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L. Giorgashvili, G. Karseladze,
G. Sadunishvili, and Sh. Zazashvili

THE BOUNDARY VALUE PROBLEMS OF STATIONARY OSCILLATIONS IN THE THEORY OF TWO-TEMPERATURE ELASTIC MIXTURES

Abstract. We derive Green's formulas for the system of differential equations of stationary oscillations in the theory of elastic mixtures, which enable us to prove the uniqueness theorems for solutions of the boundary value problems. The jump formulas for single and double-layer potentials are derived. Using the theories of potentials and integral equations the existence of solutions is proved.

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რეზიუმე. სტატიაში მიღებულია დრეკად ნარევთა თეორიის სტაციონალური რხევის დიფერენციალურ განტოლებათა სისტემისათვის გრინის ფორმულები, რომელთა დახმარებით დამტკიცებულია სასაზღვრო ამოცანების ამონახსნის ერთადერთობის თეორემები. მიღებულია მარტივი და ორმაგი ფენის პოტენციალების წყვეტის ფორმულები. პოტენციალთა და ინტეგრალურ განტოლებათა თეორიის გამოყენებით დამტკიცებულია ამოცანების ამონახსნის არსებობის თეორემები.

1. Introduction

Elastic composite materials with complex structures, as well as with structures composed of substantially differing materials are widely applied in the modern technological processes. Hemitropic elastic materials, mixtures produced from two or more elastic materials, etc., belong to the class of such composite materials and structures. The study of practical problems of mechanical properties of such materials naturally results in the necessity to develop mathematical models, which would allow to get more precise description of actual processes ongoing during the experiments. Mathematical modeling for such materials commenced as early as in the sixties of the past century. The first mathematical model of an elastic mixture (solid with solid), the so-called diffuse model, was developed by A. Green and T. Steel in 1966. In this model, the interaction force between components depends upon the difference of displacement vectors of components. In the same year they have developed the single-temperature thermoelasticity theory diffuse model of the elastic mixtures. Mathematical model of the linear theory of thermoelasticity of two-temperature elastic mixtures for the composites of granular, fibrous and layered structures was developed in 1984 by L. Khoroshun and N. Soltanov. Normally, the study of processes ongoing in the body is reduced, in the relevant mathematical model described by the system of differential equations with partial derivatives, to the study of boundary value problems (BVPs), mixed type BVPs and boundary-contact problems, and also the fundamental matrix for solving the system of differential equations playing a substantial role. For the diffuse and displacement models of the two-component mixtures (single-temperature) thermoelasticity theory, the issue of steadiness and correctness, identification of the asymptotic behavior of problem solution, proving of the uniqueness and existence theorems, solution of the BVPs for the domains bounded by the specific surfaces, as absolutely and uniformly convergent series, are studied by many scientists, among them: Alves, Munoz Rivera, Quintanilla [2], Basheleishvili [3], Basheleishvili, Zazashvili [4], Burchuladze, Svanadze [6], Gales [9], Giorgashvili, Skhvitaridze [13], [12], Giorgashvili, Karseladze, Sadunishvili [11], Iesan [18], Nappa [29], Natroshvili, Jaghmaidze, Svanadze [36], Svanadze [42], Quintanilla [41], Pompei [40], etc.

In this paper we derive Green's formulas for the system of differential equations of stationary oscillations in the theory of elastic mixtures, which enable us to prove the uniqueness theorems for solutions of the boundary value problems. Further, we establish mapping properties and jump formulas for the single and double-layer potentials, and analyse the Fredholm properties of the corresponding boundary operators. Using the potential method and the theory of singular integral equations, the existence of solutions to the basic boundary value problems is proved.

We treat here only the classical setting of basic boundary value problems for smooth domains, however applying the results obtained in the references: Agranovich [1], Buchukuri, Chkadua, Duduchava, Natroshvili [5], Duduchava, Natroshvili [8], Gao [10], Jentsch, Natroshvili [19–21], Jentsch, Natroshvili, Wendland [22, 23], Kupradze, Gegelia, Basheleishvili, Burchuladze [25], Mitrea, Mitrea, Pipher [28], Natroshvili [30–32], Natroshvili, Giorgashvili, Stratis [33], Natroshvili, Giorgashvili, Zazashvili [34], Natroshvili, Kharibegashvili, Tediashvili [37], Natroshvili, Sadunishvili [38], Natroshvili, Stratis [39], and using the same type approaches and reasonings, one can analyze the generalized basic and mixed type boundary value problems, as well as crack type and interface problems in Sobolev–Slobodetskii and Bessel potential spaces for smooth and Lipschitz domains.

2. Basic Differential Equations

The basic dynamical relationships for the two-component elastic mixtures, taking two-temperature thermal field into consideration, are mathematically described by the following system of partial differential equations [24]

$$\begin{gathered}
a_1\Delta u'(x,t)+b_1\,\mathrm{grad}\,\mathrm{div}\,u'(x,t)+c\Delta u''(x,t)+\\
+d\,\mathrm{grad}\,\mathrm{div}\,u''(x,t)-\varkappa\big[u'(x,t)-u''(x,t)\big]-\\
-\eta_1\,\mathrm{grad}\,\vartheta_1(x,t)-\eta_2\,\mathrm{grad}\,\vartheta_2(x,t)+\rho_1F'(x,t)=\rho_1\partial^2_{tt}u'(x,t),\\
c\Delta u'(x,t)+d\,\mathrm{grad}\,\mathrm{div}\,u'(x,t)+a_2\Delta u''(x,t)+\\
+b_2\,\mathrm{grad}\,\mathrm{div}\,u''(x,t)+\varkappa\big[u'(x,t)-u''(x,t)\big]-\\
-\zeta_1\,\mathrm{grad}\,\vartheta_1(x,t)-\zeta_2\,\mathrm{grad}\,\vartheta_2(x,t)+\rho_2F''(x,t)=\rho_2\partial^2_{tt}u''(x,t),\\
\varkappa_1\Delta\vartheta_1(x,t)+\varkappa_2\Delta\vartheta_2(x,t)-\alpha\big[\vartheta_1(x,t)-\vartheta_2(x,t)\big]-\\
-\eta_1\,\mathrm{div}\,\partial_tu'(x,t)-\zeta_1\,\mathrm{div}\,\partial_tu''(x,t)+G'(x,t)=\varkappa'\partial_t\vartheta_1(x,t),\\
\varkappa_2\Delta\vartheta_1(x,t)+\varkappa_3\Delta\vartheta_2(x,t)+\alpha\big[\vartheta_1(x,t)-\vartheta_2(x,t)\big]-\\
-\eta_2\,\mathrm{div}\,\partial_tu'(x,t)-\zeta_2\,\mathrm{div}\,\partial_tu''(x,t)+G''(x,t)=\varkappa''\partial_t\vartheta_2(x,t),
\end{gathered}\tag{2.1}$$

where Δ is the three-dimensional Laplace operator, $u'=(u'_1,u'_2,u'_3)^\top$, $u''=(u''_1,u''_2,u''_3)^\top$ are partial displacement vectors, ϑ_1 and ϑ_2 are temperatures of each component of the mixture, $F'=(F'_1,F'_2,F'_3)^\top$, $F''=(F''_1,F''_2,F''_3)^\top$ are the mass forces, G', G'' are the thermal sources located in the components, a_j, b_j, c, d are the elasticity coefficients, $\varkappa$, η_j, ζ_j, $\varkappa_j$, $\varkappa_3$, $\varkappa'$, $\varkappa''$, α, $j=1,2$, are the mechanical and thermal constants of the elastic mixture, ρ_1, ρ_2 are the densities of mixture components, t is a time variable, $x=(x_1,x_2,x_3)$ is a point in the three-dimensional Cartesian space, $\top$ denotes transposition.

In the system (2.1), a_j, b_j, c, d, $j=1,2$, are the constants given as follows [15, 17]

$$a_1=\mu_1-\lambda_5,\ \ b_1=\mu_1+\lambda_5+\lambda_1-\frac{\rho_2}{\rho}\,\alpha_0,$$

$$a_2 = \mu_2 - \lambda_5, \;\; b_2 = \mu_2 + \lambda_5 + \lambda_2 + \frac{\rho_1}{\rho}\,\alpha_0,$$
$$c = \mu_3 + \lambda_5, \;\; d = \mu_3 - \lambda_5 + \lambda_3 - \frac{\rho_1}{\rho}\,\alpha_0, \;\; \alpha_0 = \lambda_3 - \lambda_4, \;\; \rho = \rho_1 + \rho_2,$$

where $\lambda_1,\ \lambda_2, \dots, \lambda_5,\ \mu_1,\ \mu_2,\ \mu_3$ are elastic constants satisfying the conditions

$$\mu_1 > 0, \;\; \lambda_5 < 0, \;\; \mu_1\mu_2 - \mu_3^2 > 0, \;\; \lambda_1 + \frac{2}{3}\,\mu_1 - \frac{\rho_2}{\rho}\,\alpha_0 > 0,$$
$$\Big(\lambda_1 + \frac{2}{3}\,\mu_1 - \frac{\rho_2}{\rho}\,\alpha_0\Big)\Big(\lambda_2 + \frac{2}{3}\,\mu_2 - \frac{\rho_1}{\rho}\,\alpha_0\Big) > \Big(\lambda_3 + \frac{2}{3}\,\mu_3 - \frac{\rho_1}{\rho}\,\alpha_0\Big)^2.$$

From these inequalities it follows that

$$\begin{gathered} a_1 > 0, \quad a_1 + b_1 > 0, \\ d_1 := a_1a_2 - c^2 > 0, \quad d_2 := (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0. \end{gathered} \tag{2.2}$$

In addition, from physical considerations it follows that

$$\begin{gathered} \rho_1 > 0, \;\; \rho_2 > 0, \;\; \alpha > 0, \;\; \varkappa > 0, \;\; \varkappa' > 0, \;\; \varkappa'' > 0, \\ \varkappa_j > 0, \; j = 1, 2, 3, \quad d_3 := \varkappa_1\varkappa_3 - \varkappa_2^2 > 0. \end{gathered} \tag{2.3}$$

If all the functions involved in the system (2.1) are harmonic time dependent, i.e., $u'(x,t) = u'(x)\exp(-i\sigma t)$, $u''(x,t) = u''(x)\exp(-i\sigma t)$, $\vartheta_1(x,t) = \vartheta_1(x)\exp(-i\sigma t)$, $\vartheta_2(x,t) = \vartheta_2(x)\exp(-i\sigma t)$, $F'(x,t) = F'(x)\exp(-i\sigma t)$, $F''(x,t) = F''(x)\exp(-i\sigma t)$, $G'(x,t) = G'(x)\exp(-i\sigma t)$, $G''(x,t) = G''(x)\exp(-i\sigma t)$, where $\sigma \in \mathbb{R}$ is oscillation frequency, $i = \sqrt{-1}$, then from the system (2.1) we obtain the following system of differential equations of the theory of stationary oscillations of two-temperature elastic mixture:

$$\begin{gathered} a_1\Delta u'(x) + b_1 \operatorname{grad}\operatorname{div} u'(x) + c\Delta u''(x) + d \operatorname{grad}\operatorname{div} u''(x) - \\ -\varkappa\big[u'(x) - u''(x)\big] - \eta_1 \operatorname{grad}\vartheta_1(x) - \eta_2 \operatorname{grad}\vartheta_2(x) + \\ +\rho_1\sigma^2 u'(x) = -\rho_1 F'(x), \\ c\Delta u'(x) + d \operatorname{grad}\operatorname{div} u'(x) + a_2\Delta u''(x) + b_2 \operatorname{grad}\operatorname{div} u''(x) + \\ +\varkappa\big[u'(x) - u''(x)\big] - \zeta_1 \operatorname{grad}\vartheta_1(x) - \zeta_2 \operatorname{grad}\vartheta_2(x) + \\ +\rho_2\sigma^2 u''(x) = -\rho_2 F''(x), \\ \varkappa_1\Delta\vartheta_1(x) + \varkappa_2\Delta\vartheta_2(x) - \alpha\big[\vartheta_1(x) - \vartheta_2(x)\big] + i\sigma\eta_1 \operatorname{div} u'(x) + \\ +i\sigma\zeta_1 \operatorname{div} u''(x) + i\sigma\varkappa'\vartheta_1(x) = -G'(x), \\ \varkappa_2\Delta\vartheta_1(x) + \varkappa_3\Delta\vartheta_2(x) + \alpha\big[\vartheta_1(x) - \vartheta_2(x)\big] + i\sigma\eta_2 \operatorname{div} u'(x) + \\ +i\sigma\zeta_2 \operatorname{div} u''(x) + i\sigma\varkappa''\vartheta_2(x) = -G''(x); \end{gathered} \tag{2.4}$$

here u', u'', F', F'' are the complex vector-functions and ϑ_1, ϑ_2, G', G'', are the complex scalar functions.

If $\sigma = \sigma_1 + i\sigma_2$ is a complex parameter and $\sigma_2 \neq 0$, then (2.4) is called the system of differential equations of pseudooscillations, and if $\sigma = 0$, then (2.4) is the system of differential equations of statics.

Let us introduce the matrix differential operator of order 8×8, generated by the left hand side expressions in system (2.4),

$$L(\partial,\sigma):=\begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma) & L^{(5)}(\partial,\sigma) & L^{(6)}(\partial,\sigma)\\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma) & L^{(7)}(\partial,\sigma) & L^{(8)}(\partial,\sigma)\\ L^{(9)}(\partial,\sigma) & L^{(10)}(\partial,\sigma) & L^{(13)}(\partial,\sigma) & L^{(14)}(\partial,\sigma)\\ L^{(11)}(\partial,\sigma) & L^{(12)}(\partial,\sigma) & L^{(15)}(\partial,\sigma) & L^{(16)}(\partial,\sigma)\end{bmatrix}_{8\times 8},$$

where

$$\begin{aligned} L^{(1)}(\partial,\sigma) &:= (a_1\Delta+\alpha')I_3+b_1Q(\partial),\\ L^{(2)}(\partial,\sigma)=L^{(3)}(\partial,\sigma) &:= (c\Delta+\varkappa)I_3+dQ(\partial),\\ L^{(4)}(\partial,\sigma) &:= (a_2\Delta+\alpha'')I_3+b_2Q(\partial),\\ L^{(4+j)}(\partial,\sigma) &:= -\eta_j\nabla^\top,\ L^{(6+j)}(\partial,\sigma)=-\zeta_j\nabla^\top,\ \ j=1,2,\\ L^{(9)}(\partial,\sigma) &:= i\sigma\eta_1\nabla,\ L^{(10)}(\partial,\sigma):=i\sigma\zeta_1\nabla,\\ L^{(11)}(\partial,\sigma) &:= i\sigma\eta_2\nabla,\ L^{(12)}(\partial,\sigma):=i\sigma\zeta_2\nabla,\\ L^{(13)}(\partial,\sigma) &:= \varkappa_1\Delta+\alpha_1,\ L^{(16)}(\partial,\sigma):=\varkappa_3\Delta+\alpha_2,\\ L^{(14)}(\partial,\sigma)=L^{(15)}(\partial,\sigma) &:= \varkappa_2\Delta+\alpha; \end{aligned}$$

here $\alpha'=-\varkappa+\rho_1\sigma^2$, $\alpha''=-\varkappa+\rho_2\sigma^2$ $\alpha_1=-\alpha+i\sigma\varkappa'$, $\alpha_2=-\alpha+i\sigma\varkappa''$, $\nabla\equiv\nabla(\partial):=[\partial_1,\partial_2,\partial_3]$, $\partial=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$, $j=1,2,3$, I_3 is the 3×3 unit matrix, $Q(\partial):=[\partial_k\partial_j]_{3\times 3}$.

Applying these notation, the system (2.4) can be written as

$$L(\partial,\sigma)U(x)=\Phi(x),$$

where $U=(u',u'',\vartheta_1,\vartheta_2)^\top$, $\Phi=(-\rho_1F',-\rho_2F'',-G',-G'')^\top$.

In what follows, we apply the following differential operators:

$$L_0(\partial):=\begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial) & [0]_{3\times 1} & [0]_{3\times 1}\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial) & [0]_{3\times 1} & [0]_{3\times 1}\\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_1\Delta & \varkappa_2\Delta\\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_2\Delta & \varkappa_3\Delta \end{bmatrix}_{8\times 8}, \tag{2.5}$$

$$\widetilde{L}_0(\partial):=\begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial)\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial)\end{bmatrix}_{6\times 6},$$

where

$$\begin{aligned} L_0^{(1)}(\partial) &:= a_1I_3\Delta+b_1Q(\partial),\\ L_0^{(2)}(\partial)=L_0^{(3)}(\partial) &:= cI_3\Delta+dQ(\partial),\\ L_0^{(4)}(\partial) &:= a_2I_3\Delta+b_2Q(\partial). \end{aligned}$$

Further let us introduce the operators

$$\begin{aligned} T(\partial,n) &:= \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) \end{bmatrix}_{6\times 6}, \\ T^{(l)}(\partial,n) &= \Big[T^{(l)}_{kj}(\partial,n) \Big]_{3\times 3}, \quad l=\overline{1,4}, \end{aligned} \tag{2.6}$$

where [15, 16]

$$\begin{aligned} T^{(1)}_{kj}(\partial,n) &:= (\mu_1-\lambda_5)\delta_{kj}\partial_n + (\mu_1+\lambda_5)n_j\partial_k + \\ &\quad + \Big(\lambda_1 - \frac{\rho_2}{\rho}\,\alpha_0\Big) n_k\partial_j, \\ T^{(2)}_{kj}(\partial,n) = T^{(3)}_{kj}(\partial,n) &:= (\mu_3+\lambda_5)\delta_{kj}\partial_n + (\mu_3-\lambda_5)n_j\partial_k + \\ &\quad + \Big(\lambda_3 - \frac{\rho_1}{\rho}\,\alpha_0\Big) n_k\partial_j, \\ T^{(4)}_{kj}(\partial,n) &:= (\mu_2-\lambda_5)\delta_{kj}\partial_n + (\mu_2+\lambda_5)n_j\partial_k + \\ &\quad + \Big(\lambda_2 + \frac{\rho_1}{\rho}\,\alpha_0\Big) n_k\partial_j, \end{aligned}$$

where $\partial_n = \partial/\partial_n$ is the normal derivative, $n=(n_1,n_2,n_3)$;

$$\widetilde{T}(\partial,n) := \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & [0]_{3\times 1} & [0]_{3\times 1} \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & [0]_{3\times 1} & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_1\partial_n & \varkappa_2\partial_n \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_2\partial_n & \varkappa_3\partial_n \end{bmatrix}_{8\times 8},$$

$$\mathcal{P}(\partial,n) := \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta_1 n^\top & -\eta_2 n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta_1 n^\top & -\zeta_2 n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_1\partial_n & \varkappa_2\partial_n \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_2\partial_n & \varkappa_3\partial_n \end{bmatrix}_{8\times 8},$$

$$\mathcal{P}^*(\partial,n) := \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -i\sigma\eta_1 n^\top & -i\sigma\eta_2 n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -i\sigma\zeta_1 n^\top & -i\sigma\zeta_2 n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_1\partial_n & \varkappa_2\partial_n \\ [0]_{1\times 3} & [0]_{1\times 3} & \varkappa_2\partial_n & \varkappa_3\partial_n \end{bmatrix}_{8\times 8}, \tag{2.7}$$

where $T^{(l)}(\partial,n)$, $l=1,2,3,4$, are given by (2.6), $n^\top = (n_1,n_2,n_3)^\top$.

