence of real numbers (x_{mnk}) with the property that $X_{mnk}-x_{mnk} \to^{S^P}_{r} 0$. If $m(X_{mnk})$ is a median of X_{mnk} then $X_{mnk}-m(X_{mnk}) \to^{S^P}_{r} 0$ and $x_{mnk}-m(X_{mnk}) \to^{S^P}_{r} 0$.*

Proof. Proof is straight forward hence omitted. □

Theorem 2.10. *Let $r>0$. Then $X_{mnk} \to^{S^P}_{r} l \Longleftrightarrow$ there exists a triple sequence of random variables (Y_{mnk}) such that $Y_{mnk} \to^{S^P}_{r} l$ and $S-\lim_{mnk\to\infty} P(|X_{mnk}-Y_{mnk}| > r) = 0$.*

Proof. Let $X_{mnk} \to^{S^P}_{r} l$ and $A = \big\{(m,n,k)\in\mathbb{N}^3 : P(|X_{mnk}-l| \geq r+\epsilon) \geq \delta\big\}$. Then $\delta(A) = 0$.

Now we define

$$Y_{mnk} = \begin{cases} l & \text{if } (m,n,k)\in\mathbb{N}^3\backslash A \\ X_{mnk}+Z & \text{otherwise} \end{cases},$$

where X is a random variable and $Z\in(-r,r)$ with probability $P(X_{mnk}=-r) = P(X_{mnk}=r)$. Then it is very obvious that

$$d\Big(\Big\{(m,n,k)\in\mathbb{N}^3 : P\big(|Y_{mnk}-l| \geq \in\big) \geq \delta\Big\}\Big) = 0 \text{ and}$$
$$d\Big(\Big\{(m,n,k)\in\mathbb{N}^3 : P\big(|X_{mnk}-Y_{mnk}| \geq r+\in\big) \geq \delta\Big\}\Big)$$
$$\leq d\Big(\Big\{(m,n,k)\in\mathbb{N}^3 : P\Big(|X_{mnk}-l| \geq r+\frac{\in}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big)$$
$$+\, d\Big(\Big\{(m,n,k)\in\mathbb{N}^3 : P\Big(|Y_{mnk}-l| \geq \frac{\in}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big) = 0.$$

Conversely, let $Y_{mnk} \rightarrow_r^{S^P} l$ and $S - \lim_{mnk\rightarrow\infty} P\big(|X_{mnk} - Y_{mnk}| > r\big) = 0$. Then for each $\in, \delta > 0$,

$$\lim_{uvw\rightarrow\infty} \frac{1}{(uvw)} \Big|\Big\{m \leq u, n \leq v, k \leq w : P\Big(|Y_{mnk} - l| \geq \frac{\in}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| = 0 \text{ and}$$

$$\lim_{uvw\rightarrow\infty} \frac{1}{(uvw)} \Big|\Big\{m \leq u, n \leq v, k \leq w : P\Big(|X_{mnk} - Y_{mnk}| \geq r + \frac{\in}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| = 0.$$

We know the inequality

$$P\Big(|X_{mnk} - l| \geq r+ \in\Big) \leq P\Big(|Y_{mnk} - l| > \frac{\in}{2}\Big) + P\Big(|X_{mnk} - Y_{mnk}| \geq r + \frac{\in}{2}\Big).$$

$$\Longrightarrow \Big\{(m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - l| \geq r+ \in\Big) \geq \delta\Big\}$$

$$\subseteq \Big\{(m,n,k) \in \mathbb{N}^3 : P\Big(|Y_{mnk} - l| \geq \frac{\in}{2}\Big) \geq \frac{\delta}{2}\Big\} \bigcup$$

$$\Big\{(m,n,k) \in \mathbb{N}^3 : P\Big(|X_{mnk} - Y_{mnk}| \geq r + \frac{\in}{2}\Big) \geq \frac{\delta}{2}\Big\}.$$