3. Green's Formulas

Let Ω^+ be a finite three-dimensional region bounded by the Lyapunov surface $\partial\Omega$; $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

Definition 3.1. *A vector $U=(u',u'',\vartheta_1,\vartheta_2)^\top$ will be called regular in a domain $\Omega\subset\mathbb{R}^3$ if $U\in C^2(\Omega)\cap C^1(\overline{\Omega})$.*

Let

$$U=(u,\vartheta)^\top,\quad V=(v,\vartheta')^\top,\quad u=(u',u'')^\top,\quad v=(v',v'')^\top,$$
$$\vartheta=(\vartheta_1,\vartheta_2)^\top,\quad \vartheta'=(\vartheta_1',\vartheta_2')^\top.$$

It can be proved that for regular vectors u and v, the following Green's formula is valid [36]

$$\int\limits_{\Omega^+} v\cdot\widetilde{L}_0(\partial)u\,dx=\int\limits_{\partial\Omega}[v(z)]^+\cdot[T(\partial,n)u(z)]^+\,ds-\int\limits_{\Omega^+}E(u,v)\,dx, \tag{3.1}$$

where the differential operator $T(\partial,n)$ is given by formula (2.6), $n(z)$ is the outward unit normal vector w.r.t. Ω^+ at the point $z\in\partial\Omega$, $a\cdot b=\sum\limits_{j=1}^3 a_jb_j$ is the scalar product of vectors a and b, and $E(u,v)$ is a quatratic form defined as follows:

$$\begin{aligned}E(u,v)&=\Big(\lambda_1-\frac{\varrho_2}{\varrho}\,\alpha_0\Big)\operatorname{div}v'\operatorname{div}u'+\Big(\lambda_2+\frac{\varrho_1}{\varrho}\,\alpha_0\Big)\operatorname{div}v''\operatorname{div}u''+\\
&\quad+\Big(\lambda_3-\frac{\varrho_1}{\varrho}\,\alpha_0\Big)(\operatorname{div}v'\operatorname{div}u''+\operatorname{div}v''\operatorname{div}u')+\\
&+\frac{\mu_1}{2}\sum_{k,j=1}^3(\partial_jv_k'+\partial_kv_j')(\partial_ju_k'+\partial_ku_j')+\frac{\mu_2}{2}\sum_{k,j=1}^3(\partial_jv_k''+\partial_kv_j'')(\partial_ju_k''+\partial_ku_j'')+\\
&\quad+\frac{\mu_3}{2}\sum_{k,j=1}^3\Big[(\partial_jv_k'+\partial_kv_j')(\partial_ju_k''+\partial_ku_j'')+(\partial_jv_k''+\partial_kv_j'')(\partial_ju_k'+\partial_ku_j')\Big]-\\
&\quad-\frac{\lambda_5}{2}\sum_{k,j=1}^3(\partial_jv_k'-\partial_kv_j'-\partial_jv_k''+\partial_kv_j'')(\partial_ju_k'-\partial_ku_j'-\partial_ju_k''+\partial_ku_j'').\end{aligned} \tag{3.2}$$

Rewrite the vector $L(\partial,\sigma)U$ as

$$L(\partial,\sigma)U=L_0(\partial)U+L_0'(\partial,\sigma)U, \tag{3.3}$$

where

$$L_0'(\partial,\sigma)U=\begin{bmatrix}\alpha'u'+\varkappa u''-\eta_1\nabla^\top\vartheta_1-\eta_2\nabla^\top\vartheta_2\\ \varkappa u'+\alpha''u''-\zeta_1\nabla^\top\vartheta_1-\zeta_2\nabla^\top\vartheta_2\\ i\sigma\eta_1\nabla u'+i\sigma\zeta_1\nabla u''+\alpha_1\vartheta_1+\alpha\vartheta_2\\ i\sigma\eta_2\nabla u'+i\sigma\zeta_2\nabla u''+\alpha\vartheta_1+\alpha_2\vartheta_2\end{bmatrix}_{8\times1}. \tag{3.4}$$

Note that

$$V\cdot L_0(\partial)U=v\cdot\widetilde{L}_0(\partial)u+\vartheta_1'(\varkappa_1\Delta\vartheta_1+\varkappa_2\Delta\vartheta_2)+\vartheta_2'(\varkappa_2\Delta\vartheta_1+\varkappa_3\Delta\vartheta_2). \tag{3.5}$$

The following equality is valid [43]

$$\int\limits_{\Omega^+} \vartheta'_k \Delta\vartheta_j\,dx = \\ = \int\limits_{\partial\Omega} \big[\vartheta'_k(z)\partial_n\vartheta_j(z)\big]^+\,ds - \int\limits_{\Omega^+} \big(\nabla^\top\vartheta'_k\cdot\nabla^\top\vartheta_j\big)\,dx, \ \ k,j=1,2. \tag{3.6}$$

Using equalities (3.1) and (3.6), from (3.5) we have

$$\int\limits_{\Omega^+} V\cdot L_0(\partial)U\,dx = \int\limits_{\partial\Omega}\big[V(z)\cdot\widetilde{T}(\partial,n)U(z)\big]^+\,ds - \int\limits_{\Omega^+} E(U,V)\,dx, \tag{3.7}$$

where

$$E(U,V) = E(u,v) + \varkappa_1(\nabla^\top\vartheta'_1\cdot\nabla^\top\vartheta_1) + \\ + \varkappa_2\big(\nabla^\top\vartheta'_1\cdot\nabla^\top\vartheta_2 + \nabla^\top\vartheta'_2\cdot\nabla^\top\vartheta_1\big) + \varkappa_3(\nabla^\top\vartheta'_2\cdot\nabla^\top\vartheta_2)$$

and $E(u,v)$ is given by (3.2).

Multiplying both sides of equality (3.4) by vector $V=(v,\vartheta')^\top$ and taking into consideration the equality

$$\int\limits_{\Omega^+} v'\cdot\nabla^\top\vartheta_j\,dx = \int\limits_{\partial\Omega}\big[\vartheta_j(z)(n(z)\cdot v'(z))\big]^+\,ds - \int\limits_{\Omega^+}\vartheta_j\nabla v'\,dx, \ \ j=1,2, \tag{3.8}$$

we obtain

$$\int\limits_{\Omega^+} V\cdot L'_0(\partial,\sigma)U\,dx = -\int\limits_{\partial\Omega}\Big[(\eta_1\vartheta_1+\eta_2\vartheta_2)(n\cdot v')+(\zeta_1\vartheta_1+\zeta_2\vartheta_2)(n\cdot v'')\Big]^+\,ds + \\ + \int\limits_{\Omega^+}\Big[v'(\alpha' u' + \varkappa u'') + v''(\varkappa u' + \alpha'' u'') + \\ + i\sigma\big(\eta_1\vartheta'_1\nabla u' + \zeta_1\vartheta'_1\nabla u'' + \eta_2\vartheta'_2\nabla u' + \zeta_2\vartheta'_2\nabla u''\big) + \\ + \vartheta'_1(\alpha_1\vartheta_1 + \alpha\vartheta_2) + \vartheta'_2(\alpha\vartheta_1 + \alpha_2\vartheta_2)\Big]\,dx. \tag{3.9}$$

Combining equalities (3.7) and (3.9) we get

$$\int\limits_{\Omega^+} V\cdot L(\partial,\sigma)U\,dx = \int\limits_{\partial\Omega}\Big[V(z)\cdot\mathcal{P}(\partial,n)U(z)\Big]^+\,ds - \\ \int\limits_{\Omega^+}\Big[E(U,V) - v'\cdot(\alpha' u' + \varkappa u'') - v''\cdot(\varkappa u' + \alpha'' u'') - i\sigma\vartheta'_1(\eta_1\nabla u' + \zeta_1\nabla u'') - \\ - i\sigma\vartheta'_2(\eta_2\nabla u' + \zeta_2\nabla u'') - \vartheta'_1(\alpha_1\vartheta_1 + \alpha\vartheta_2) - \vartheta'_2(\alpha\vartheta_1 + \alpha_2\vartheta_2)\Big]\,dx. \tag{3.10}$$

With the help of equality (3.10), we derive

$$\int\limits_{\Omega^+} \Big[V\cdot L(\partial,\sigma)U - U\cdot L^*(\partial,\sigma)V\Big]\,dx = \\ = \int\limits_{\partial\Omega} \Big[V(z)\cdot \mathcal{P}(\partial,n)U(z) - U(z)\cdot \mathcal{P}^*(\partial,n)V(z)\Big]^+ ds, \quad (3.11)$$

where $L^*(\partial,\sigma)=\big[L(-\partial,\sigma)\big]^\top$ and $\mathcal{P}^*(\partial,n)$ is given by (2.7). The formulas (3.10) and (3.11) are Green's formulas.

Assume that a vector $U=(u,\vartheta)^\top$ is e solution of equation $L(\partial,\sigma)U=0$. According to (3.3) we obtain

$$L_0(\partial)U + L_0'(\partial,\sigma)U = 0, \quad (3.12)$$

where $L_0(\partial)$ is given by formula (2.5) and $L_0'(\partial,\sigma)U$ is defined by equality (3.4).

Let us multiply the first equation of (3.12) by the vector $\overline{u}'$, the second one by the vector $\overline{u}''$ and the complex conjugates of the third and fourth equations, respectively, by the functions $\frac{1}{i\overline{\sigma}}\vartheta_1$ and $\frac{1}{i\overline{\sigma}}\vartheta_2$ and sum up. In addition, taking into consideration equalities (3.1) and (3.8), we obtain

$$\int\limits_{\Omega^+}\bigg[-E(u,\overline{u}) + \frac{i}{\varkappa_3\overline{\sigma}}\Big(d_3|\nabla^\top\vartheta_1|^2 + |\varkappa_2\nabla^\top\vartheta_1 + \varkappa_3\nabla^\top\vartheta_2|^2\Big) - \varkappa|u'-u''|^2 + \\ + \rho_1\sigma^2|u'|^2 + \rho_2\sigma^2|u''|^2 + \frac{\alpha i}{\overline{\sigma}}\,|\vartheta_1-\vartheta_2|^2 - \big(\varkappa'|\vartheta_1|^2 + \varkappa''|\vartheta_2|^2\big)\bigg]\,dx + \\ + \int\limits_{\partial\Omega}\bigg[\overline{u}(z)T(\partial,n)u(z) - (\eta_1\vartheta_1+\eta_2\vartheta_2)(n\cdot\overline{u}') - (\zeta_1\vartheta_1+\zeta_2\vartheta_2)(n\cdot\overline{u}'') - \\ - \frac{i}{\varkappa_3\overline{\sigma}}\Big(d_3\vartheta_1\partial_n\overline{\vartheta}_1 + (\varkappa_2\vartheta_1+\varkappa_3\vartheta_2)(\varkappa_2\partial_n\overline{\vartheta}_1 + \varkappa_3\partial_n\overline{\vartheta}_2)\Big)\bigg]^+ ds = 0. \quad (3.13)$$

Here $\overline{u}$ is the complex conjugate of u and

$$E(u,\overline{u}) = \frac{d_2}{a_1+b_1}\,|\operatorname{div}u''|^2 + \frac{1}{a_1+b_1}\,\big|(a_1+b_1)\operatorname{div}u' + (c+d)\operatorname{div}u''\big|^2 + \\ + \frac{d_4}{2\mu_1}\sum_{k\neq j=1}^{3}|\partial_j u_k'' + \partial_k u_j''|^2 + \frac{1}{2\mu_1}\sum_{k\neq j=1}^{3}\big|\mu_1(\partial_j u_j' + \partial_k u_j') + \mu_3(\partial_j u_k'' + \partial_k u_j'')\big|^2 - \\ - \frac{\lambda_5}{2}\sum_{k,j=1}^{3}\big|\partial_j u_k' - \partial_k u_j' - \partial_j u_k'' + \partial_k u_j''\big|^2 > 0, \quad (3.14)$$

where $d_4=\mu_1\mu_2-\mu_3^2>0$. The sesquilinear form $E(u,\overline{u})$ is obtained from formula (3.2) by substituting the vectors v' and v'' by the vectors $\overline{u}'$ and $\overline{u}''$, respectively, and taking into consideration that $\lambda_1 - \frac{\rho_2}{\rho}\,\alpha_0 = a_1+b_1-2\mu_1$, $\lambda_2 + \frac{\rho_1}{\rho}\,\alpha_0 = a_2+b_2-2\mu_2$, $\lambda_3 - \frac{\rho_1}{\rho}\,\alpha_0 = c+d-2\mu_3$.

4. Formulation of Problems. Uniqueness Theorems

Problem $(\mathrm{I}^{(\sigma)})^{\pm}$ **(Dirichlet's problem).** Find a regular vector $U = (u', u'', \vartheta_1, \vartheta_2)^\top$ satisfying the system of differential equations

$$L(\partial, \sigma)U(x) = \Phi^{\pm}(x), \quad x \in \Omega^{\pm}, \tag{4.1}$$

and the boundary conditions

$$\{U(z)\}^{\pm} = f(z), \quad z \in \partial\Omega; \tag{4.2}$$

Problem $(\mathrm{II}^{(\sigma)})^{\pm}$ **(Neumann's problem).** Find a regular vector $U = (u', u'', \vartheta_1, \vartheta_2)^\top$ satisfying (4.1) and the boundary conditions

$$\{\mathcal{P}(\partial, n)\, U(z)\}^{\pm} = F(z), \quad z \in \partial\Omega; \tag{4.3}$$

here $\Phi^{\pm}$ are eight-component given vectors in $\Omega^{\pm}$, respectively while

$$f = (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})^\top, \quad F = (F^{(1)}, F^{(2)}, F^{(3)}, F^{(4)})^\top,$$
$$f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})^\top, \quad F^{(j)} = (F_1^{(j)}, F_2^{(j)}, F_3^{(j)})^\top, \quad j = 1, 2,$$

with $f^{(j)}$, $F^{(j)}$, $j = 3, 4$, being scalar function are assumed to be given on the boundary $\partial\Omega^{\pm}$; $n(z)$ is the outward unit normal vector w.r.t. Ω^+ at the point $z \in \partial\Omega$.

In the case of the exterior problems for the domain Ω^-, a vector $U(x)$ in a neighbourhood of infinity has to satisfy some sufficient vanishing conditions allowing one to write Green's formula (3.13) for the domain Ω^-.

Theorem 4.1. *If* $\sigma = \sigma_1 + i\sigma_2$, *where* $\sigma_1 \in R$, $\sigma_2 > 0$, *then the homogeneous problems* $(\mathrm{I}^{(\sigma)})_0^+$ *and* $(\mathrm{II}^{(\sigma)})_0^+$ $(\Phi^+ = 0,\ f = 0,\ F = 0)$ *have only the trivial solution.*

Proof. If in equation (3.13) we take into consideration the homogeneous boundary conditions, we obtain

$$\int\limits_{\Omega^+} \bigg[- E(u, \overline{u}) + \frac{i}{\varkappa_3 \overline{\sigma}} \Big(d_3 |\nabla^\top \vartheta_1|^2 + |\varkappa_2 \nabla^\top \vartheta_1 + \varkappa_3 \nabla^\top \vartheta_2|^2 \Big) - \varkappa |u' - u''|^2 +$$
$$+ \rho_1 \sigma^2 |u'|^2 + \rho_2 \sigma^2 |u''|^2 + \frac{\alpha i}{\overline{\sigma}} \, |\vartheta_1 - \vartheta_2|^2 - \big(\varkappa' |\vartheta_1|^2 + \varkappa'' |\vartheta_2|^2 \big) \bigg] \, dx = 0. \tag{4.4}$$

Separating the imaginary part of the equation (4.4), we obtain

$$\sigma_1 \int\limits_{\Omega^+} \bigg[\frac{1}{\varkappa_3 |\sigma|} \Big(d_3 |\nabla^\top \vartheta_1|^2 + |\varkappa_2 \nabla^\top \vartheta_1 + \varkappa_3 \nabla^\top \vartheta_2|^2 \Big) +$$
$$+ 2\rho_1 \sigma_2 |u'|^2 + 2\rho_2 \sigma_2 |u''|^2 + \frac{\alpha}{|\sigma|^2} \, |\vartheta_1 - \vartheta_2|^2 \bigg] \, dx = 0. \tag{4.5}$$

Assuming that $\sigma_1 \neq 0$, from (4.5) we get $u'(x) = 0$, $u''(x) = 0$, $\vartheta_1(x) = \vartheta_2(x) = const$, $x \in \Omega^+$. Taking these data into account in (4.4), we obtain $\vartheta_1(x) = \vartheta_2(x) = 0$, $x \in \Omega^+$. If $\sigma_1 = 0$, then from (4.4) we have

$$\int\limits_{\Omega^+} \bigg[E(u,\overline{u}) + \frac{1}{\varkappa_3\sigma_2} \Big(d_3|\nabla^\top\vartheta_1|^2 + |\varkappa_2\nabla^\top\vartheta_1 + \varkappa_3\nabla^\top\vartheta_2|^2 \Big) + \varkappa|u'-u''|^2 + \\ + \rho_1\sigma_2^2|u'|^2 + \rho_2\sigma_2^2|u''|^2 + \frac{\alpha}{\sigma_2}\,|\vartheta_1-\vartheta_2|^2 + \big(\varkappa'|\vartheta_1|^2 + \varkappa''|\vartheta_2|^2\big) \bigg]\,dx = 0.$$

From this equation we easily deduce $u'(x) = 0$, $u''(x) = 0$, $\vartheta_1(x) = 0$, $\vartheta_2(x) = 0$, $x \in \Omega^+$. □

5. Integral Representation Formulas

The fundamental matrix of solutions of the homogeneous system of differential equations of pseudo-oscillations of the two-temperature elastic mixtures theory reads as ([14, 42]):

$$\Gamma(x,\sigma) = \\ = \frac{1}{4\pi d_1 d_2 d_3} \begin{bmatrix} \widetilde{\Psi}_1(x,\sigma) & \widetilde{\Psi}_2(x,\sigma) & \nabla^\top\Psi_{13}(x,\sigma) & \nabla^\top\Psi_{14}(x,\sigma) \\ \widetilde{\Psi}_3(x,\sigma) & \widetilde{\Psi}_4(x,\sigma) & \nabla^\top\Psi_{15}(x,\sigma) & \nabla^\top\Psi_{16}(x,\sigma) \\ \nabla\Psi_{17}(x,\sigma) & \nabla\Psi_{18}(x,\sigma) & \Psi_5(x,\sigma) & \Psi_6(x,\sigma) \\ \nabla\Psi_{19}(x,\sigma) & \nabla\Psi_{20}(x,\sigma) & \Psi_7(x,\sigma) & \Psi_8(x,\sigma) \end{bmatrix}, \quad (5.1)$$

where d_1, d_2 are given by (2.2) and d_3 is given by (2.3),

$$\begin{aligned} \widetilde{\Psi}_1(x,\sigma) &= \Psi_1(x,\sigma)I_3 + Q(\partial)\Psi_9(x,\sigma), \\ \widetilde{\Psi}_2(x,\sigma) &= \Psi_2(x,\sigma)I_3 + Q(\partial)\Psi_{10}(x,\sigma), \\ \widetilde{\Psi}_3(x,\sigma) &= \Psi_3(x,\sigma)I_3 + Q(\partial)\Psi_{11}(x,\sigma), \\ \widetilde{\Psi}_4(x,\sigma) &= \Psi_4(x,\sigma)I_3 + Q(\partial)\Psi_{12}(x,\sigma), \\ \Psi_l(x,\sigma) &= \sum_{j=1}^{2} p_j\beta_{lj}^*\,\frac{e^{ik_j|x|}}{|x|}, \quad l = 1,2,3,4, \\ \Psi_{l-8}(x,\sigma) &= \sum_{j=3}^{6} p_j\beta_{lj}^*\,\frac{e^{ik_j|x|}}{|x|}, \quad l = 13,14,15,16, \\ \Psi_{l+8}(x,\sigma) &= -\sum_{j=1}^{6} p_j\gamma_{lj}^*\,\frac{e^{ik_j|x|}}{|x|}, \quad l = 1,2,3,4, \\ \Psi_{l+8}(x,\sigma) &= i\sum_{j=3}^{6} p_j\delta_{lj}^*\,\frac{e^{ik_j|x|}}{|x|}, \quad l = 5,6,\dots,12. \end{aligned} \quad (5.2)$$

k_j^2, $j=1,2$, and k_j^2, $j=3,4,5,6$, are, respectively, the solutions of the following equations

$$a(z):=d_1z^2-(a_1\alpha''+a_2\alpha'-2c\varkappa)z+\alpha'\alpha''-\varkappa^2=0,$$

$$\begin{aligned}\Lambda(z):=\big[d_3z^2-(\alpha_1\varkappa_3+\alpha_2\varkappa_1-2\alpha\varkappa_2)z+\alpha_1\alpha_2-\alpha^2\big](a(z)+zb(z))-\\ -i\sigma z\big[(\varkappa_3\varepsilon_1(z)+\varkappa_1\varepsilon_3(z)-2\varkappa_2\varepsilon_2(z))z+2\alpha\varepsilon_2(z)-\alpha_2\varepsilon_1(z)-\\ -\alpha_1\varepsilon_3(z)\big]-\sigma^2(\eta_1\zeta_2-\eta_2\zeta_1)^2z^2=0,\end{aligned}$$

where

$$b(z):=(d_2-d_1)z-(b_1\alpha''+b_2\alpha'-2\varkappa d),$$

$$\varepsilon_1(z):=\eta_1\delta_1''(z)+\zeta_1\delta_1'(z),\ \ \varepsilon_3(z):=\eta_2\delta_2''(z)+\zeta_2\delta_2'(z),$$

$$\varepsilon_2(z):=\eta_1\delta_2''(z)+\zeta_1\delta_2'(z)=\eta_2\delta_1''(z)+\zeta_2\delta_1'(z),$$

$$\delta_j'(z):=\eta_j\big[\varkappa-(c+d)z\big]+\zeta_j\big[(a_1+b_1)z-\alpha'\big],\ \ j=1,2,$$

$$\delta_j''(z):=\zeta_j\big[\varkappa-(c+d)z\big]+\eta_j\big[(a_2+b_2)z-\alpha''\big],\ \ j=1,2;$$

$$\beta_{1j}^*:=\Lambda_j^*(\alpha''-a_2k_j^2),\ \ \beta_{2j}^*=\beta_{3j}^*:=\Lambda_j^*(ck_j^2-\varkappa),$$

$$\beta_{4j}^*:=\Lambda_j^*(\alpha'-a_1k_j^2),\ \ \beta_{13j}^*:=a_j^*\Big[i\sigma k_j^2\varepsilon_{3j}^*+(\alpha_2-\varkappa_3k_j^2)(a_j^*+b_j^*k_j^2)\Big],$$

$$\beta_{14j}^*=\beta_{15j}^*:=-a_j^*\Big[i\sigma k_j^2\varepsilon_{2j}^*+(\alpha-\varkappa_2k_j^2)(a_j^*+b_j^*k_j^2)\Big],$$

$$\beta_{16j}^*:=a_j^*\big[i\sigma k_j^2\varepsilon_{1j}^*+(\alpha_1-\varkappa_1k_j^2)(a_j^*+b_j^*k_j^2)\big],$$

$$\begin{aligned}\gamma_{1j}^*:=a_2\Lambda_j^*-\big[a_j^*(a_2+b_2)+b_j^*\alpha''\big]H_j^*-\alpha''\sigma^2(\eta_1\zeta_2-\eta_2\zeta_1)^2k_j^2-\\ -i\sigma\Big[(a_j^*\zeta_1^2+\alpha''\varepsilon_{1j}^*)(\alpha_2-\varkappa_3k_j^2)+(a_j^*\zeta_2^2+\alpha''\varepsilon_{3j}^*)(\alpha_1-\varkappa_1k_j^2)-\\ -2(a_j^*\zeta_1\zeta_2+\alpha''\varepsilon_{2j}^*)(\alpha-\varkappa_2k_j^2)\Big],\end{aligned}$$

$$\begin{aligned}\gamma_{2j}^*=\gamma_{3j}^*:=-c\Lambda_j^*+\big[a_j^*(c+d)+b_j^*\varkappa\big]H_j^*-\varkappa\sigma^2(\eta_1\zeta_2-\eta_2\zeta_1)^2k_j^2+\\ +i\sigma\Big[(a_j^*\eta_1\zeta_1+\varkappa\varepsilon_{1j}^*)(\alpha_2-\varkappa_3k_j^2)+(a_j^*\eta_2\zeta_2+\varkappa\varepsilon_{3j}^*)(\alpha_1-\varkappa_1k_j^2)+\\ +\big(2\varkappa\varepsilon_{2j}^*+(\eta_1\zeta_2+\eta_2\zeta_1)a_j^*\big)(\alpha-\varkappa_2k_j^2)\Big],\end{aligned}$$

$$\begin{aligned}\gamma_{4j}^*:=a_1\Lambda_j^*-\big[a_j^*(a_1+b_1)+b_j^*\alpha'\big]H_j^*+\alpha'\sigma^2(\eta_1\zeta_2-\eta_2\zeta_1)^2k_j^2-\\ -i\sigma\Big[(a_j^*\eta_1^2+\alpha'\varepsilon_{1j}^*)(\alpha_2-\varkappa_3k_j^2)+(a_j^*\eta_2^2+\alpha'\varepsilon_{3j}^*)(\alpha_1-\varkappa_1k_j^2)-\\ -2(a_j^*\eta_1\eta_2+\alpha'\varepsilon_{2j}^*)(\alpha-\varkappa_2k_j^2)\Big],\end{aligned}$$

$$\delta_{5j}^*:=ia_j^*\Big[i\sigma\zeta_2(\eta_1\zeta_2-\eta_2\zeta_1)k_j^2+\delta_{1j}''(\alpha_2-\varkappa_3k_j^2)-\delta_{2j}''(\alpha-\varkappa_2k_j^2)\Big],$$

$$\delta_{6j}^*:=ia_j^*\Big[-i\sigma\zeta_1(\eta_1\zeta_2-\eta_2\zeta_1)k_j^2-\delta_{1j}''(\alpha-\varkappa_2k_j^2)+\delta_{2j}''(\alpha_1-\varkappa_1k_j^2)\Big],$$

$$\delta_{7j}^*:=ia_j^*\Big[-i\sigma\eta_2(\eta_1\zeta_2-\eta_2\zeta_1)k_j^2+\delta_{1j}'(\alpha_2-\varkappa_3k_j^2)-\delta_{2j}'(\alpha-\varkappa_2k_j^2)\Big],$$

$$\delta_{8j}^*:=ia_j^*\Big[i\sigma\eta_1(\eta_1\zeta_2-\eta_2\zeta_1)k_j^2-\delta_{1j}'(\alpha-\varkappa_2k_j^2)+\delta_{2j}'(\alpha_1-\varkappa_1k_j^2)\Big],$$

$$\delta^*_{9j}=-i\sigma\delta^*_{5j},\ \ \delta^*_{10j}=-i\sigma\delta^*_{7j},\ \ \delta^*_{11j}=-i\sigma\delta^*_{6j},\ \ \delta^*_{12j}=-i\sigma\delta^*_{8j}.$$

$$a^*_j:=d_1\prod_{j\neq q=1}^{2}(k_j^2-k_q^2),\quad b^*_j:=(d_2-d_1)k_j^2-b_2\alpha'-b_1\alpha''+2\varkappa d,$$

$$\Lambda^*_j:=d_2d_3\prod_{j\neq q=3}^{6}(k_j^2-k_q^2),\ \ H^*_j:=d_3k_j^4-(\alpha_1\varkappa_3+\alpha_2\varkappa_1-2\alpha\varkappa_2)k_j^2+\alpha_1\alpha_2-\alpha^2;$$

$$\delta'_{lj}:=\eta_l\big[\varkappa-(c+d)k_j^2\big]+\zeta_l\big[(a_1+b_1)k_j^2-\alpha'\big],\ \ l=1,2,$$
$$\delta''_{lj}:=\zeta_l\big[\varkappa-(c+d)k_j^2\big]+\eta_l\big[(a_2+b_2)k_j^2-\alpha''\big],\ \ l=1,2,$$
$$\varepsilon^*_{1j}=\eta_1\delta''_{lj}+\zeta_1\delta'_{1j},\ \ \varepsilon^*_{2j}=\eta_1\delta''_{2j}+\zeta_1\delta'_{2j},\ \ \varepsilon^*_{3j}=\eta_2\delta''_{2j}+\zeta_2\delta'_{2j},$$

$$p_j=\prod_{j\neq q=1}^{6}\big(k_j^2-k_q^2\big)^{-1}.$$

Remark 5.1. *Using formulas (5.1) and (5.2), and the equalities*

$$k_1^{2m}p_1+k_2^{2m}p_2+\cdots+k_6^{2m}p_6=0,\ \ m=\overline{0,\,4},$$
$$k_1^{10}p_1+k_2^{10}p_2+\cdots+k_6^{10}p_6=1,$$

we conclude that in a vicinity of the origin the functions $\Psi_j(x,\sigma)$, $j=\overline{1,8}$, *and* $\Psi_j(x,\sigma)$, $j=\overline{9,20}$, *are, respectively, of order const +* $O(|x|^{-1})$ *and* $O(|x|^{-1})$.

Hereinafter, we shall always assume that $k_j\neq k_p$, $j\neq p$, $\Im k_j>0$, $j=\overline{1,6}$. According to these requirements regarding to equalities (5.2), all entries of $\Gamma(x,\sigma)$ exponentially decay at infinity.

Let us introduce the generalized single and double-layer potentials, and the Newton type volume potential

$$V(\varphi)(x)=\int\limits_S\Gamma(x-y,\sigma)\,\varphi(y)\,dS_y,\ \ x\in\mathbb{R}^3\setminus S, \tag{5.3}$$

$$W(\varphi)(x)=\int\limits_S\big[\mathcal{P}^*(\partial,n)\Gamma^\top(x-y,\sigma)\big]^\top\varphi(y)\,dS_y,\ \ x\in\mathbb{R}^3\setminus S, \tag{5.4}$$

$$N_{\Omega^\pm}(\psi)(x)=\int\limits_{\Omega^\pm}\Gamma(x-y,\sigma)\,\psi(y)\,dy,\ \ x\in\mathbb{R}^3,$$

where $\mathcal{P}^*(\partial,n)$ is the boundary differential operator defined by (2.7), $\Gamma(\cdot,\sigma)$ is the fundamental matrix given by (5.1), $\varphi=(\varphi_1,\cdots,\varphi_8)^\top$ is a density vector-function defined on S, while a density vector-function $\psi=(\psi_1,\cdots,\psi_8)^\top$ is defined on $\Omega^\pm$, and we assume that in the case of Ω^- the support of the density vector-function ψ of the Newtonian potential is a compact set.

Due to the equality

$$\sum_{j=1}^{8} L_{kj}(\partial_x,\sigma)\big([\mathcal{P}^*(\partial,n)\Gamma^\top(x-y,\sigma)]^\top\big)_{jp} =$$

$$= \sum_{j,\,q=1}^{8} L_{kj}(\partial_x,\sigma)\mathcal{P}^*_{pq}(\partial,n)\Gamma_{jq}(x-y,\sigma) =$$

$$= \sum_{j,\,q=1}^{8} \mathcal{P}^*_{pq}(\partial,n)L_{kj}(\partial_x,\sigma)\Gamma_{jq}(x-y,\sigma) = 0, \quad x\neq y,\ k,p=\overline{1,8},$$

it can be easily checked that the potentials defined by (5.3) and (5.4) are C^∞–smooth in $\mathbb{R}^3\setminus S$ and solve the homogeneous equation $L(\partial,\sigma)U(x)=0$ in $\mathbb{R}^3\setminus S$ for an arbitrary L_p-summable vector-function φ. The Newtonian potential solves the nonhomogeneous equation

$$L(\partial,\sigma)N_{\Omega^\pm}(\psi)=\psi \ \text{ in } \ \Omega^\pm \ \text{ for } \ \psi\in[C^{0,k}(\overline{\Omega^\pm})]^8.$$

This relation holds true for an arbitrary $\psi\in[L_p(\Omega^\pm)]^8$ with $1<p<\infty$. It is easy to show that $\Gamma(-x,\sigma)$ is a fundamental matrix of the formally adjoint operator $L^*(\partial,\sigma)$, i.e.

$$L^*(\partial,\sigma)\big[\Gamma(-x,\sigma)\big]^\top = I_8\delta(x). \tag{5.5}$$

With the help of Green's formulas (3.11) and (5.5) by standard arguments we can prove the following assertions (cf., e.g., [7, 26, 27] and [36, Ch. I, Lemma 2.1; Ch. II, Lemma 8.2]).

Theorem 5.2. *Let $S=\partial\Omega^+$ be $C^{1,k}$-smooth with $0<k\leq 1$, either $\sigma=0$ or $\sigma=\sigma_1+i\,\sigma_2$ with $\sigma_2>0$, and let U be a regular vector of the class $[C^2(\overline{\Omega^+})]^8$. Then there holds the integral representation formula*

$$W\big(\{U\}^+\big)(x) - V\big(\{\mathcal{P}U\}^+\big)(x) + N_{\Omega^+}\big(L(\partial,\sigma)U\big)(x) = \\ = \begin{cases} U(x) & \text{for } x\in\Omega^+, \\ 0 & \text{for } x\in\Omega^-. \end{cases}$$

Proof. For the smooth case it easily follows from Green's formula (3.11) with the domain of integration $\Omega^+\setminus B(x,\varepsilon')$, where $x\in\Omega^+$ is treated as a fixed parameter, $B(x,\varepsilon')$ is a ball with the centre at the point x and radius $\varepsilon'>0$ and $\overline{B(x,\varepsilon')}\subset\Omega^+$. One needs to take the j-th column of the fundamental matrix $\Gamma^*(y-x,\sigma)$ for $V(y)$, calculate the surface integrals over the sphere $\Sigma(x,\varepsilon'):=\partial B(x,\varepsilon')$ and pass to the limit as $\varepsilon'\to 0$. □

Similar representation formula holds in the exterior domain Ω^- if a vector U and its derivatives possess some asymptotic properties at infinity. In particular, the following assertion holds.

Theorem 5.3. *Let $S=\partial\Omega^-$ be $C^{1,k}$-smooth with $0<k\leq 1$ and let U be a regular vector of the class $[C^2(\overline{\Omega^-})]^8$ such that for any multi-index*

$\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $0 \le |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \le 2$, the function $\partial^\alpha U_j$ is polynomially bounded at infinity, i.e., for sufficiently large $|x|$

$$\big|\partial^\alpha U_j(x)\big| \le C_0\,|x|^m, \ \ j = \overline{1,8}, \tag{5.6}$$

with some constants m and $C_0 > 0$. Then there holds the integral representation formula

$$-W\big(\{U\}^-\big)(x) + V\big(\{\mathcal{P}U\}^-\big)(x) + N_{\Omega^-}\big(L(\partial,\sigma)U\big)(x) = \\ = \begin{cases} 0 & \text{for } x \in \Omega^+, \\ U(x) & \text{for } x \in \Omega^-, \end{cases} \tag{5.7}$$

with $\sigma = \sigma_1 + i\,\sigma_2$, where $\sigma_2 > 0$.

Proof. The proof immediately follows from Theorem 5.2 and Remark 3.1 (cf. [14]). Indeed, one needs to write the integral representation formula (5.2) for the bounded domain $\Omega^- \cap B(0,R)$, and then send R to $+\infty$ and take into consideration that the surface integral over $\Sigma(0,R)$ tends to zero due to the conditions (5.6) and the exponential decay of the fundamental matrix at infinity. □

Corollary 5.4. *Let $\sigma = \sigma_1 + i\,\sigma_2$ with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$, and U be a solution to the homogeneous equation $L(\partial,\sigma)U = 0$ in $\Omega^\pm$ satisfying the condition (5.6) and $U \in [C^{1,k}(\overline{\Omega^\pm})]^8$ for some $0 < k \le 1$. Then the representation formula*

$$U(x) = W([U]_S)(x) - V([\mathcal{P}U]_S)(x), \ \ x \in \Omega^\pm,$$

holds, where $[U]_S = \{U\}^+ - \{U\}^-$ and $[\mathcal{P}U]_S = \{\mathcal{P}U\}^+ - \{\mathcal{P}U\}^-$ on S.

Proof. It Immediately follows from Theorems 5.2 and 5.3. □

Theorem 5.5. *Assume that $S = \partial\Omega \in C^{m,k}$, $m \ge 1$ and $0 < k \le 1$. If $g \in [C^{0,k'}(S)]^8$, $h \in [C^{0,k'}(S)]^8$, $0 < k' < k$, then for each $z \in S$,*

$$[V(g)(z)]^\pm = V(g)(z) = \mathcal{H}g(z), \tag{5.8}$$

$$\big[\mathcal{P}(\partial,n)V(g)(z)\big]^\pm = [\mp 2^{-1}I_8 + \mathcal{K}]g(z), \tag{5.9}$$

$$[W(h)(z)]^\pm = \big[\pm 2^{-1}I_8 + \mathcal{N}\big]h(z), \tag{5.10}$$

$$\big[\mathcal{P}(\partial,n)W(h)(z)\big]^+ = \big[\mathcal{P}(\partial,n)W(h)(z)\big]^- = \mathcal{L}h(z), \tag{5.11}$$

where

$$\mathcal{H}g(z) := \int\limits_S \Gamma(z-y,\sigma)g(y)\,dS_y,$$

$$\mathcal{L}h(z) := \lim_{\Omega^\pm \ni x \to z \in S} \mathcal{P}(\partial_x, n(x)) \int\limits_S \big[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(x-y,\sigma)\big]^\top h(y)\,dS_y,$$

$$\mathcal{K}g(z) := \int\limits_S \big[\mathcal{P}(\partial,n)\Gamma(z-y,\sigma)\big]g(y)\,dS_y,$$

$$\mathcal{N}h(z) := \int\limits_S \big[\mathcal{P}^*(\partial, n)\Gamma^\top(z-y,\sigma)\big]^\top h(y)\,dS_y.$$

The prove of this theorem is analogous to that given in [25, 35].