Hence $X_{mnk} \rightarrow_r^{S^P} l$. □

Theorem 2.11. *If $X_{mnk} \rightarrow_r^{S^P} l$ and $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a continuous function on $\mathbb{R}^3$, then there exists a triple sequence of random variables (Y_{mnk}) such that $g(Y_{mnk}) \rightarrow_r^{S^P} g(l)$ and $g\big(P(|X_{mnk} - Y_{mnk}| > r)\big) \rightarrow_r^{S^P} 0$.*

Proof. The proof is similar to Theorem 2.4 in [7] and hence omitted. □

3. Strong CesÀro Summable of a Triple Sequence of Real Numbers

Definition 3.1. A triple sequence (x_{mnk}) is said to be strong Cesàro summable to l if

$$\lim_{uvw\rightarrow\infty} \frac{1}{(uvw)} \sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} |x_{mnk} - l| = 0.$$

In this case we write $x_{mnk} \rightarrow^{[C,1,1]} l$. The set of all strong Cesàro summable triple squences is denoted by either $|C,1|$ or $|C,1,1|$.

Definition 3.2. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough strong Cesàro summable in probability to a random variable $X : W \rightarrow \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r-$ strong Cesàro summable in probability to l) of for each $\epsilon > 0$,

$$\lim_{uvw\rightarrow\infty} \frac{1}{(uvw)} \sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P\left(|X_{mnk} - l| \geq r + \epsilon\right) = 0.$$

In this case we write $X_{mnk} \rightarrow_r^{[C,1,1]^P} l$.

The class of all $r-$ strong Cesàro summable triple sequences of random variables in probability will be simply denoted by $r\,[C,1,1]^P$.

Theorem 3.3. *The followings are equivalent:* (i) $X_{mnk} \rightarrow_r^{S^P} l$ (ii) $X_{mnk} \rightarrow_r^{[C,1,1]^P} l$.

Proof. (i) $\Longrightarrow$ (ii). First suppose that $X_{mnk} \rightarrow_r^{S^P} l$. Then we can write

$$\frac{1}{(uvw)} \sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P\big(|X_{mnk} - l| \geq r+ \in\big)$$

$$= \frac{1}{(uvw)} \sum_{m=1}^{u}\sum_{n=1}^{v} \sum_{k=1, P(|X_{mnk}-l|\geq r+\epsilon)\geq\frac{\delta}{2}}^{w} P\big(|X_{mnk} - l| \geq r+ \in\big)$$

$$+ \frac{1}{(uvw)} \sum_{m=1}^{u}\sum_{n=1}^{v} \sum_{k=1, P(|X_{mnk}-l|\geq r+\epsilon)<\frac{\delta}{2}}^{w} P\big(|X_{mnk} - l| \geq r+ \in\big)$$

$$\leq \frac{1}{(uvw)}\Big|\Big\{m \leq u,\ n \leq v,\ k \leq w : P\Big(\,|X_{mnk} - l| \geq r + \epsilon\Big) > \frac{\delta}{2}\Big\}\Big| + \frac{\delta}{2}.$$

(ii) $\Longrightarrow$ (i). Next suppose that condition (ii) holds. Then

$$\begin{aligned}&\sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P\big(\,|X_{mnk} - l| \geq r + \epsilon\big)\\ &\geq \sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1, P(|X_{mnk}-l| \geq r+\epsilon) \geq \delta}^{w} P\big(\,|X_{mnk} - l| \geq r + \in\big)\\ &\geq \delta\Big|\Big\{m \leq u,\ n \leq v,\ k \leq w : P\big(\,|X_{mnk} - l| \geq r + \in\big) > \delta\Big\}\Big|.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{(uvw)}\sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P\big(\,|X_{mnk} - l| \geq r + \in\big)\\ &\geq \frac{1}{(uvw)}\Big|\Big\{m \leq u, n \leq v, k \leq w : P\big(\,|X_{mnk} - l| \geq r + \in\big) > \delta\Big\}\Big|.\end{aligned}$$