Theorem 5.6. *Assume that $S = \partial\Omega \in C^{m,k}$, $m \geq 2$, $0 < k' < k \leq 1$, $l \leq m-1$, $\sigma = \sigma_1 + i\sigma_2$, $\sigma_2 > 0$. If $g \in [C^{0,k'}(S)]^8$, $h \in [C^{1,k'}(S)]^8$, then*

$$\begin{aligned}
V &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l+1,k'}(\overline{\Omega^\pm})]^8,\\
W &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l,k'}(\overline{\Omega^\pm})]^8,\\
\mathcal{H} &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l+1,k'}(S)]^8,\\
\mathcal{K} &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l,k'}(S)]^8,\\
\mathcal{N} &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l,k'}(S)]^8,\\
\mathcal{L} &: [C^{l,k'}(S)]^8 \longrightarrow [C^{l-1,k'}(S)]^8.
\end{aligned}$$

Remark 5.7. *Assume that $\sigma = \sigma_1 + i\sigma_2$, $\sigma_2 > 0$ and $\Im k_j > 0$. From equation (5.7) it follows that if $L(\partial,\sigma)U(x) = 0$, $x \in \Omega^-$, then U is exponentially decaying at infinity and therefore in the unbounded domain Ω^- Green's formula (3.13) holds true,:*

$$\begin{aligned}
&\int\limits_{\Omega^-} \bigg[-E(u,\overline{u}) + \frac{i}{\varkappa_3\overline{\sigma}}\Big(d_3|\nabla^\top\vartheta_1|^2 + |\varkappa_2\nabla^\top\vartheta_1 + \varkappa_3\nabla^\top\vartheta_2|^2\Big) - \varkappa|u'-u''|^2 + \\
&\quad + \rho_1\sigma^2|u'|^2 + \rho_2\sigma^2|u''|^2 + \frac{\alpha i}{\overline{\sigma}}\,|\vartheta_1-\vartheta_2|^2 - \big(\varkappa'|\vartheta_1|^2 + \varkappa''|\vartheta_2|^2\big)\bigg]\,dx - \\
&- \int\limits_{\partial\Omega} \Big[\overline{u}(z)\cdot T(\partial,n)u(z) - (\eta_1\vartheta_1+\eta_2\vartheta_2)(n\cdot\overline{u'}) - (\zeta_1\vartheta_1+\zeta_2\vartheta_2)(n\cdot\overline{u''}) - \\
&- \frac{i}{\varkappa_3\overline{\sigma}}\Big(d_3\vartheta_1\partial_n\overline{\vartheta_1} + (\varkappa_2\vartheta_1+\varkappa_3\vartheta_2)\big(\varkappa_2\partial_n\overline{\vartheta_1}+\varkappa_3\partial_n\overline{\vartheta_2}\big)\Big)\Big]^-\,ds = 0, \qquad (5.12)
\end{aligned}$$

where the sesquilinear form $E(u,\overline{u})$ is given by (3.14) and the operator $T(\partial,n)$ by formula (2.6).

Similarly to Theorem 4.1 in view of formula (5.12) the following theorem takes place.

Theorem 5.8. *If $\sigma = \sigma_1 + i\sigma_2$, where $\sigma_1 \in \mathbb{R}$, $\sigma_2 > 0$, then the homogeneous problems $(\mathrm{I}^{(\sigma)})_0^-$ and $(\mathrm{II}^{(\sigma)})_0^-$ ($\Phi^\pm$, $f = 0$, $F = 0$) have only the trivial solution.*

The following theorem is valid.

Theorem 5.9. *Let $S = \partial\Omega \in C^{m,k}$ with integer $m \geq 2$ and $0 < k \leq 1$. Then:*

(a) *The principal homogenous symbol matrices of the singular integral operators $\mp 2^{-1}I_8 + \mathcal{K}$ and $\pm 2^{-1}I_8 + \mathcal{N}$ are non-degenerate, while* *the principal homogenous symbol matrices of the operators $\mathcal{H}$ and $\mathcal{L}$ are positive definite;*

(b) *the operators $\mathcal{H}$, $\mp 2^{-1}I_8+\mathcal{K}$, $\pm 2^{-1}I_8+\mathcal{N}$ and $\mathcal{L}$ are elliptic pseudo-differential operators (of order -1, 0, 0 and 1, respectively) with zero index;*

(c) *the following equalities hold in appropriate function spaces:*

$$\begin{gathered}\mathcal{N}\mathcal{H}=\mathcal{H}\mathcal{K}, \quad \mathcal{L}\mathcal{N}=\mathcal{K}\mathcal{L}, \\ \mathcal{H}\mathcal{L}=-4^{-1}I_8+\mathcal{N}^2, \;\; \mathcal{L}\mathcal{H}=-4^{-1}I_8+\mathcal{K}^2. \end{gathered} \tag{5.13}$$

The proof of this theorem is word for word of the proof of its counterparts in [31, 33, 35, 36].

6. Existence of Classical Solutions of the Boundary Value Problems

This section provides the study of problems stated in Section 4 using the theory of potentials and theory of integral equations. We seek solutions of the problems in the form of single or double-layer potentials allowing one to reduce the BVPs to the correspond boundary integral equations. Simultaneously, the question of invertibility of the obtained integral operators will be considered.

6.1. **Investigation of Dirichlet's problem by the double-layer potential.** We seek solutions of problems $(\mathrm{I}^{(\sigma)})^+$ and $(\mathrm{I}^{(\sigma)})^-$ (see (4.1), $\Phi^\pm=0$, (4.2)) by means of the double-layer potential $W(h)(x)$ (see (5.4)), where $h\in C^{1,\beta}(S)$ is the sought for vector-function. Taking into consideration the boundary condition (4.2) and the jump formulas (5.10), for the density h we obtain the following integral equations of second kind

$$\text{BVP } (\mathrm{I}^{(\sigma)})^+ : \; \big[2^{-1}I_8+\mathcal{N}\big]h=f \;\text{ on }\; S, \tag{6.1}$$

$$\text{BVP } (\mathrm{I}^{(\sigma)})^- : \; \big[-2^{-1}I_8+\mathcal{N}\big]h=f \;\text{ on }\; S. \tag{6.2}$$

In the left hand side of(6.1) and (6.2) we have singular integral operators of normal type with the index equal to zero (see Theorem 5.9).

Theorem 6.1. *If $S\in C^{2,\alpha}$ and $f\in C^{1,\beta}$, $0<\beta<\alpha\le 1$, then the problem $(\mathrm{I}^{(\sigma)})^+$ has a unique solution representable by the double-layer potential $W(h)$, where h is determined from uniquely solvable integral equation* (6.1).

Proof. Uniqueness follows from Theorem 4.1. Now, let us show that the operator

$$2^{-1}I_8+\mathcal{N}: C^{1,\beta}(S)\longrightarrow C^{1,\beta}(S) \tag{6.3}$$

is invertible. Note that the operator $-2^{-1}I_8+\mathcal{N}$ the arguments are verbatim. By virtue of Theorem 5.9,operator (6.3) is Fredholm with zero index and therefore for proving its invertibility it is sufficient to show that its kernel $ker(2^{-1}I_8+\mathcal{N})$ is trivial, i.e. we have to show that the homogeneous equation

$$\big[2^{-1}I_8+\mathcal{N}\big]h=0 \;\text{ on } S \tag{6.4}$$

has only the trivial solution. Indeed, assume that h is a solution of (6.4) and construct the double-layer potential $W(h)$. In view of the inclusion $h \in C^{1.\beta}(S)$, we have $W(h) \in C^{1,\beta}\overline{(\Omega^{\pm})}$. It easy to see that equation (6.4) corresponds to Dirichlet's interior homogeneous problem $[W(h)(z)]^+ = 0$, $z \in S$. Since this problem has only the trivial solution, we conclude $W(h)(x) = 0$, $x \in \Omega^+$. Therefore we have $[\mathcal{P}(\partial, n)W(h)(z)]^+ = 0$, $z \in S$, and according to the Lyapunov-Tauber theorem we deduce $[\mathcal{P}(\partial, n)W(h)(z)]^+ = [\mathcal{P}(\partial, n)W(h)(z)]^- = 0$, $z \in S$ (see Theorem 5.6). This means that $W(h)(x)$ is a solution to the homogeneous problem $(\mathrm{II}^{(\sigma)})^-$ which possesses only the trivial solution. Thus $W(h)(x) = 0$, $x \in \Omega^-$ and by virtue of formula (5.10) we conclude that $[W(h)(z)]^+ - [W(h)(z)]^- = h(z) = 0$, $z \in S$, i.e. integral equation (6.4) has only the trivial solution. Hence, the operator (6.3) is invertible and therefore the equation (6.1) is unique solvable for arbitrary vector-function $f \in C^{1,\beta}(S)$, which proves the theorem. □

The following theorem can be proved similarly.

Theorem 6.2. *If $S \in C^{2,\alpha}$ and $f \in C^{1,\beta}(S)$, $0 < \beta < \alpha \leq 1$, then the problem $(\mathrm{I}^{(\sigma)})^-$ has a unique solution, which is representable by the double-layer potential $W(h)$, where h is determined from unique by solvable integral equation* (6.2).

6.2. **Investigation of Neumann's problem by single-layer potential.** Solutions to the problems $(\mathrm{II}^{(\sigma)})^+$ and $(\mathrm{II}^{(\sigma)})^-$ (see (4.1), $\Phi^{\pm} = 0$, (4.3)) are sought by single-layer potential $V(g)(x)$, where $g \in C^{0,\beta}(S)$ (see (5.3)). Taking into consideration the boundary conditions (4.3) and the jump formulas (5.9) for the density g we obtain, the following integral equations of second kind respectively

$$\text{BVP } (\mathrm{II}^{(\sigma)})^+ : \ \big[-2^{-1}I_8 + \mathcal{K}\big]g = F \ \text{ on } \ S, \tag{6.5}$$

$$\text{BVP } (\mathrm{II}^{(\sigma)})^- : \ \big[2^{-1}I_8 + \mathcal{K}\big]g = F \ \text{ on } \ S. \tag{6.6}$$

The operators in the left hand side of(6.5) and (6.6) are singular integral operators of normal type with the index equal to zero (see Theorem 5.9).

Theorem 6.3. *If $S \in C^{1,\alpha}$ and $F \in C^{0,\beta}(S)$, $0 < \beta < \alpha \leq 1$, then the problem $(\mathrm{II}^{(\sigma)})^+$ has a unique solution, which is representable by the single-layer potential $V(g)(x)$, where g is determined from uniquely solvable integral equation* (6.5).

Proof. Uniqueness follows from Theorem 4.1. Now, let us show that the operator

$$-2^{-1}I_8 + \mathcal{K} : C^{0,\beta}(S) \longrightarrow C^{0,\beta}(S) \tag{6.7}$$

is invertible. Note that the invertibility of the operator $2^{-1}I_8 + \mathcal{K}$ can be performed by word for word arguments. By virtue of Theorem 5.9, the operator (6.7) is Fredholm with zero index and therefore for proving its invertibility it is sufficient to show that its kernel $ker(-2^{-1}I_8+\mathcal{K})$ is trivial, i.e. we have to show that the homogeneous equation

$$\big[-2^{-1}I_8+\mathcal{K}\big]g=0 \ \text{ on } \ S \tag{6.8}$$

has only the trivial solution. Indeed, assume that g is e solution of (6.8). Construct the single-layer potential $V(g)$. Since $g\in C^{0,\beta}(S)$, we have $V(g)\in C^{1,\beta}(\overline{\Omega^{\pm}})$. The equation (6.8) corresponds to Neumann's interior homogeneous problem $[\mathcal{P}(\partial,n)V(g)(z)]^{+}=0$, $z\in S$. Since this problem has only the trivial solution, we get $V(g)(x)=0$, $x\in\Omega^{+}$. Since $[V(g)(z)]^{-}=[V(g)(z)]^{+}=0$, $z\in S$, we have that $V(g)(x)$ is a solution of Dirichlet's exterior homogeneous problem and hence $V(g)(x)=0$, $x\in\Omega^{-}$. On the other hand, by virtue of formula (5.9) we obtain that $[\mathcal{P}(\partial_z,n(z))V(g)(z)]^{-}-[\mathcal{P}(\partial_z,n(z))V(g)(z)]^{+}=g(z)=0$, $z\in S$, i.e. the integral equation (6.8) has only the trivial solution. Consequently, the operator (6.7) is invertible and therefore equation (6.5) is solvable for arbitrary vector-function $F\in C^{0,\beta}(S)$, which proves the theorem. □

The following theorem can be proved similarly.

Theorem 6.4. *If $S\in C^{1,\alpha}$ and $F\in C^{0,\beta}(S)$, $0<\beta<\alpha\le 1$, then the problem* $(\mathrm{II}^{(\sigma)})^{-}$ *has a unique solution, which is representable by the single-layer potential $V(g)$, where g is determined from unique by solvable integral equation* (6.6).

6.3. **Investigation of Dirichlet's problem by single-layer potential.** We seek solutions of the problems $(\mathrm{I}^{(\sigma)})^{+}$ and $(\mathrm{I}^{(\sigma)})^{-}$ (see (4.1), $\Phi^{\pm}=0$, (4.2)) by means of the single-layer potential $V(g)(x)$ (see (5.3)), where $g\in C^{0,\beta}(S)$ is the sought for vector-function. Taking into consideration the boundary condition (4.2) and the jump formula (5.8), for the density g we obtain the following integral equation of the first kind:

$$\mathcal{H}g=f \ \text{ on } \ S. \tag{6.9}$$

Theorem 6.5. *If $S\in C^{2,\alpha}$ and $f\in C^{1,\beta}(S)$, $0<\beta<\alpha\le 1$, then the problem* $(\mathrm{I}^{(\sigma)})^{\pm}$ *has a unique solution, which can be represented by the single-layer potential $V(g)$, where g is determined from uniquely solvable integral equation* (6.9).

Proof. Uniqueness follows from Theorems 4.1 and 5.9. Now, let us show that the operator

$$\mathcal{H}:C^{0,\beta}(S)\longrightarrow C^{1,\beta}(S) \tag{6.10}$$

is invertible. Applying the operator $\mathcal{L}$ to both sides of the equation (6.9), we obtain (see (5.13)) the singular integral equation

$$\mathcal{L}\mathcal{H}g=(-4^{-1}I_8+\mathcal{K}^2)g=(-2^{-1}I_8+\mathcal{K})(2^{-1}I_8+\mathcal{K})g=\mathcal{L}f, \tag{6.11}$$

where $\mathcal{L}f\in C^{0,\beta}(S)$ and the operator

$$\mathcal{L}\mathcal{H}=(-2^{-1}I_8+\mathcal{K})(2^{-1}I_8+\mathcal{K}):C^{0,\alpha}(S)\longrightarrow C^{0,\alpha}(S)$$

is a singular operator of normal type with the index equal to zero.By the same arguments applied in [33], it can be shown that the operator (6.11) is invertible. Therefore we can write

$$g = (2^{-1}I_8 + \mathcal{K})^{-1}(-2^{-1}I_8 + \mathcal{K})^{-1}\mathcal{L}f.$$

Let us show that (6.9) and (6.11) are equivalent integral equations. Indeed, if $g \in C^{0,\beta}(S)$ is a solution to the equation (6.9), then it will be a solution to the equation (6.11) as well. Assume now that g is a solution to the equation (6.11). Introduce notation

$$\varphi := (\mathcal{H}g - f) \in C^{1,\beta}(S). \tag{6.12}$$

Then equation (6.11) can be rewritten as

$$\mathcal{L}\varphi = 0 \ \text{ on } \ S. \tag{6.13}$$

Construct the double-layer potential $W(\varphi)$ with the density φ determined by equation (6.12). Then it follows that $W(\varphi)$ solves Neumann's homogeneous problem $[\mathcal{P}(\partial_z, n(z))W(\varphi)(z)]^{\pm} = 0$, $z \in S$, in view of equation (6.13). Since this problem has only the trivial solution, we infer $W(\varphi)(x) = 0$, $x \in \Omega^{\pm}$. According to (5.10) we have $[W(\varphi)(z)]^{+} - [W(\varphi)(z)]^{-} = \varphi(z) = 0$, $z \in S$, i.e. g is a solution to equation (6.9). Hence operator (6.10) is invertible. □

Corollary 6.6. *Solution to problem* $(\mathrm{I}^{(\sigma)})^{\pm}$ *is presentable in the following form:*

$$U(x) = V(\mathcal{H}^{-1}f)(x), \ \ x \in \Omega^{\pm},$$

where $[U(z)]^{\pm} = f(z)$, $z \in S$.

This representation plays a crucial role in the study of mixed boundary value problems, when on a part of the boundary $\partial\Omega$ the Dirichlet condition is given, while on the remainder part the Neumann condition is prescribed

6.4. **Investigation of Neumann's problem by double-layer potential.** We seek a solution to problem $(\mathrm{II}^{(\sigma)})^{\pm}$ (see (4.1), $\Phi^{\pm} = 0$, (4.3)) in the form of double-layer potential $W(h)$, where $h \in C^{1,\beta}(S)$ is the sought vector (see (5.4)). Taking into consideration the boundary conditions (4.3) and formula (5.11), for the density h we obtain the following integral equation of the "first kind":

$$\mathcal{L}h = F \ \text{ on } \ S. \tag{6.14}$$

Theorem 6.7. *If* $S \in C^{1,\alpha}$ *and* $F \in C^{0,\beta}(S)$, $0 < \beta < \alpha \le 1$, *then the problem* $(\mathrm{II}^{(\sigma)})^{\pm}$ *has a unique solution, which is representable by double-layer potential* $W(h)$, *where* h *is determined from uniquely solvable integral equation* (6.14).

Proof. Uniqueness follows from Theorems 4.1 and 5.9. Now, let us show that the operator

$$\mathcal{L} : C^{1,\beta}(S) \longrightarrow C^{0,\beta}(S) \tag{6.15}$$

is invertible. Apply the operator $\mathcal{H}$ to both sides of equation (6.14) to obtain the singular integral equation

$$\mathcal{H}\mathcal{L}h = (-4^{-1}I_8 + \mathcal{N}^2)h = (-2^{-1}I_8 + \mathcal{N})(2^{-1}I_8 + \mathcal{N})h = \mathcal{H}F, \quad (6.16)$$

where $\mathcal{H}F \in C^{1,\beta}(S)$ and the operator

$$\mathcal{H}\mathcal{L} = (-2^{-1}I_8 + \mathcal{N})(2^{-1}I_8 + \mathcal{N}) : C^{1,\beta}(S) \longrightarrow C^{1,\beta}(S) \quad (6.17)$$

is a singular operator of normal type with zero index. Again, applying the arguments as in [33] we can shown that (6.17) is invertible, and therefore we can write

$$h = (2^{-1}I_8 + \mathcal{N})^{-1}(-2^{-1}I_8 + \mathcal{N})^{-1}\mathcal{H}F.$$

Note that the operators $(-2^{-1}I_8 + \mathcal{N})$ and $(2^{-1}I_8 + \mathcal{N})$ commute.

Let us show that (6.14) and (6.16) are equivalent integral equations. Indeed, if $h \in C^{1,\beta}(S)$ is e solution to equation (6.14), then it will be solution to equation (6.16) as well. Introduce notation

$$\psi := (\mathcal{L}h - F) \in C^{0,\beta}(S). \quad (6.18)$$

Then equation (6.16) can be rewritten as

$$\mathcal{H}\psi = 0 \ \text{ on } \ S. \quad (6.19)$$

Construct the single-layer potential $V(\psi)$ with the density ψ determined by equation (6.18). Dirichlet's problem $[V(\psi)(z)]^{\pm} = 0$, $z \in S$, corresponds to the equation (6.19). As this problem has only the trivial solution, we have $V(\psi)(x) = 0$, $x \in \Omega^{\pm}$, from which we obtain that $\psi(z) = 0$, $z \in \Omega^{\pm}$, i.e. h is a solution to equation (6.14) and hence the operator (6.15) is invertible. □

Corollary 6.8. *The solution to the problem* $(\mathrm{II}^{(\sigma)})^{\pm}$ *is represented in the following form:*

$$U(x) = W(\mathcal{L}^{-1}F)(x), \quad x \in \Omega^{\pm},$$

where $F(z) = [P(\partial_z, n(z))U(z)]^{\pm}$, $z \in S$.

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Authors' address:

Department of Mathematics, Georgian Technical University, 77 Kostava St., Tbilisi 0175, Georgia.

E-mail: lgiorgashvili@gmail.com; gkarseladze@gmail.com; g.saduni@mail.ru; zaza-ude@hotmail.com

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L. Giorgashvili, D. Natroshvili, and Sh. Zazashvili

TRANSMISSION AND INTERFACE CRACK PROBLEMS OF THERMOELASTICITY FOR HEMITROPIC SOLIDS

Abstract. The purpose of this paper is to investigate basic transmission and interface crack problems for the differential equations of the theory of elasticity of hemitropic materials with regard to thermal effects. We consider the so called pseudo-oscillation equations corresponding to the time harmonic dependent case. Applying the potential method and the theory of pseudodifferential equations first we prove uniqueness and existence theorems of solutions to the Dirichlet and Neumann type transmission-boundary value problems for piecewise homogeneous hemitropic composite bodies. Afterwards we investigate the interface crack problems and study regularity properties of solution.

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რეზიუმე. სტატიის მიზანია ჰემიტროპული სხეულების დრეკადობის თეორიის დიფერენციალური განტოლებებისათვის ძირითადი საკონტაქტო და ბზარის ტიპის ამოცანების გამოკვლევა თერმული ეფექტების გათვალისწინებით. განხილულია ე. წ. ფსევდო-რხევის განტოლებები, რომლებიც შეესაბამება დროზე ჰარმონიულად დამოკიდებულ შემთხვევას. პოტენციალთა მეთოდისა და ფსევდოდიფერენციალურ განტოლებათა თეორიის გამოყენებით ჯერ დამტკიცებულია დირიხლესა და ნეიმანის ტიპის სასაზღვრო-საკონტაქტო ამოცანების ამონახსნების არსებობისა და ერთადერთობის თეორემები უბნობრივ ერთგვაროვანი ჰემიტროპული სხეულებისათვის, ხოლო შემდეგ გამოკვლეულია ბზარის ტიპის ამოცანა, როდესაც ბზარი მდებარეობს საკონტაქტო ზედაპირზე, და შესწავლილია ამონახსნის რეგულარობა.

1. Introduction

Technological and industrial developments, and also recent important progress in biological and medical sciences require the use of more general and refined models for elastic bodies. In a generalized solid continuum, the usual displacement field has to be supplemented by a microrotation field. Such materials are called micropolar or Cosserat solids. They model continua with a complex inner structure whose material particles have 6 degree of freedom (3 displacement components and 3 microrotation components). Recall that the classical elasticity theory allows only 3 degrees of freedom (3 displacement components).

Experiments have shown that micropolar materials possess quite different properties in comparison with the classical elastic materials (see, e.g., [3], [4], [7], [15], [23], [25], [26], and the references therein). For example, in non-centrosymmetric micropolar materials the propagation of left-handed and right-handed elastic waves is observed. Moreover, the twisting behaviour under an axial stress is a purely hemitropic (chiral) phenomenon and has no counterpart in classical elasticity. Such solids are called *hemitropic non-centrosymmetric, acentric*, or *chiral*. Throughout the paper we use the term *hemitropic*.

Hemitropic solids are not isotropic with respect to inversion, i.e., they are isotropic with respect to all proper orthogonal transformations but not with respect to mirror reflections.

Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules - DNA, as well as on a large scale, as in composites with helical or screw–shaped inclusions, certain types of nanotubes, fabricated structures such as foams, chiral sculptured thin films and twisted fibers. For more details see the references [3], [4], [14], [15], [20], [23], [24], [26]–[30], [34], [35], [46]–[50], [53], [56], [57].

Mathematical models describing the chiral properties of elastic hemitropic materials have been proposed by Aéro and Kuvshinski [3], [4] (for historical notes see also [14], [15], [46], and the references therein).

In the present paper we deal with the model of micropolar elasticity for hemitropic solids when the thermal effects are taken into consideration.

In the mathematical theory of hemitropic thermoelasticity there are introduced the asymmetric force stress tensor and couple stress tensor, which are kinematically related with the asymmetric strain tensor, torsion (curvature) tensor and the temperature function via the constitutive equations. All these quantities along with the heat flux vector are expressed in terms of the components of the displacement and microrotation vectors, and the temperature function. In turn, the displacement and microrotation vectors, and the temperature satisfy a coupled complex system of second order partial differential equations of dynamics. When the mechanical and thermal characteristics (displacements, microrotations, temperature, body force, body couple vectors, and heat source) do not depend on the time variable t we have the differential equations of statics. If time dependence is harmonic (i.e., the pertinent fields are represented as the product of the time dependent exponential function $\exp\{-i\sigma t\}$ and a function of the spatial variable $x \in \mathbb{R}^3$) then we have the steady state oscillation equations. Here σ is a real frequency parameter. Note that if $\sigma = 0$, then we obtain the equations of statics. If $\sigma = \sigma_1 + i\sigma_2$ is a complex parameter, then we have the so called *pseudo-oscillation equations* (which are related to the dynamical equations via the Laplace transform). All the above equations generate a strongly elliptic, formally non-self-adjoint 7×7 matrix differential operator.

The Dirichlet, Neumann and mixed type boundary value problems (BVP) corresponding to this model are well investigated for homogeneous bodies of arbitrary shape and the uniqueness and existence theorems are proved, and regularity results for solutions are established by the potential method, as well as by variational methods (see [39]–[43] and the references therein).

The main goal of our investigation is to study the Dirichlet and Neumann type transmission and interface crack problems of the theory of elasticity with regard to thermal effects for piecewise homogeneous hemitropic composite bodies of arbitrary geometrical shape. We develop the boundary integral equations method to obtain the existence and uniqueness results in various Hölder ($C^{k,\alpha}$), Sobolev–Slobodetski (W_p^s) and Besov ($B_{p,q}^s$) functional spaces. We study regularity properties of solutions at the crack edges and characterize the corresponding stress singularity exponents.

2. Field Equations

2.1. **Constitutive relations and basic differential equations.** Denote by $\mathbb{R}^3$ the three-dimensional Euclidean space and let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with a boundary $S := \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$. Further, let $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. We assume that $\overline{\Omega} \in \{\overline{\Omega^+}, \overline{\Omega^-}\}$ is filled with an elastic material possessing the hemitropic properties.

Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the displacement vector and the microrotation vector, respectively. By ϑ we denote the temperature increment – temperature distribution function. Here and in what follows the symbol $(\cdot)^\top$ denotes transposition. Note that the microrotation vector in the hemitropic elasticity theory is kinematically distinct from the macrorotation vector $\frac{1}{2}$ curl u.

Throughout the paper the central dot denotes the real scalar product, i.e., $a \cdot b := \sum\limits_{k=1}^{N} a_k b_k$ for complex-valued N-dimensional vectors $a, b \in \mathbb{C}^N$.

The force stress $\{\tau_{pq}\}$ and the couple stress $\{\mu_{pq}\}$ tensors in the linear theory of hemitropic thermoelasticity read as follows (the constitutive equations)

$$\tau_{pq} = \tau_{pq}(U) := (\mu + \alpha)\partial_p u_q + (\mu - \alpha)\partial_q u_p + \lambda\delta_{pq} \operatorname{div} u + \delta\delta_{pq} \operatorname{div} \omega +$$

$$+(\varkappa+\nu)\partial_p\omega_q+(\varkappa-\nu)\partial_q\omega_p-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k-\delta_{pq}\eta\vartheta, \tag{2.1}$$

$$\mu_{pq}=\mu_{pq}(U):=\delta\delta_{pq}\operatorname{div}u+(\varkappa+\nu)\Big[\partial_p u_q-\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k\Big]+\beta\delta_{pq}\operatorname{div}\omega+$$

$$+(\varkappa-\nu)\Big[\partial_q u_p-\sum_{k=1}^{3}\varepsilon_{qpk}\omega_k\Big]+(\gamma+\varepsilon)\partial_p\omega_q+(\gamma-\varepsilon)\partial_q\omega_p-\delta_{pq}\zeta\vartheta, \tag{2.2}$$

where $U=(u,\omega,\vartheta)^\top$, δ_{pq} is the Kronecker delta, ε_{pqk} is the permutation (Levi–Civitá) symbol, and α, β, γ, δ, λ, μ, ν, $\varkappa$, and ε are the material constants, while $\eta>0$ and $\zeta>0$ are constants describing the coupling of mechanical and thermal fields (see [3], [14]), $\partial=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$, $j=1,2,3$.

The linear equations of dynamics of the thermoelasticity theory of hemitropic materials have the form (see, e.g., [14])

$$\sum_{p=1}^{3}\partial_p\tau_{pq}(x,t)+\varrho F_q(x,t)=\varrho\partial_{tt}^2u_q(x,t),\quad q=1,2,3,$$

$$\sum_{p=1}^{3}\partial_p\mu_{pq}(x,t)+\sum_{l,r=1}^{3}\varepsilon_{qlr}\tau_{lr}(x,t)+\varrho G_q(x,t)=\mathcal{I}\partial_{tt}^2\omega_q(x,t),\quad q=1,2,3,$$

$$\kappa'\Delta\vartheta(x,t)-\eta\partial_t\operatorname{div}u(x,t)-\zeta\partial_t\operatorname{div}\omega(x,t)-\kappa''\partial_t\vartheta(x,t)+Q(x,t)=0,$$

where t is the time variable, $\partial_t=\partial/\partial t$, $F=(F_1,F_2,F_3)^\top$ and $G=(G_1,G_2,G_3)^\top$ are the body force and body couple vectors per unit volume, Q is the heat source density, ϱ is the mass density of the elastic material, and $\mathcal{I}$ is a constant characterizing the so called spin torque corresponding to the microrotations (i.e., the moment of inertia per unit volume); here $\kappa'=\frac{\lambda_0}{T_0}$ and $\kappa''=\frac{c_0}{T_0}$, where $\lambda_0>0$ is the heat conduction coefficient, $T_0>0$ is an initial natural state temperature and $c_0>0$ is the specific heat coefficient.

Using the relations (2.1)–(2.2) we can rewrite the above dynamic equations as

$$\begin{gathered}(\mu+\alpha)\Delta u(x,t)+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}u(x,t)+(\varkappa+\nu)\Delta\omega(x,t)+\\+(\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}\omega(x,t)+2\alpha\operatorname{curl}\omega(x,t)-\\-\eta\operatorname{grad}\vartheta(x,t)+\varrho F(x,t)=\varrho\partial_{tt}^2u(x,t),\\(\varkappa+\nu)\Delta u(x,t)+(\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}u(x,t)+2\alpha\operatorname{curl}u(x,t)+\\+(\gamma+\varepsilon)\Delta\omega(x,t)+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x,t)+4\nu\operatorname{curl}\omega(x,t)-\\-4\alpha\omega(x,t)-\zeta\operatorname{grad}\vartheta(x,t)+\varrho G(x,t)=\mathcal{I}\partial_{tt}^2\omega(x,t),\\\kappa'\Delta\vartheta(x,t)-\eta\partial_t\operatorname{div}u(x,t)-\zeta\partial_t\operatorname{div}\omega(x,t)-\kappa''\partial_t\vartheta(x,t)+Q(x,t)=0,\end{gathered}$$

where Δ is the Laplace operator.

If all the quantities involved in these equations are harmonic time dependent, i.e., $u(x,t) = u(x)e^{-it\sigma}$, $\omega(x,t) = \omega(x)e^{-it\sigma}$, $\vartheta(x,t) = \vartheta(x)e^{-it\sigma}$, $F(x,t) = F(x)e^{-it\sigma}$, $G(x,t) = G(x)e^{-it\sigma}$ and $Q(x,t) = Q(x)e^{-it\sigma}$ with $\sigma \in \mathbb{R}$ and $i = \sqrt{-1}$, we obtain the *steady state oscillation equations* of the hemitropic theory of thermoelasticity:

$$\begin{gathered}
(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div} u(x) + \varrho\sigma^2 u(x)+\\
+(\varkappa+\nu)\Delta\omega(x) + (\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}\omega(x) + 2\alpha\operatorname{curl}\omega(x)-\\
-\eta\operatorname{grad}\vartheta(x) = -\varrho F(x),\\
(\varkappa+\nu)\Delta u(x) + (\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div} u(x) + 2\alpha\operatorname{curl} u(x)+\\
+(\gamma+\varepsilon)\Delta\omega(x) + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x) + 4\nu\operatorname{curl}\omega(x)-\\
-\zeta\operatorname{grad}\vartheta(x) + (\mathcal{I}\sigma^2 - 4\alpha)\omega(x) = -\varrho G(x),\\
(\kappa'\Delta + i\sigma\kappa'')\vartheta(x) + i\eta\sigma\operatorname{div} u(x) + i\zeta\sigma\operatorname{div}\omega(x) = -Q(x),
\end{gathered} \tag{2.3}$$

here u, ω, F, and G are complex-valued vector functions, while ϑ and Q are complex-valued scalar functions, and σ is a frequency parameter.

If $\sigma = \sigma_1 + i\sigma_2$ is a complex parameter with $\sigma_2 \neq 0$, then the above equations are called the *pseudo–oscillation equations*, while for $\sigma = 0$ they represent the *equilibrium equations of statics.*

Let us introduce the block wise 7×7 matrix differential operator corresponding to the system (2.3):

$$L(\partial,\sigma) := \begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma) & L^{(5)}(\partial,\sigma)\\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma) & L^{(6)}(\partial,\sigma)\\ L^{(7)}(\partial,\sigma) & L^{(8)}(\partial,\sigma) & L^{(9)}(\partial,\sigma)\end{bmatrix}_{7\times 7}, \tag{2.4}$$

where

$$\begin{gathered}
L^{(1)}(\partial,\sigma) := \big[(\mu+\alpha)\Delta + \varrho\sigma^2\big] I_3 + (\lambda+\mu-\alpha)Q(\partial),\\
L^{(2)}(\partial,\sigma) = L^{(3)}(\partial,\sigma) := (\varkappa+\nu)\Delta I_3 + (\delta+\varkappa-\nu)Q(\partial) + 2\alpha R(\partial),\\
L^{(4)}(\partial,\sigma) := [(\gamma+\varepsilon)\Delta + (\mathcal{I}\sigma^2 - 4\alpha)] I_3 + (\beta+\gamma-\varepsilon)Q(\partial) + 4\nu R(\partial),\\
L^{(5)}(\partial,\sigma) := -\eta\nabla^\top, \quad L^{(6)}(\partial,\sigma) := -\zeta\nabla^\top, \quad L^{(7)}(\partial,\sigma) := i\eta\sigma\nabla,\\
L^{(8)}(\partial,\sigma) := i\zeta\sigma\nabla, \quad L^{(9)}(\partial,\sigma) := \kappa'\Delta + i\sigma\kappa''.
\end{gathered}$$

Here and in the sequel I_k stands for the $k\times k$ unit matrix and

$$R(\partial) := [-\varepsilon_{kjl}\partial_l]_{3\times 3}, \;\; Q(\partial) := [\partial_k\partial_j]_{3\times 3}, \;\; \nabla := [\partial_1, \partial_2, \partial_3]. \tag{2.5}$$

Throughout the paper summation over repeated indexes is meant from one to three if not otherwise stated. It is easy to see that for $v = (v_1, v_2, v_3)^\top$

$$\begin{gathered}
R(\partial)v = \operatorname{curl} v, \quad Q(\partial)v = \operatorname{grad}\operatorname{div} v, \\
R(-\partial) = -R(\partial) = [R(\partial)]^\top, \quad Q(\partial)R(\partial) = R(\partial)Q(\partial) = 0,\\
Q(\partial) = [Q(\partial)]^\top, \quad [R(\partial)]^2 = Q(\partial) - \Delta I_3, \quad [Q(\partial)]^2 = Q(\partial)\Delta.
\end{gathered} \tag{2.6}$$

Due to the above notation, the system (2.3) can be rewritten in matrix form as

$$L(\partial,\sigma)U(x)=\Phi(x),\quad U=(u,\omega,\vartheta)^\top,\ \ \Phi=(-\varrho F,-\varrho G,-Q)^\top.$$

Note that $L(\partial,\sigma)$ is not formally self-adjoint. Further, let us remark that the differential operator

$$L(\partial):=L(\partial,0) \tag{2.7}$$

corresponds to the static equilibrium case, while the formally self-adjoint differential operator

$$L_0(\partial):=\begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial) & [0]_{3\times1}\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial) & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & \kappa'\Delta \end{bmatrix}_{7\times7} \tag{2.8}$$

with

$$\begin{aligned} L_0^{(1)}(\partial)&:=(\mu+\alpha)\Delta I_3+(\lambda+\mu-\alpha)Q(\partial),\\ L_0^{(2)}(\partial)=L_0^{(3)}(\partial)&:=(\varkappa+\nu)\Delta I_3+(\delta+\varkappa-\nu)Q(\partial),\\ L_0^{(4)}(\partial)&:=(\gamma+\varepsilon)\Delta I_3+(\beta+\gamma-\varepsilon)Q(\partial), \end{aligned}$$

represents the principal homogeneous part of the operators (2.4) and (2.7). Denote

$$\begin{aligned} \widetilde{L}(\partial,\sigma)&:=\begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma)\\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma)\end{bmatrix}_{6\times6},\\ \widetilde{L}_0(\partial)&:=\begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial)\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial)\end{bmatrix}_{6\times6}. \end{aligned} \tag{2.9}$$

The operators (2.9) correspond to the equations of hemitropic elasticity when thermal effects are not taken into consideration ([40]). It is clear that the operator $L_0(\partial)$, $\widetilde{L}(\partial,\sigma)$ and $\widetilde{L}_0(\partial)$ are formally self-adjoint.

2.2. **Generalized stress operators.** The components of the force stress vector $\tau^{(n)}$ and the couple stress vector $\mu^{(n)}$, acting on a surface element with a unite normal vector $n=(n_1,n_2,n_3)$, read as

$$\tau^{(n)}=\big(\tau_1^{(n)},\tau_2^{(n)},\tau_3^{(n)}\big)^\top,\quad \mu^{(n)}=\big(\mu_1^{(n)},\mu_2^{(n)},\mu_3^{(n)}\big)^\top,$$

where

$$\tau_q^{(n)}=\sum_{p=1}^3\tau_{pq}n_p,\quad \mu_q^{(n)}=\sum_{p=1}^3\mu_{pq}n_p,\ \ q=1,2,3.$$

It is also well known that the normal component of the heat flux vector across a surface element with a normal vector $n=(n_1,n_2,n_3)$ is expressed with the help of the normal derivative of the temperature function

$$\kappa' n\cdot\nabla\vartheta=\kappa'\sum_{p=1}^3 n_p\partial_p\vartheta=\kappa'\partial_n\vartheta,$$

where $\partial_n = \partial/\partial n$ denotes the usual normal derivative.

Throughout the paper we will refer the six vector $(\tau^{(n)}, \mu^{(n)})^\top$ as the *mechanical thermo-stress vector*, while the seven vector $(\tau^{(n)}, \mu^{(n)}, \kappa'\partial_n\vartheta)^\top$ as *the generalized thermo-stress vector*.

Let us introduce the generalized thermo-stress operators

$$\mathcal{T}(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta n^\top \end{bmatrix}_{6\times 7}, \tag{2.10}$$

$$\mathcal{P}(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\partial_n \end{bmatrix}_{7\times 7}, \tag{2.11}$$

where

$$T^{(j)} = [T^{(j)}_{pq}]_{3\times 3}, \;\; j=\overline{1,4}, \;\;\; n^\top = (n_1,n_2,n_3)^\top,$$

$$T^{(1)}_{pq}(\partial,n) = (\mu+\alpha)\delta_{pq}\partial_n + (\mu-\alpha)n_q\partial_p + \lambda n_p\partial_q,$$

$$T^{(2)}_{pq}(\partial,n) = (\varkappa+\nu)\delta_{pq}\partial_n + (\varkappa-\nu)n_q\partial_p + \delta n_p\partial_q - 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,$$

$$T^{(3)}_{pq}(\partial,n) = (\varkappa+\nu)\delta_{pq}\partial_n + (\varkappa-\nu)n_q\partial_p + \delta n_p\partial_q,$$

$$T^{(4)}_{pq}(\partial,n) = (\gamma+\varepsilon)\delta_{pq}\partial_n + (\gamma-\varepsilon)n_q\partial_p + \beta n_p\partial_q - 2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.$$

One can easily check that for an arbitrary vector $U=(u,\omega,\vartheta)^\top$,

$$\mathcal{T}(\partial,n)U = \big(\tau^{(n)},\mu^{(n)}\big)^\top, \quad \mathcal{P}(\partial,n)U = \big(\tau^{(n)},\mu^{(n)},\kappa'\partial_n\vartheta\big)^\top,$$

i.e., the six vector $\mathcal{T}(\partial,n)U$ corresponds to the mechanical thermo-stress vector and the seven vector $\mathcal{P}(\partial,n)U$ corresponds to the generalized thermo-stress vector.