Hence $X_{mnk} \rightarrow_{r}^{S^P} l$. ◻

4. Triple Rough Lacunary Statistical Convergence in Probability

Definition 4.1. The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$\begin{aligned} m_0 &= 0,\ ]; h_i = m_i - m_{r-1} \rightarrow \infty \text{ as } i \rightarrow \infty \text{ and}\\ n_0 &= 0,\ \overline{h_\ell} = n_\ell - n_{\ell-1} \rightarrow \infty \text{ as } \ell \rightarrow \infty,\\ k_0 &= 0,\ \overline{h_j} = k_j - k_{j-1} \rightarrow \infty \text{ as } j \rightarrow \infty.\end{aligned}$$

Let $m_{i,\ell,j} = m_i n_\ell k_j$, $h_{i,\ell,j} = h_i \overline{h_\ell h_j}$ and $\theta_{i,\ell,j}$ is determine by $I_{i,\ell,j} = \big\{\,(m,n,k) : m_{i-1} < m < m_i$ and $n_{\ell-1} < n \leq n_\ell$ and $k_{j-1} < k \leq k_j\big\}$, $q_i = \frac{m_i}{m_{i-1}}$, $\overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}$, $\overline{q_j} = \frac{k_j}{k_{j-1}}$.

Definition 4.2. Let $\theta = \{m_r n_s k_t\}_{(rst) \in \mathbb{N} \bigcup 0}$ be the triple lacunary sequence. A number triple sequence (X_{mnk}) is said to be triple lacunary statistically convergent to a real number l (or shortly: $S_\theta-$ convergent to l) if for any $\in > 0$,

$$\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}}\Big|\Big\{\,(m,n,k) \in I_{rst} : |X_{mnk} - l| \geq \epsilon\Big\}\Big| = 0$$

and it is denoted by $X_{mnk} \rightarrow^{S_\theta} l$, where $I_{r,s,t} = \big\{\,(m,n,k) : m_{r-1} < m < m_r$ and $n_{s-1} < n \leq n_s$ and $k_{t-1} < k \leq k_t\big\}$, $q_r = \frac{m_r}{m_{r-1}}$, $\overline{q_s} = \frac{n_s}{n_{s-1}}$, $\overline{q_t} = \frac{k_t}{k_{t-1}}$.

Definition 4.3. Let $\theta = \{m_r n_s k_t\}$ be the triple lacunary sequence. A number triple seqeunce (x_{mnk}) is said to be $N_\theta-$ convergent to a real number l if for any $\in > 0$,

$$\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} \sum_{m \in I_r}\sum_{n \in I_s}\sum_{k \in I_t} |x_{mnk} - l| = 0.$$

In this case we write $x_{mnk} \rightarrow^{N_\theta} l$.

Definition 4.4. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough lacunary statistically convergent in probability to $X : W \rightarrow \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r-$ lacunary statistically convergent in probability to l) if for any $\epsilon, \delta > 0$

$$\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}}\Big|\Big\{\,(m,n,k) \in I_{rst} : P\big(\big|X_{mnk} - l\big| \geq r + \epsilon\big) \geq \delta\Big\}\Big| = 0,$$

and we write $X_{mnk} \rightarrow_{r}^{S_\theta^P} l$. The class of all $r-$ triple lacunary statistically convergent sequences of random variables in probability will be denoted simply by rS_θ^P.

Definition 4.5. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough $N_\theta-$ convergent in probability to $X : W \to \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r - N_\theta-$ convergent in probability to l) if for any $\epsilon > 0$,

$$\lim_{rst\to\infty} \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} P\big(|x_{mnk} - l| \geq r + \epsilon\big) = 0$$

and we write $X_{mnk} \to_r^{N_\theta^P} l$. The class of all $r - N_\theta-$ convergent triple sequence of random variables in probability will be denoted simply by rN_θ^P.