Further, let us introduce the boundary differential operators which occur in Green's formulas and are associated with the adjoint differential operator $L^*(\partial,\sigma) := L^\top(-\partial,\sigma)$:

$$\begin{gathered} \mathcal{T}^*(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -i\sigma\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -i\sigma\zeta n^\top \end{bmatrix}_{6\times 7}, \\ \mathcal{P}^*(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -i\sigma\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -i\sigma\zeta n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\partial_n \end{bmatrix}_{7\times 7}. \end{gathered} \tag{2.12}$$

It is easy to see that the principal homogeneous parts of the operators $\mathcal{T}(\partial,n)$ and $\mathcal{T}^*(\partial,n)$ are the same, as well as the principal homogeneous parts of the operators $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$.

Note that when the thermal effects are not taken into consideration the hemitropic stress operator reads as [40]

$$T(\partial,n)=\begin{bmatrix}T^{(1)}(\partial,n) & T^{(2)}(\partial,n)\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n)\end{bmatrix}_{6\times 6}. \tag{2.13}$$

Evidently, for $U=(u,\omega,0)^\top$ and $\widetilde{U}=(u,\omega)^\top$ we have $\mathcal{T}(\partial,n)U=T(\partial,n)\widetilde{U}$ in view of (2.10) and (2.13).

2.3. **Green's identities.** For vector functions

$$\widetilde{U}=(u,\omega)^\top,\widetilde{U}'=(u',\omega')^\top\in[C^2(\overline{\Omega^+})]^6,$$

we have the following Green formula [40]

$$\int\limits_{\Omega^+}\Big[\widetilde{U}'\cdot\widetilde{L}(\partial,0)\widetilde{U}+E(\widetilde{U}',\widetilde{U})\Big]\,dx=\int\limits_{\partial\Omega^+}\{\widetilde{U}'\}^+\cdot\{T(\partial,n)\widetilde{U}\}^+\,dS, \tag{2.14}$$

where the operators $\widetilde{L}(\partial,0)$ and $T(\partial,n)$ are given by (2.9) and (2.13) respectively, $\partial\Omega^+$ is a piecewise smooth manifold, n is the outward unit normal vector to $\partial\Omega^+$, the symbols $\{\,\cdot\,\}^\pm$ denote the limiting values on $\partial\Omega^\pm$ from $\Omega^\pm$ respectively, $E(\cdot\,,\,\cdot)$ is the so called *energy bilinear form*,

$$\begin{aligned}E(\widetilde{U}',\widetilde{U})=E(\widetilde{U},\widetilde{U}')=\sum_{p,q=1}^{3}\Big\{&(\mu+\alpha)u'_{pq}u_{pq}+(\mu-\alpha)u'_{pq}u_{qp}+\\&+(\varkappa+\nu)(u'_{pq}\omega_{pq}+\omega'_{pq}u_{pq})+(\varkappa-\nu)(u'_{pq}\omega_{qp}+\omega'_{pq}u_{qp})+(\gamma+\varepsilon)\omega'_{pq}\omega_{pq}+\\&+(\gamma-\varepsilon)\omega'_{pq}\omega_{qp}+\delta(u'_{pp}\omega_{qq}+\omega'_{qq}u_{pp})+\lambda u'_{pp}u_{qq}+\beta\omega'_{pp}\omega_{qq}\Big\}\end{aligned} \tag{2.15}$$

with

$$u_{pq}=\partial_p u_q-\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k,\quad \omega_{pq}=\partial_p\omega_q,\quad p,q=1,2,3. \tag{2.16}$$

In what follows the over bar denotes complex conjugation. The necessary and sufficient conditions for the quadratic form $E(\widetilde{U},\overline{\widetilde{U}})$ to be positive definite with respect to the variables u_{pq} and ω_{pq}, read as (see [4], [14], [18])

$$\begin{gathered}\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ \lambda+2\mu>0,\ \ \mu\gamma-\varkappa^2>0,\ \ \alpha\varepsilon-\nu^2>0,\\ (\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2>0,\ \ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0,\\ (\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\ \ (\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0,\\ \mu\big[(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2\big]+(\lambda+\mu)(\mu\gamma-\varkappa^2)>0,\\ \mu\big[(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2\big]+(3\lambda+2\mu)(\mu\gamma-\varkappa^2)>0.\end{gathered}$$

Let us note that, if the condition $3\lambda+2\mu>0$ is fulfilled, which is very natural in the classical elasticity, then the above conditions are equivalent to the following simultaneous inequalities

$$\begin{gathered}\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ 3\lambda+2\mu>0,\ \ \mu\gamma-\varkappa^2>0,\\ \alpha\varepsilon-\nu^2>0,\ \ (\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\\ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0.\end{gathered}\tag{2.17}$$

For simplicity in what follows we assume that $3\lambda+2\mu>0$ and therefore the conditions (2.17) imply positive definiteness of the energy quadratic form $E(\widetilde{U},\overline{\widetilde{U}})$ defined by (2.15). From (2.17) it follows that

$$\begin{gathered}\gamma>0,\quad \varepsilon>0,\quad \lambda+\mu>0,\quad \beta+\gamma>0,\\ d_1:=(\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\\ d_2:=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0.\end{gathered}$$

Formula (2.15) can be rewritten as

$$\begin{aligned}E(\widetilde{U},\widetilde{U}')=&\ \frac{3\lambda+2\mu}{3}\Big(\operatorname{div}u+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega\Big)\Big(\operatorname{div}u'+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega'\Big)+\\ &+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\varkappa)^2}{3\lambda+2\mu}\Big)(\operatorname{div}\omega)(\operatorname{div}\omega')+\\ &+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)\operatorname{curl}\omega\cdot\operatorname{curl}\omega'+\\ &+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big]\times\\ &\qquad\times\Big[\frac{\partial u_k'}{\partial x_j}+\frac{\partial u_j'}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k'}{\partial x_j}+\frac{\partial\omega_j'}{\partial x_k}\Big)\Big]+\\ &+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big]\times\\ &\qquad\times\Big[\frac{\partial u_k'}{\partial x_k}-\frac{\partial u_j'}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k'}{\partial x_k}-\frac{\partial\omega_j'}{\partial x_j}\Big)\Big]+\\ &+\Big(\gamma-\frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{1}{2}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big(\frac{\partial\omega_k'}{\partial x_j}+\frac{\partial\omega_j'}{\partial x_k}\Big)+\\ &\qquad+\frac{1}{3}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big(\frac{\partial\omega_k'}{\partial x_k}-\frac{\partial\omega_j'}{\partial x_j}\Big)\Big]+\\ &+\alpha\Big(\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\Big)\cdot\Big(\operatorname{curl}u'+\frac{\nu}{\alpha}\operatorname{curl}\omega'-2\omega'\Big).\end{aligned}$$

In particular,

$$\begin{aligned}E(\widetilde{U},\overline{\widetilde{U}})=&\ \frac{3\lambda+2\mu}{3}\Big|\operatorname{div}u+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega\Big|^2+\\ &+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\varkappa)^2}{3\lambda+2\mu}\Big)|\operatorname{div}\omega|^2+\end{aligned}$$

$$+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\left|\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\Big)\right|^2+$$
$$+\frac{\mu}{3}\sum_{k,j=1}^{3}\left|\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\Big)\right|^2+$$
$$+\Big(\gamma-\frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{1}{2}\left|\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right|^2+\frac{1}{3}\left|\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right|^2\right]+$$
$$+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)|\operatorname{curl}\omega|^2+\alpha\Big|\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\Big|^2.$$

We formulate here the following technical lemma.

Lemma 2.1. *Let $\widetilde{U}=(u,\omega)^\top\in[C^1(\Omega^+)]^6$ and $E(\widetilde{U},\overline{\widetilde{U}})=0$ in Ω^+. Then*

$$u(x)=[a\times x]+b,\quad \omega(x)=a,\ \ x\in\Omega^+, \tag{2.18}$$

where a and b are arbitrary three-dimensional constant complex vectors.

Moreover,

(i) *for an arbitrary vector $\widetilde{U}=(u,\omega)^\top$ defined by formulas* (2.18) *and an arbitrary unit vector $n=(n_1,n_2,n_3)$ the generalized hemitropic stress vector $T(\partial,n)\widetilde{U}$ vanishes identically, i.e., $T(\partial,n)\widetilde{U}(x)=0$ for all $x\in\Omega^+$.*

(ii) *for an arbitrary vector $U:=(\widetilde{U},0)^\top=(u,\omega,0)^\top$, where u and ω are given by formulas* (2.18), *and for an arbitrary unit vector $n=(n_1,n_2,n_3)$ the generalized hemitropic thermo-stress vector $\mathcal{P}(\partial,n)U$ vanishes identically, i.e., $\mathcal{P}(\partial,n)U(x)=0$ for all $x\in\Omega^+$.*

Proof. The first part of the lemma is shown in [40]. The second part easily follows from the first part and from the formulas (2.10), (2.11), (2.13). □

Throughout the paper L_p, W_p^s, H_p^s, and $B_{p,q}^s$ (with $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$) denote the well–known Lebesgue, Sobolev–Slobodetski, Bessel potential, and Besov spaces, respectively (see, e.g., [54], [55], [31]). We recall that $H_2^s=W_2^s=B_{2,2}^s$, $W_p^t=B_{p,p}^t$, and $H_p^k=W_p^k$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Further, let $\mathcal{M}_0$ be a Lipschitz surface without boundary. For a Lipschitz sub-manifold $\mathcal{M}\subset\mathcal{M}_0$ we denote by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ the subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\widetilde{H}_p^s(\mathcal{M})=\Big\{g:\ g\in H_p^s(\mathcal{M}_0),\ \operatorname{supp}g\subset\overline{\mathcal{M}}\Big\},$$
$$\widetilde{B}_{p,q}^s(\mathcal{M})=\Big\{g:\ g\in B_{p,q}^s(\mathcal{M}_0),\ \operatorname{supp}g\subset\overline{\mathcal{M}}\Big\},$$

while $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ denote the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$H_p^s(\mathcal{M})=\big\{r_{\mathcal{M}}f:\ f\in H_p^s(\mathcal{M}_0)\big\},\quad B_{p,q}^s(\mathcal{M})=\big\{r_{\mathcal{M}}f\ :\ f\in B_{p,q}^s(\mathcal{M}_0)\big\}.$$

Here $r_{\mathcal{M}}$ is the restriction operator.

If $\widetilde{U} = \widetilde{U}^{(1)} + i\widetilde{U}^{(2)}$ is a complex–valued vector, where $\widetilde{U}^{(j)} = (u^{(j)}, \omega^{(j)})^\top$ $(j = 1, 2)$ are real–valued vectors, then

$$E(\widetilde{U}, \overline{\widetilde{U}}) = E(\widetilde{U}^{(1)}, \widetilde{U}^{(1)}) + E(\widetilde{U}^{(2)}, \widetilde{U}^{(2)}),$$

and, due to the positive definiteness of the energy form for real–valued vector functions, we have

$$E(\widetilde{U}, \overline{\widetilde{U}}) \geq c^* \sum_{p,q=1}^{3} \Big[(u_{pq}^{(1)})^2 + (u_{pq}^{(2)})^2 + (\omega_{pq}^{(1)})^2 + (\omega_{pq}^{(2)})^2 \Big],$$

where c^* is a positive constant depending only on the material constants, and $u_{pq}^{(j)}$ and $\omega_{pq}^{(j)}$ are defined by formulae (2.16) with $u^{(j)}$ and $\omega^{(j)}$ for u and ω.

From the positive definiteness of the energy form $E(\cdot, \cdot)$ with respect to the variables (2.16) it follows that there exist positive constants c_1 and c_2 such that for an arbitrary real–valued vector $\widetilde{U} \in [C^1(\overline{\Omega^+})]^6$

$$\widetilde{\mathcal{B}}(\widetilde{U}, \widetilde{U}) := \int\limits_{\Omega^+} E(\widetilde{U}, \widetilde{U})\, dx \geq$$

$$\geq c_1 \int\limits_{\Omega^+} \bigg\{ \sum_{p,q=1}^{3} \big[(\partial_p u_q)^2 + (\partial_p \omega_q)^2 \big] + \sum_{p=1}^{3} [u_p^2 + \omega_p^2] \bigg\}\, dx -$$

$$- c_2 \int\limits_{\Omega^+} \sum_{p=1}^{3} [u_p^2 + \omega_p^2]\, dx,$$

i.e., the following Korn's type inequality holds (cf. [17, Part I, § 12], [32, Ch. 10])

$$\widetilde{\mathcal{B}}(\widetilde{U}, \widetilde{U}) \geq c_1 \|\widetilde{U}\|^2_{[H_2^1(\Omega^+)]^6} - c_2 \|\widetilde{U}\|^2_{[H_2^0(\Omega^+)]^6}, \tag{2.19}$$

where $\|\cdot\|_{[H_2^s(\Omega^+)]^6}$ denotes the norm in the Sobolev space $[H_2^s(\Omega^+)]^6$. Clearly, the counterpart of (2.19) holds for an arbitrary complex-valued vector $\widetilde{U} \in [H_2^1(\Omega^+)]^6$ as well,

$$\widetilde{\mathcal{B}}(\widetilde{U}, \overline{\widetilde{U}}) \geq c_1 \|\widetilde{U}\|^2_{[H_2^1(\Omega^+)]^6} - c_2 \|\widetilde{U}\|^2_{[H_2^0(\Omega^+)]^6}. \tag{2.20}$$

These results imply that the differential operators $\widetilde{L}(\partial, \sigma)$ and $\widetilde{L}_0(\partial)$ are *strongly elliptic* and the following inequality (*the accretivity condition*) holds (cf., e.g., [17, Part I, § 5], [32, Ch. 4, Lemma 4.5])

$$c_2' |\xi|^2 |\eta|^2 \geq \widetilde{L}_0(\xi)\eta \cdot \eta = \sum_{k,j=1}^{6} \widetilde{L}_{0kj}(\xi) \eta_j \overline{\eta_k} \geq c_1' |\xi|^2 |\eta|^2 \tag{2.21}$$

with some constants $c_k' > 0$, $k = 1, 2$, for arbitrary $\xi \in \mathbb{R}^3$ and arbitrary complex vector $\eta \in \mathbb{C}^6$.

Consequently, in view of (2.8) and (2.21) the differential operator $L(\partial,\sigma)$ is strongly elliptic as well, since

$$C_2'|\xi|^2|\eta|^2 \geq L_0(\xi)\eta\cdot\overline{\eta} = \sum_{k,j=1}^{6} L_{0kj}(\xi)\eta_j\overline{\eta_k} \geq C_1'|\xi|^2|\eta|^2$$

with some constants $C_k' > 0$, $k = 1, 2$, for arbitrary $\xi \in \mathbb{R}^3$ and for arbitrary complex vector $\eta \in \mathbb{C}^7$.

Now let $U = (\widetilde{U},\vartheta)^\top = (u,\omega,\vartheta)^\top$ and $U' = (\widetilde{U}',\vartheta')^\top = (u',\omega',\vartheta')^\top$ be vector functions of the class $[C^2(\overline{\Omega^+})]^7$. With the help of relation (2.14) and standard manipulations we can show that the following Green's formulas hold

$$\begin{aligned}\int\limits_{\Omega^+} U'\cdot L(\partial,\sigma)U\,dx = \int\limits_{\partial\Omega^+}\{U'\}^+\cdot\big\{\mathcal{P}(\partial,n)U\big\}^+\,dS - \\ -\int\limits_{\Omega^+}\Big[E(\widetilde{U}',\widetilde{U}) - \varrho\sigma^2 u'\cdot u - \mathcal{I}\sigma^2\omega'\cdot\omega - \eta\vartheta\,\mathrm{div}\,u' - \zeta\vartheta\,\mathrm{div}\,\omega' - \\ - i\eta\sigma\vartheta'\,\mathrm{div}\,u - i\zeta\sigma\vartheta'\,\mathrm{div}\,\omega - i\sigma\kappa''\vartheta\vartheta' + \kappa'\,\mathrm{grad}\,\vartheta'\cdot\mathrm{grad}\,\vartheta\Big]\,dx,\end{aligned} \tag{2.22}$$

$$\begin{aligned}\int\limits_{\Omega^+}\Big[U'\cdot L(\partial,\sigma)U - L^*(\partial,\sigma)U'\cdot U\Big]\,dx = \\ = \int\limits_{\partial\Omega^+}\Big[\{U'\}^+\cdot\big\{\mathcal{P}(\partial,n)U\big\}^+ - \big\{\mathcal{P}^*(\partial,n)U'\big\}^+\cdot\{U\}^+\Big]\,dS,\end{aligned} \tag{2.23}$$

where $L^*(\partial,\sigma) = L^\top(-\partial,\sigma)$ is the operator formally adjoint to $L(\partial,\sigma)$, the differential operators $L(\partial,\sigma)$, $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$ are defined by (2.4), (2.11) and (2.12) respectively. The proof of (2.22) and (2.23) easily follows from (2.14) in view of the identity

$$\begin{aligned}U'\cdot L(\partial,\sigma)U - \widetilde{U}'\cdot\widetilde{L}(\partial,0)\widetilde{U} = \varrho\sigma^2 u'\cdot u - \eta\,\mathrm{grad}\,\vartheta\cdot u' + \mathcal{I}\sigma^2\omega'\cdot\omega - \\ - \zeta\,\mathrm{grad}\,\vartheta\cdot\omega' + \kappa'\vartheta'\Delta\vartheta + i\eta\sigma\vartheta'\,\mathrm{div}\,u + i\sigma\zeta\vartheta'\,\mathrm{div}\,\omega + i\sigma\kappa''\vartheta\vartheta'.\end{aligned}$$

By the standard limiting approach, Green's formula (2.22) can be extended to Lipschitz domains (see, e.g., [45], [32]) and to the case of complex–valued vector functions $U \in [W_p^1(\Omega^+)]^7$ and $U' \in [W_{p'}^1(\Omega^+)]^7$ with $1/p + 1/p' = 1$, $1 < p < \infty$, and $L(\partial,\sigma)U \in [L_p(\Omega^+)]^7$ (cf. [31], [10], [32])

$$\begin{aligned}\Big\langle\{U'\}^+,\big\{\mathcal{P}(\partial,n)U\big\}^+\Big\rangle_{\partial\Omega^+} = \int\limits_{\Omega^+} U'\cdot L(\partial,\sigma)U\,dx + \\ + \int\limits_{\Omega^+}\Big[E(\widetilde{U}',\widetilde{U}) - \varrho\sigma^2 u'\cdot u - \mathcal{I}\sigma^2\omega'\cdot\omega - \eta\vartheta\,\mathrm{div}\,u' - \zeta\vartheta\,\mathrm{div}\,\omega' - \\ - i\eta\sigma\vartheta'\,\mathrm{div}\,u - i\zeta\sigma\vartheta'\,\mathrm{div}\,\omega - i\sigma\kappa''\vartheta\vartheta' + \kappa'\,\mathrm{grad}\,\vartheta'\cdot\mathrm{grad}\,\vartheta\Big]\,dx,\end{aligned} \tag{2.24}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^+}$ denotes the duality between the spaces $[B^{\frac{1}{p}}_{p,p}(\partial\Omega^+)]^7$ and $[B^{-\frac{1}{p}}_{p',p'}(\partial\Omega^+)]^7$, which extends the usual real L_2-scalar product, i.e., for $f,g \in [L_2(S)]^7$

$$\langle f,g\rangle_S = \sum_{k=1}^{7}\int_S f_k g_k\,dS = (f,g)_{[L_2(S)]^7}.$$

Clearly, the generalized trace functional $\{\mathcal{P}(\partial,n)U\}^+ \in [B^{-\frac{1}{p}}_{p,p}(\partial\Omega^+)]^7$ is well defined by the relation (2.24).

Let us introduce the sesquilinear form related to the operator $L(\partial,\sigma)$

$$\mathcal{B}(U,U') := \int_{\Omega^+}\Big[E(\widetilde{U},\overline{\widetilde{U}'}) - \varrho\sigma^2 u\cdot\overline{u'} - \mathcal{I}\sigma^2\omega\cdot\overline{\omega'} - \eta\vartheta\,\mathrm{div}\,\overline{u'} - \zeta\vartheta\,\mathrm{div}\,\overline{\omega'} - \\ - i\eta\sigma\overline{\vartheta'}\,\mathrm{div}\,u - i\zeta\sigma\overline{\vartheta'}\,\mathrm{div}\,\omega - i\sigma\kappa''\vartheta\overline{\vartheta'} + \kappa'\,\mathrm{grad}\,\vartheta\cdot\mathrm{grad}\,\overline{\vartheta'}\Big]\,dx. \tag{2.25}$$

With the help of (2.20) and (2.25) we derive the inequality

$$\mathrm{Re}\,\mathcal{B}(U,U) \geq C_1\|U\|^2_{[H^1_2(\Omega^+)]^7} - C_2\|U\|^2_{[H^0_2(\Omega^+)]^7}, \tag{2.26}$$

with some positive constants C_1 and C_2. This inequality plays a crucial role in the study of boundary value problems of the micropolar elasticity theory for hemitropic continua by means of the variational methods based on the well known Lax–Milgram theorem.

3. Formulation of Transmission Problems and Uniqueness Theorems

Let Ω be a bounded region in $\mathbb{R}^3$ with the smooth connected boundary $\partial\Omega = S_0$. Let $\overline{\Omega}_1 \subset \Omega$ be a sub-domain of Ω with a smooth simply connected boundary $\partial\Omega_1 = S_1 \subset \Omega$. Put $\Omega_0 := \Omega \setminus \overline{\Omega}_1$. In what follows, by $n(z)$, $z \in S_0 \cup S_1$, we denote the outward unit normal vector with respect to the domains Ω_1 and Ω, at the point z. We assume that $S_\ell \in C^{2,\gamma'}$, $0 < \gamma' \leq 1$, $\ell = 0,1$, if not otherwise stated. Let the domains Ω_ℓ be filled up by elastic continua heaving different hemitropic material constants, $\alpha^{(\ell)}$, $\beta^{(\ell)}$, $\gamma^{(\ell)}$, $\delta^{(\ell)}$, $\lambda^{(\ell)}$, $\mu^{(\ell)}$, $\nu^{(\ell)}$, $\varkappa^{(\ell)}$ and $\varepsilon^{(\ell)}$, $\ell = 0,1$; $\eta^{(\ell)} > 0$ and $\zeta^{(\ell)} > 0$, $\ell = 0,1$, are constants describing the coupling of mechanical and thermal fields in Ω_ℓ (see [3], [14]), $\partial = (\partial_1,\partial_2,\partial_3)$, $\partial_j = \partial/\partial x_j$, $j = 1,2,3$.

Analogously, for the mechanical characteristics, e.g., the displacement and microrotation vectors, the force stress and couple stress vectors, and also for the differential operators, fundamental matrices and potentials related to the hemitropic material occupying the domain Ω_ℓ, $\ell = 0,1$, we also employ the superscript (ℓ). In particular, $u^{(\ell)} = (u_1^{(\ell)}, u_2^{(\ell)}, u_3^{(\ell)})^T$, $\omega^{(\ell)} = (\omega_1^{(\ell)}, \omega_2^{(\ell)}, \omega_3^{(\ell)})^T$ and $\vartheta^{(\ell)}$ denote the displacement and microrotation vectors and temperature function in the domain Ω_ℓ; $E^{(\ell)}(U^{(\ell)}, U^{(\ell)})$ designates the appropriate potential energy density, $L^{(\ell)}(\partial,\sigma)$, $L^{(\ell)}(\partial)$, $L_0^{(\ell)}(\partial)$,

$\mathcal{P}^{(\ell)}(\partial, n)$ and $\mathcal{P}_0^{(\ell)}(\partial, n)$ are the corresponding differential operators given by the formulae (2.4), (2.7), (2.8), (2.5) and (2.6).

In what follows we treat transmission problems for the differential equations of pseudo-oscillations, i.e., we assume that

$$\sigma = \sigma_1 + i\sigma_2 \ \text{ with } \ \sigma_2 > 0. \tag{3.1}$$

It is clear that the nonhomogeneous differential equation $L^{(\ell)}(\partial, \sigma)U^{(\ell)} = \Psi^{(\ell)}$ in Ω_ℓ we can reduce to the homogeneous one, $L^{(\ell)}(\partial, \sigma)V^{(\ell)} = 0$, with the help of the volume Newtonian potential $N_{\Omega_\ell}(\Psi^{(\ell)})$ (see Appendix A). Therefore, without loss of generality we can assume that the body force and body couple vectors absent.

We will study the following boundary-transmission problems:

Find regular complex-valued vector-functions $U^{(\ell)} \in [C^1(\overline{\Omega}_\ell)]^7 \cap [C^2(\Omega_\ell)]^7$, $\ell = 0, 1$, satisfying the differential equations

$$L^{(\ell)}(\partial, \sigma)U^{(\ell)}(x) = 0 \ \text{ in } \ \Omega_\ell, \ \ \ell = 0, 1, \tag{3.2}$$

the transmission conditions on S_1

$$\{U^{(1)}(z)\}^+ - \{U^{(0)}(z)\}^- = f(z) \ \text{ on } \ S_1, \tag{3.3}$$

$$\big\{\mathcal{P}^{(1)}(\partial, n)U^{(1)}(z)\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial, n)U^{(0)}(z)\big\}^- = F(z) \ \text{ o } \ S_1, \tag{3.4}$$

and either the Dirichlet boundary condition on S_0

$$\{U^{(0)}(z)\}^+ = f^{(D)}(z) \ \text{ n } \ S_0, \tag{3.5}$$

or the Neumann boundary condition on S_0

$$\big\{\mathcal{P}^{(0)}(\partial, n)U^{(0)}(z)\big\}^+ = F^{(N)}(z) \ \text{ on } \ S_0. \tag{3.6}$$

We assume that the given transmission and boundary data are complex-valued vectors and

$$f \in [C^{1,\beta'}(S_0)]^7, \quad F \in [C^{0,\beta'}(S_0)]^7,$$
$$f^{(D)} \in [C^{1,\beta'}(S_1)]^7, \quad F^{(N)} \in [C^{0,\beta'}(S_1)]^7,$$

with $0 < \beta' < \gamma' \le 1$. We refer to the boundary-transmission problem (3.2)–(3.5) as Problem (TD) and the boundary-transmission problem (3.2)–(3.4) and (3.6) as Problem (TN).

The above problem setting is a *classical* one in the space of continuously differentiable vector-functions.

In the case of a *weak setting* of the problems we look for a solution pair $(U^{(0)}, U^{(1)})$ in the Sobolev spaces, $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$, $\ell = 0, 1$, with $L^{(\ell)}(\partial, \sigma)U^{(\ell)} \in [L_p(\Omega_\ell)]^7$. Therefore, equations (3.2) are understood in the distributional sense. However, we remark that solutions to these homogeneous equations actually are analytical vector-functions of the real spatial variable x in the open domains Ω_0 and Ω_1, since the differential operators $L^{(\ell)}(\partial, \sigma)$ are strongly elliptic.

The Dirichlet type boundary and transmission conditions are understood in the usual trace sense, while the Neumann type conditions are understood in the generalized trace sense defined by Green's identity (2.24) (for details see [37], [42]).

We start with the study of uniqueness of solutions to these problems.

Theorem 3.1. *Problems* (TD) *and* (TN) *may have at most one solution in the space of regular vector-functions.*

Proof. Due to linearity of the problems under consideration, it suffices to show that the corresponding homogeneous problems have only the trivial solutions. Let a pair of regular vectors

$$(U^{(0)},U^{(1)})\in\big([C^1(\overline{\Omega}_0)]^7\cap[C^2(\Omega_0)]^7\big)\times\big([C^1(\overline{\Omega}_1)]^7\cap[C^2(\Omega_1)]^7\big)$$

be a solution of either the homogeneous Problem (TD) or Problem (TN). Using Green's formulae for the vector-functions $U^{(0)}$ and $U^{(1)}$ and taking into account the chosen direction of the normal vector on the boundaries S_0 and S_1, we get

$$\int\limits_{\Omega_1}\Big[-E^{(1)}\big(\widetilde{U}^{(1)},\overline{\widetilde{U}^{(1)}}\big)+\varrho_1\sigma^2|u^{(1)}|^2+\mathcal{I}_1\sigma^2|\omega^{(1)}|^2-C_0\kappa_1'|\nabla\vartheta^{(1)}|^2-\kappa_1''|\vartheta^{(1)}|^2\Big]dx+ \\ +\int\limits_{S_1}\Big\{\mathcal{T}^{(1)}(\partial,n)U^{(1)}\cdot\overline{\widetilde{U}^{(1)}}+C_0\kappa_1'\vartheta^{(1)}\partial_n\overline{\vartheta^{(1)}}\Big\}^+dS=0, \tag{3.7}$$

$$\int\limits_{\Omega_0}\Big[-E^{(0)}\big(\widetilde{U}^{(0)},\overline{\widetilde{U}^{(0)}}\big)+\varrho_0\sigma^2|u^{(0)}|^2+\mathcal{I}_0\sigma^2|\omega^{(0)}|^2-C_0\kappa_0'|\nabla\vartheta^{(0)}|^2-\kappa_0''|\vartheta^{(0)}|^2\Big]dx+ \\ +\int\limits_{S_0}\Big\{\mathcal{T}^{(0)}(\partial,n)U^{(0)}\cdot\overline{\widetilde{U}^{(0)}}+C_0\kappa_0'\vartheta^{(0)}\partial_n\overline{\vartheta^{(0)}}\Big\}^+dS- \\ -\int\limits_{S_1}\Big\{\mathcal{T}^{(0)}(\partial,n)U^{(0)}\cdot\overline{\widetilde{U}^{(0)}}+C_0\kappa_0'\vartheta^{(0)}\partial_n\overline{\vartheta^{(0)}}\Big\}^-dS=0, \tag{3.8}$$

where

$$C_0=-\frac{i}{\sigma}\,,\ \ \kappa_\ell'=\frac{\lambda_0^{(\ell)}}{T_0^{(\ell)}},\ \ \kappa_\ell''=\frac{c_0^{(\ell)}}{T_0^{(\ell)}},\ \ \widetilde{U}^{(\ell)}=(u^{(\ell)},\ \ \omega^{(\ell)})^\top,\ \ \ell=0,1.$$

The homogeneous boundary and transmission conditions, $f^{(\ell)}=F^{(\ell)}=0$, yield

$$\sum_{\ell=0}^{1}\int\limits_{\Omega_\ell}\Big[E^{(\ell)}\big(\widetilde{U}^{(\ell)},\overline{\widetilde{U}^{(\ell)}}\big)-\varrho_\ell\sigma^2|u^{(\ell)}|^2-\mathcal{I}_\ell\sigma^2|\omega^{(\ell)}|^2+ \\ +C_0\kappa_\ell'|\nabla\vartheta^{(\ell)}|^2+\kappa_\ell''|\vartheta^{(\ell)}|^2\Big]dx=0. \tag{3.9}$$

Separating the imaginary part leads to the relation

$$\sigma_1 \sum_{\ell=0}^{1} \int\limits_{\Omega_\ell} \Big[2\sigma_2 \varrho_\ell |u^{(\ell)}|^2 + 2\sigma_2 \mathcal{I}_\ell |\omega^{(\ell)}|^2 + \frac{\kappa'_\ell}{|\sigma|^2} |\nabla \vartheta^{(\ell)}|^2 \Big]\, dx = 0.$$

If $\sigma_1 \neq 0$, we then conclude $u^{(\ell)} = 0$, $\omega^{(\ell)} = 0$, $\vartheta^{(\ell)} = const$. But from (3.9) we have $\vartheta^{(\ell)} = 0$ and consequently $U^{(\ell)} = 0$ in Ω_ℓ. If $\sigma_1 = 0$, then from (3.9) we have

$$E^{(\ell)}(\widetilde{U}^{(\ell)}, \overline{\widetilde{U}^{(\ell)}}) + \sigma_2^2 \varrho_\ell |u^{(\ell)}|^2 + \sigma_2^2 \mathcal{I}_\ell |\omega^{(\ell)}|^2 + \frac{\kappa'_\ell}{\sigma_2} |\nabla \vartheta^{(\ell)}|^2 + \kappa''_\ell |\vartheta^{(\ell)}|^2 = 0$$

for $\ell = 0, 1$, whence $u^{(\ell)} = 0$, $\omega^{(\ell)} = 0$, $\vartheta^{(\ell)} = 0$ in Ω_ℓ follow. □

By the quite similar arguments one can prove the following uniqueness theorem for the same transmission problems in the weak formulation.

Theorem 3.2. *Problems* (TD) *and* (TN) *may have at most one solution in the space* $(U^{(0)}, U^{(1)}) \in [W_2^1(\Omega_0)]^7 \times [W_2^1(\Omega_1)]^7$.

4. Existence Results for Problem (TD)

Here we develop the so called indirect boundary integral equations approach. We look for a solution pair of Problem (TD) in the form of single layer potentials, see Appendix A,

$$U^{(1)}(x) = V_{S_1}^{(1)}\big([\mathcal{H}_{S_1}^{(1)}]^{-1}\varphi\big)(x) \equiv$$
$$\equiv \int\limits_{S_1} \Gamma^{(1)}(x-y,\sigma)\big([\mathcal{H}_{S_1}^{(1)}]^{-1}\varphi\big)(y)\, dS_y, \quad x \in \Omega_1, \tag{4.1}$$
$$U^{(0)}(x) = V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\psi\big)(x) + V_{S_1}^{(0)}\big([\mathcal{H}_{S_1}^{(0)}]^{-1}\chi\big)(x) \equiv$$
$$\equiv \int\limits_{S_0} \Gamma^{(0)}(x-y,\sigma)\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\psi\big)(y)\, dS_y +$$
$$+ \int\limits_{S_1} \Gamma^{(0)}(x-y,\sigma)\big([\mathcal{H}_{S_1}^{(0)}]^{-1}\chi\big)(y)\, dS_y, \quad x \in \Omega_0, \tag{4.2}$$

where $\varphi = (\varphi_1, \dots, \varphi_7)^\top$, $\psi = (\psi_1, \dots, \psi_7)^\top$ and $\chi = (\chi_1, \dots, \chi_7)^\top$ are unknown densities; $\Gamma^{(\ell)}(x-y,\sigma)$ is the fundamental matrix of the operator $L^{(\ell)}(\partial, \sigma)$, $\ell = 0, 1$; $[\mathcal{H}_{S_j}^{(\ell)}]^{-1}$ stands for the operator inverse to $\mathcal{H}_{S_j}^{(\ell)}$, $\ell, j = 0, 1$, which is well defined due to Theorems A.5 and A.6 in Appendix A.

Recall that for the potentials and the boundary operators generated by them, the superscript (ℓ) shows the correspondence to the type of hemitropic material in Ω_ℓ.

Taking into consideration the transmission and boundary conditions of Problem (TD) and using the properties of the single-layer potentials we arrive at the system of boundary integral (pseudodifferential) equations:

$$\begin{aligned}
&\varphi(z)-\chi(z)-\int\limits_{S_0}\Gamma^{(0)}(z-y,\sigma)\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)(y)\,dS_y=f(z),\;\; z\in S_1,\\
&\Big[(-2^{-1}I_7+\mathcal{K}^{(1)}_{S_1})[\mathcal{H}^{(1)}_{S_1}]^{-1}\varphi\Big](z)-\Big[(2^{-1}I_7+\mathcal{K}^{(0)}_{S_1})[\mathcal{H}^{(0)}_{S_1}]^{-1}\chi\Big](z)-\\
&-\int\limits_{S_0}\mathcal{P}^{(0)}(\partial_z,n(z))\Gamma^{(0)}(z-y,\sigma)\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)(y)\,dS_y=F(z),\;\; z\in S_1,\\
&\int\limits_{S_1}\Gamma^{(0)}(z-y,\sigma)\big([\mathcal{H}^{(0)}_{S_1}]^{-1}\chi\big)(y)\,dS_y+\psi(z)=f^{(D)}(z),\;\; z\in S_0.
\end{aligned}\tag{4.3}$$

The operators $\mathcal{K}^{(\ell)}_{S_\ell}$, $\ell=0,1$, are defined in Appendix A (see Theorem A.1).

Introduce the so called Steklov–Poincaré type operators

$$\mathcal{A}^{(0)}_{S_1}:=(2^{-1}I_7+\mathcal{K}^{(0)}_{S_1})[\mathcal{H}^{(0)}_{S_1}]^{-1},\quad \mathcal{A}^{(1)}_{S_1}:=(-2^{-1}I_7+\mathcal{K}^{(1)}_{S_1})[\mathcal{H}^{(1)}_{S_1}]^{-1},\tag{4.4}$$

and rewrite system (4.3) as

$$\begin{aligned}
\varphi-\chi-V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)&=f\ \text{ on } S_1,\\
\mathcal{A}^{(1)}_{S_1}\varphi-\mathcal{A}^{(0)}_{S_1}\chi-\mathcal{P}^{(0)}(\partial_z,n)V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)&=F\ \text{ on } S_1,\\
V^{(0)}_{S_1}([\mathcal{H}^{(0)}_{S_1}]^{-1}\chi)+\psi&=f^{(D)}\ \text{ on } S_0.
\end{aligned}\tag{4.5}$$

Denote by r_Σ the restriction operator onto Σ. Clearly, the operators $r_{S_1}V^{(0)}_{S_0}$, $r_{S_1}\mathcal{P}^{(0)}V^{(0)}_{S_0}$ and $r_{S_0}V^{(0)}_{S_1}$, involved in the above equations are smoothing operators, since the surfaces S_1 and S_0 are disjoint.