Theorem 4.6. *Let $\theta = \{m_r, n_s, k_t\}$ be a triple lacunary sequence. Then the followings are equivalent:*
(i) *(X_{mnk}) is a $r-$ triple lacunary statistically convergent in probability to l.*
(ii) *(X_{mnk}) is $r - N_\theta-$ convergent in probability to l.*

Proof. (i) $\implies$ (ii) First suppose that $X_{mnk} \to_r^{S_\theta^P} l$. Then we can write

$$\begin{aligned} &\frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\ &= \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t, P(|x_{mnk}-l|\geq r+\epsilon)\geq \frac{\delta}{2}} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\ &+ \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t, P(|x_{mnk}-l|\geq r+\epsilon)< \frac{\delta}{2}} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\ &\leq \frac{1}{h_{rst}} \Big| \Big\{ (m,n,k) \in I_{rst} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \frac{\delta}{2} \Big\} \Big|. \end{aligned}$$

(ii) $\implies$ (i) Next suppose that condition (ii) holds. Then

$$\begin{aligned} &\sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\ &\geq \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t, P(|x_{mnk}-l|\geq r+\epsilon)\geq \delta} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\ &\geq \delta \Big| \Big\{ (m,n,k) \in I_{rst} : P\big(|X_{mnk} - l| \geq r + \epsilon\big) \geq \delta \Big\} \Big|. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{\delta\, h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} P\big(|x_{mnk} - l| \geq r + \epsilon\big) \\ &\geq \frac{1}{h_{rst}} \Big| \Big\{ (m,n,k) \in I_{rst} : P\big(|x_{mnk} - l| \geq r + \epsilon\big) \Big\} \Big|. \end{aligned}$$

Hence $X_{mnk} \to_r^{S_\theta^P} l$. □

Theorem 4.7. *If $X_{mnk} \to_r^{S_\theta^P} l_1$ and $X_{mnk} \to_r^{S_\theta^P} l_2$ then $P\left(|l_1 - l_2| \geq r\right) = 0$.*

Proof. Similar to the proof of the Theorem 2.4 and therefore omitted. □

5. Triple Rough-$\lambda-$statistical Convergence in Probability

Let $\lambda = (\lambda_{uvw})$ be a non-decreasing triple sequence of positive numbers tending to ∞ such that $\lambda_{uvw+1} \leq \lambda_{uvw} + 1$, $\lambda_{111} = 1$. The collection of all such triple sequence λ is denoted by $\mathbb{D}$.

The generalized De la valeé$-$ Pousin mean is defined for the triple sequence (x_{mnk}) of real numbers by $t_{uvw}(x) = \frac{1}{\lambda_{uvw}} \sum_{(m,n,k)\in Q_{uvw}} x_{mnk}$, where $Q_{uvw} = [(uvw) - \lambda_{uvw} + 1, (uvw)]$. A triple sequence (x_{mnk}) of real numbers is said to be $[V, \lambda]-$ summable to l, if $\lim t_{uvw}(x) = l$.

Definition 5.1. A triple sequence (x_{mnk}) is said to be strong $[V, \lambda]-$ summable (or shortly: $[V, \lambda]-$ convergent) to l, if $\lim_{uvw\to\infty} \frac{1}{\lambda_{uvw}} \sum_{(m,n,k)\in Q_{uvw}} |x_{mnk} - l| = 0$. In this case we write $x_{mnk} \to^{[V,\lambda]} l$.

Definition 5.2. A triple sequence (x_{mnk}) is said to be $\lambda-$ statistically convergent (or shortly: $S_\lambda-$ convergent) to l if for any $\epsilon > 0$,

$$\lim_{uvw\to\infty} \frac{1}{\lambda_{uvw}} \Big| \Big\{ (m,n,k) \in Q_{uvw} : |x_{mnk} - l| \geq \epsilon \Big\} \Big| = 0.$$

In this case we write $S_\lambda - \lim x_{mnk} = l$ or by $x_{mnk} \to^{S_\lambda} l$.

Definition 5.3. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough $[V,\lambda]-$ summable in probability to $X : W \to \mathbb{R}^3$ with respect to the roughness degree r (or shortly: $r - [V,\lambda] -$ summable in probability to l) if for any $\epsilon > 0$,

$$\lim_{uvw\to\infty} \frac{1}{\lambda_{uvw}} \sum_{(m,n,k)\in Q_{uvw}} : P\big(|X_{mnk} - l| \geq r + \epsilon \big) = 0.$$

In this case we write $X_{mnk} \to_r^{[V,\lambda]^P} l$. The class all rough $[V,\lambda]-$ summable sequences of random variables in probability will be denoted by $r\,[V,\lambda]^P$.

Definition 5.4. Let r be a non-negative real number. A triple sequence of random variables (X_{mnk}) is said to be rough $\lambda-$ statistically convergent in probability to $X : W \to \mathbb{R}^3$ with respect to the roughness degree r (or shortly: $r - \lambda-$ statistically convergent in probability to l) if for any $\epsilon, \delta > 0$,

$$\lim_{uvw\to\infty} \frac{1}{\lambda_{uvw}} \Big| \Big\{ (m,n,k) \in Q_{uvw} : P\left(|x_{mnk} - l| \geq r + \epsilon \right) \geq \delta \Big\} \Big| = 0.$$

In this case we write $X_{mnk} \to_r^{S_\lambda^P} l$. The class of all $r - \lambda-$ statistically convergent triple sequence of random variables in probability will be denoted simply by rS_λ^P.

Theorem 5.5. *For any triple sequence of random variables (X_{mnk}) the following are equivalent:*
(i) (X_{mnk}) *is* $r - [V,\lambda]-$ *summable in probability to* l.
(ii) (X_{mnk}) *is* $r - \lambda-$ *statistically convergent in probability to* l.

Proof. It can be established using the technique of Theorem 4.6, so omitted. □

Theorem 5.6. *If* $X_{mnk} \to_r^{S_\lambda^P} l_1$ *and* $X_{mnk} \to_r^{S_\lambda^P} l_2$ *then* $P\left(|l_1 - l_2| \geq r \right) = 0$.

Proof. It can be established using the technique of Theorem 2.4, so omitted. □

Theorem 5.7. *If* $\lambda \in \mathbb{D}$ *is such that* $\lim\left(\frac{\lambda_{uvw}}{uvw}\right) = 1$, *then* $rS_\lambda^P \subset rS^P$.

Proof. Let $0 < \eta < 1$ be given. Since $\lim\left(\frac{\lambda_{uvw}}{uvw}\right) = 1$, we can choose $(r,s,t) \in \mathbb{N}^3$ such that $\left| \frac{\lambda_{uvw}}{uvw} - 1 \right| < \frac{\eta}{2}$ for all $u \geq r$, $v \geq s$, $w \geq t$. Now observe that for $\epsilon, \delta > 0$

$$\begin{aligned}
&\frac{1}{(uvw)} \Big| \Big\{ m \leq u,\, n \leq v,\, k \leq w : P\big(|X_{mnk} - l| \geq r + \epsilon \big) \geq \delta \Big\} \Big| \\
&= \frac{1}{(uvw)} \Big| \Big\{ m \leq u, n \leq v, k \leq w - \lambda_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon \big) \geq \delta \Big\} \Big| \\
&+ \frac{1}{(uvw)} \Big| \Big\{ Big|\, (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon \big) \geq \delta \Big\} \Big| \\
&\leq \frac{(uvw) - \lambda_{uvw}}{(uvw)} + \frac{1}{(uvw)} \Big| \Big\{ (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon \big) \geq \delta \Big\} \Big| \\
&\leq 1 - \Big(1 - \frac{\eta}{2} \Big) + \frac{1}{(uvw)} \Big| \Big\{ (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon \big) \geq \delta \Big\} \Big| \\
&= \frac{\eta}{2} + \frac{\lambda_{uvw}}{(uvw)} \cdot \frac{1}{\lambda_{uvw}} \Big| \Big\{ (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon \big) \geq \delta \Big\} \Big| \\
&< \frac{\eta}{2} + \frac{2}{\lambda_{uvw}} \Big| \Big\{ (m,n,k) \in Q_{uvw} : P\big(|X_{mnk} - l| \geq r + \epsilon \big) \geq \delta \Big\} \Big|
\end{aligned}$$

hold for all $u \geq r$, $v \geq s$, $w \geq t$. □

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[1]Department of Mathematics SASTRA University, Thanjavur-613 401, India.
E-mail address: nsmaths@yahoo.com

[2]Department of Mathematics Adiyaman University, 02040, 5 Adiyaman, Turkey
E-mail address: aesi23@hotmail.com

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