Denote the operator generated by the left hand side expressions in (4.5) by $\mathcal{D}$ which acts on the triplet of the sought for vectors $(\varphi,\chi,\psi)^\top$,

$$\mathcal{D}:=\begin{bmatrix} I_7 & -I_7 & -r_{S_1}V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\big)\\ \mathcal{A}^{(1)}_{S_1} & -\mathcal{A}^{(0)}_{S_1} & -r_{S_1}\mathcal{P}^{(0)}V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\big)\\ 0 & r_{S_0}V^{(0)}_{S_1}([\mathcal{H}^{(0)}_{S_1}]^{-1}) & I_7\end{bmatrix}_{21\times21}.$$

Set

$$\Psi=(\varphi,\chi,\psi)^\top,\quad Q=(f,F,f^{(D)})^\top,$$

and rewrite (4.5) in matrix form

$$\mathcal{D}\Psi=Q.$$

Let us introduce the function spaces:

$$\begin{aligned}
\boldsymbol{X}^{k,\beta'}&:=[C^{k,\beta'}(S_1)]^7\times[C^{k,\beta'}(S_1)]^7\times[C^{k,\beta'}(S_0)]^7,\\
\boldsymbol{Y}^{k,\beta'}&:=[C^{k,\beta'}(S_1)]^7\times[C^{k-1,\beta'}(S_1)]^7\times[C^{k,\beta'}(S_0)]^7,\\
&S_0,S_1\in C^{k+1,\gamma'},\;\; k\geq1,\;\; 0<\beta'<\gamma'\leq1,
\end{aligned}\tag{4.6}$$

$$\begin{aligned} \boldsymbol{X}^s_p &:= [H^s_p(S_1)]^7 \times [H^s_p(S_1)]^7 \times [H^s_p(S_0)]^7, \\ \boldsymbol{Y}^s_p &:= [H^s_p(S_1)]^7 \times [H^{s-1}_p(S_1)]^7 \times [H^s_p(S_0)]^7, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \boldsymbol{X}^s_{p,t} &:= [B^s_{p,t}(S_1)]^7 \times [B^s_{p,t}(S_1)]^7 \times [B^s_{p,t}(S_0)]^7, \\ \boldsymbol{Y}^s_{p,t} &:= [B^s_{p,t}(S_1)]^7 \times [B^{s-1}_{p,t}(S_1)]^7 \times [B^s_{p,t}(S_0)]^7, \\ & s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ 1 \le t \le \infty, \ \ S_0, S_1 \in C^\infty. \end{aligned} \tag{4.8}$$

The results collected in Appendix A yield the following mapping properties:

$$\mathcal{D} : \boldsymbol{X}^{k,\beta'} \longrightarrow \boldsymbol{Y}^{k,\beta'}, \ \ S_0, S_1 \in C^{k+1,\gamma'}, \ \ k \ge 1, \ \ 0 < \beta' < \gamma' \le 1,$$
$$\mathcal{D} : \boldsymbol{X}^s_p \longrightarrow \boldsymbol{Y}^s_p, \ \ s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ S_0, S_1 \in C^\infty,$$
$$\mathcal{D} : \boldsymbol{X}^s_{p,t} \longrightarrow \boldsymbol{Y}^s_{p,t}, \ \ s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ 1 \le t \le \infty \ \ S_0, S_1 \in C^\infty.$$

Further, let us introduce the operator

$$\widetilde{\mathcal{D}} := \begin{bmatrix} I_7 & -I_7 & 0 \\ \mathcal{A}^{(1)}_{S_1} & -\mathcal{A}^{(0)}_{S_1} & 0 \\ 0 & 0 & I_7 \end{bmatrix}_{21\times 21}.$$

It is clear that $\widetilde{\mathcal{D}}$ has the same mapping properties as the operator $\mathcal{D}$ and the operator $\mathcal{D} - \widetilde{\mathcal{D}}$ with the same domain and range spaces is a compact operator. To establish the Fredholm properties of the operator $\mathcal{D}$ first we study the operator $\widetilde{\mathcal{D}}$.

Lemma 4.1. *The operators*

$$\widetilde{\mathcal{D}} : \boldsymbol{X}^{k,\beta'} \longrightarrow \boldsymbol{Y}^{k,\beta'}, \ \ k \ge 1, \ \ 0 < \beta' < \gamma' \le 1, \ \ S_0, S_1 \in C^{k+1,\gamma'}, \tag{4.9}$$
$$\widetilde{\mathcal{D}} : \boldsymbol{X}^s_p \longrightarrow \boldsymbol{Y}^s_p, \ \ s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ S_0, S_1 \in C^\infty, \tag{4.10}$$
$$\widetilde{\mathcal{D}} : \boldsymbol{X}^s_{p,t} \longrightarrow \boldsymbol{Y}^s_{p,t}, \ \ s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ 1 \le t \le \infty, \ \ S_0, S_1 \in C^\infty \tag{4.11}$$

are invertible.

Proof. We prove the lemma into several steps.

Step 1. First we show that the null-space of the operator (4.9) is trivial. To this end, we have to prove that the simultaneous homogeneous equations

$$\begin{aligned} \varphi(z) - \chi(z) = 0, \ \ z \in S_1, \\ [\mathcal{A}^{(1)}_{S_1}\varphi](z) - [\mathcal{A}^{(0)}_{S_1}\chi](z) = 0, \ \ z \in S_1, \\ \psi(z) = 0, \ \ z \in S_0, \end{aligned} \tag{4.12}$$

have only the trivial solution. Since $\psi = 0$ on S_0 it suffices to show that the first two equations imply $\varphi = \chi = 0$ on S_1. Indeed, let φ and χ solve the above homogeneous equations. Construct the single-layer potentials:

$$\begin{aligned} \widetilde{U}^{(1)}(x) &= V^{(1)}_{S_1}\big([\mathcal{H}^{(1)}_{S_1}]^{-1}\varphi\big)(x), \ \ x \in \Omega^+ := \Omega_1, \\ \widetilde{U}^{(0)}(x) &= V^{(0)}_{S_1}\big([\mathcal{H}^{(0)}_{S_1}]^{-1}\chi\big)(x), \ \ x \in \Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}_1. \end{aligned} \tag{4.13}$$

From the first two equations in (4.12) and the properties of the single-layer potentials it follows that the pair of vectors $(\widetilde{U}^{(0)}, \widetilde{U}^{(1)})$ solve the basic homogeneous transmission problem for the whole space with the interface S_1:

$$L^{(1)}(\partial,\sigma)\widetilde{U}^{(1)}(x)=0 \ \text{ in } \ \Omega^+, \quad L^{(0)}(\partial,\sigma)\widetilde{U}^{(0)}(x)=0 \ \text{ in } \ \Omega^-,$$
$$\{\widetilde{U}^{(1)}(z)\}^+ - \{\widetilde{U}^{(0)}(z)\}^- = 0 \ \text{ on } \ S_1,$$
$$\big\{\mathcal{P}^{(1)}(\partial,n)\widetilde{U}^{(1)}(z)\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial,n)\widetilde{U}^{(0)}(z)\big\}^- = 0 \ \text{ on } \ S_1.$$

Note that, if $\varphi,\chi\in[C^{k,\beta'}(S_1)]^7$, then the corresponding single-layer potentials are regular vectors in the $\overline{\Omega^\pm}$, i.e., $\widetilde{U}^{(1)}\in[C^{k,\beta'}(\overline{\Omega^+})]^7\cap[C^\infty(\Omega^+)]^7$ and $\widetilde{U}^{(0)}\in[C^{k,\beta'}(\overline{\Omega^-})]^7\cap[C^\infty(\Omega^-)]^7$. We recall that the entries of the fundamental matrix $\Gamma^{(\ell)}(x,\sigma)$ decay exponentially at infinity (see [44]), and therefore the vector $\widetilde{U}^{(0)}$ and its partial derivatives decay exponentially as $|x|\to+\infty$. It is clear that for such vectors the corresponding Green's formulae hold in the unbounded domain Ω^- (cf. (3.7), (3.8)).

Therefore, by virtue of the homogeneous transmission conditions, as in the proof of Theorem 3.1, we arrive at the equalities $\widetilde{U}^{(1)}=0$ in Ω^+ and $\widetilde{U}^{(0)}=0$ in Ω^-, which in view of (4.13) proves that $\ker\widetilde{\mathcal{D}}$ is trivial.

Step 2. Let us consider the vectors

$$\begin{aligned} U^{(1)}(x)&=V_{S_1}^{(1)}\big([\mathcal{H}_{S_1}^{(1)}]^{-1}\chi\big)(x), \ \ x\in\Omega^+,\\ U^{(0)}(x)&=V_{S_1}^{(0)}\big([\mathcal{H}_{S_1}^{(0)}]^{-1}\chi\big)(x), \ \ x\in\Omega^-, \end{aligned} \tag{4.14}$$

then we have

$$\{U^{(1)}\}^+=\{U^{(0)}\}^-=\chi, \tag{4.15}$$
$$\big\{\mathcal{P}^{(1)}(\partial,n)U^{(1)}\big\}^+=\mathcal{A}_{S_1}^{(1)}\chi, \quad \big\{\mathcal{P}^{(0)}(\partial,n)\{U^{(0)}\big\}^-=\mathcal{A}_{S_1}^{(0)}\chi \ \text{ on } \ S_1.$$

With the help of formulae (2.24), for vectors $U'=\overline{U^{(1)}}$ and $U=U^{(1)}$ we have

$$\Big\langle\overline{\chi},\mathcal{A}_{S_1}^{(1)}\chi\Big\rangle_{S_1}=\mathcal{B}^{(1)}(U^{(1)},U^{(1)}), \tag{4.16}$$

where

$$\begin{aligned} \mathcal{B}^{(1)}(U^{(1)},U^{(1)})=\int\limits_{\Omega^+}\Big[&E^{(1)}(\overline{\widetilde{U}^{(1)}},\widetilde{U}^{(1)})+\kappa_1'|\nabla\vartheta^{(1)}|^2-\\ &-\varrho^{(1)}\sigma^2|u^{(1)}|^2-\mathcal{I}^{(1)}\sigma^2|\omega^{(1)}|^2-i\sigma\kappa_1''|\vartheta^{(1)}|^2-\\ -\vartheta^{(1)}\operatorname{div}\big(\eta^{(1)}\overline{u^{(1)}}+\zeta^{(1)}\overline{\omega^{(1)}}\big)&-i\sigma\overline{\vartheta^{(1)}}\operatorname{div}(\eta^{(1)}u^{(1)}+\zeta^{(1)}\omega^{(1)})\Big]\,dx. \end{aligned}$$

Quite similarly from (2.24) for vectors $U'=\overline{U^{(0)}}$ and $U=U^{(0)}$ we derive

$$-\Big\langle\overline{\chi},\mathcal{A}_{S_1}^{(0)}\chi\Big\rangle_{S_1}=\mathcal{B}^{(0)}(U^{(0)},U^{(0)}), \tag{4.17}$$

where

$$\mathcal{B}^{(0)}(U^{(0)},U^{(0)}) = \int\limits_{\Omega^-} \Big[E^{(0)}(\overline{\widetilde{U}^{(0)}},\widetilde{U}^{(0)}) + \kappa_0'|\nabla\vartheta^{(0)}|^2 - \\ - \varrho^{(0)}\sigma^2|u^{(0)}|^2 - \mathcal{I}^{(0)}\sigma^2|\omega^{(0)}|^2 - i\sigma\kappa_0''|\vartheta^{(0)}|^2 - \\ - \vartheta^{(0)} \operatorname{div}\big(\eta^{(0)}\overline{u^{(0)}} + \zeta^{(0)}\overline{\omega^{(0)}}\big) - i\sigma\overline{\vartheta^{(0)}} \operatorname{div}\big(\eta^{(0)}u^{(0)} + \zeta^{(0)}\omega^{(0)}\big)\Big]\,dx.$$

Now from (4.16) and (4.17) we have

$$\Big\langle(\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)})\chi,\overline{\chi}\Big\rangle_{S_1} = \mathcal{B}^{(1)}(U^{(1)},U^{(1)}) + \mathcal{B}^{(0)}(U^{(0)},U^{(0)}).$$

Let

$$U := \begin{cases} U^{(1)} & \text{in } \Omega^+, \\ U^{(0)} & \text{in } \Omega^-. \end{cases}$$

Since $U^{(1)} \in [H_2^1(\Omega^+)]^7$ and $U^{(0)} \in [H_2^1(\Omega^-)]^7$, by relation (4.15) we easily conclude that $U \in [H_2^1(\mathbb{R}^3)]^7$. Taking into consideration the coercivity relation (2.26), we have

$$\operatorname{Re}\Big\langle(\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)})\chi,\overline{\chi}\Big\rangle_{S_1} \geq C_1\|U\|^2_{[H_2^1(\mathbb{R}^3)]^7} - C_2\|U\|^2_{[H_2^0(\mathbb{R}^3)]^7}, \tag{4.18}$$

where C_1 and C_2 are some positive constants. Note that, by the trace theorem from (4.18) we derive

$$\operatorname{Re}\Big\langle(\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)})\chi,\overline{\chi}\Big\rangle_{S_1} \geq C_1'\|\{U\}^\pm\|^2_{[H_2^{\frac12}(S_1)]^7} - C_2\|U\|^2_{[H_2^0(\mathbb{R}^3)]^7} \geq \\ \geq C_1'\|\chi\|^2_{[H_2^{\frac12}(S_1)]^7} - C_2'\|\chi\|^2_{[H_2^{-\frac12}(S_1)]^7}, \tag{4.19}$$

since by Theorem A.4 we have the estimate

$$\|U\|_{[H_2^0(\mathbb{R}^3)]^7} \leq C_2^*\|\chi\|_{[H_2^{-\frac12}(S_1)]^7}.$$

In turn, the inequality (4.19) implies that the operator

$$\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)} : [H_2^{\frac12}(S_1)]^7 \longrightarrow [H_2^{-\frac12}(S_1)]^7 \tag{4.20}$$

is Fredholm with zero index (see, e.g., [32]).

Let us show that the null space of the operator (4.20) is trivial. Indeed, if $\chi \in [H_2^{\frac12}(S_1)]^7$ is a solution of the homogeneous equation $(\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)})\chi = 0$ on S_1, then it follows that the vectors $U^{(1)}$ and $U^{(0)}$ defined by (4.14) solve the homogeneous transmission problem:

$$\begin{aligned} L^{(1)}(\partial,\sigma)U^{(1)}(x) &= 0 \ \text{ in } \ \Omega^+, \\ L^{(0)}(\partial,\sigma)U^{(0)}(x) &= 0 \ \text{ in } \ \Omega^-, \\ \{U^{(1)}(z)\}^+ - \{U^{(0)}(z)\}^- &= 0 \ \text{ on } \ S_1, \\ \big\{\mathcal{P}^{(1)}(\partial,n)U^{(1)}(z)\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial,n)U^{(0)}(z)\big\}^- &= 0 \ \text{ on } \ S_1. \end{aligned}$$

By the arguments applied in the proof of Theorem 3.1, we conclude that $U^{(1)} = 0$ in Ω^+ and $U^{(0)} = 0$ in Ω^-, implying $\chi = 0$ on S_1. Consequently, the null space of the operator (4.20) is trivial. Thus the operator (4.20) is invertible. Then from the general theory of pseudodifferential operators on manifolds without boundary it follows that

$$\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(0)}_{S_1} : [H^s_p(S_1)]^7 \longrightarrow [H^{s-1}_p(S_1)]^7,$$
$$: [B^s_{p,t}(S_1)]^7 \longrightarrow [B^{s-1}_{p,t}(S_1)]^7$$

are also invertible operators for arbitrary $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$ (see, e.g., [1], [2], [19], [51], [52]).

Step 3. In turn, this yields that the operator (4.10) is invertible for $s = 1/2$ and $p = 2$, i.e., the system of equations for the triplet $(\varphi, \chi, \psi) \in \boldsymbol{X}^{\frac{1}{2}}_2$,

$$\varphi - \chi = f \;\text{ on }\; S_1,$$
$$\mathcal{A}^{(1)}_{S_1}\varphi - \mathcal{A}^{(0)}_{S_1}\chi = F \;\text{ on }\; S_1,$$
$$\psi = f^{(D)} \;\text{ on }\; S_0,$$

is uniquely solvable for arbitrary $(f, F, f^{(D)}) \in \boldsymbol{Y}^{\frac{1}{2}}_2$.

Applying again the results from the general theory of pseudodifferential operators on manifolds without boundary we conclude that all the operators in (4.9)–(4.11) are invertible. □

Now we are in a position to prove the following invertibility results.

Theorem 4.2. *The operators*

$$\mathcal{D} : \boldsymbol{X}^{k,\beta'} \longrightarrow \boldsymbol{Y}^{k,\beta'}, \;\; k \ge 1, \;\; 0 < \beta' < \gamma' \le 1, \;\; S_0, S_1 \in C^{k+1,\gamma'}, \tag{4.21}$$
$$: \boldsymbol{X}^s_p \longrightarrow \boldsymbol{Y}^s_p, \;\; s \in \mathbb{R}, \;\; 1 < p < \infty, \;\; S_0, S_1 \in C^\infty, \tag{4.22}$$
$$: \boldsymbol{X}^s_{p,t} \longrightarrow \boldsymbol{Y}^s_{p,t}, \;\; s \in \mathbb{R}, \;\; 1 < p < \infty, \;\; 1 \le t \le \infty, \;\; S_0, S_1 \in C^\infty, \tag{4.23}$$

are invertible.

Proof. First let us note that by Lemma 4.1 the operators (4.21)–(4.23) are Fredholm with zero index, since they are compact perturbations of the invertible operators, due to the compactness of the difference $\mathcal{D} - \widetilde{\mathcal{D}}$ in the corresponding function spaces. Thus, for invertibility we need only to show that their null-spaces are trivial. Let the triplet $\Psi = (\varphi, \chi, \psi)^\top$ belonging to one of the spaces $\boldsymbol{X}^{k,\beta'}$ or $\boldsymbol{X}^s_p$ or $\boldsymbol{X}^s_{p,t}$ be a solution of the homogeneous equation $\mathcal{D}\Psi = 0$, i.e., the homogeneous equation (4.5). Due to the regularity theorem for solutions to the elliptic pseudodifferential equations on manifolds without boundary we conclude that actually $\Psi \in \boldsymbol{X}^{k,\beta'}$. Further, with the help of the solution triplet (φ, χ, ψ) we construct the vectors $U^{(0)}$ and $U^{(1)}$ by formulae (4.1)–(4.2). Clearly, the pair $(U^{(0)}, U^{(1)})$ is a regular solution to the homogeneous Problem (TD). Consequently, by the uniqueness Theorem 3.1 we have $U^{(1)} = 0$ in Ω_1 and $U^{(0)} = 0$ in Ω_0. Since $[U^{(1)}]^+ = \varphi$ on S_1 we get $\varphi = 0$.

The vector $U^{(0)}$ defined by formula (4.2) solves the homogeneous differential equation $L^{(0)}(\partial, \sigma)U^{(0)} = 0$ in $R^3 \setminus [S_0 \cup S_1]$ and is identical zero in Ω_0.

Since the single layer potentials are continuous in $\mathbb{R}^3$ we have that $[U^{(0)}]^- = [U^{(0)}]^+ = 0$ on S_1 and $[U^{(0)}]^+ = [U^{(0)}]^- = 0$ on S_0. So $U^{(0)}$ solves the homogeneous Dirichlet problems for the operator $L^{(0)}(\partial, \sigma)$ in the domain Ω_1 and in the unbounded domain $\mathbb{R}^3 \setminus [\overline{\Omega}_0 \cup \overline{\Omega}_1]$. Moreover, $U^{(0)}$ decays exponentially at infinity. By the uniqueness theorem for the Dirichlet interior and exterior problems, which can be easily proved with the help of Green's formulae (2.22), we establish that $U^{(0)}$ vanishes in $\mathbb{R}^3$. Now, the jump relations for the singlelayer potential imply $[\mathcal{P}^{(0)}U^{(0)}]^- - [\mathcal{P}^{(0)}U^{(0)}]^+ = \chi = 0$ on S_1 and $[\mathcal{P}^{(0)}U^{(0)}]^- - [\mathcal{P}^{(0)}U^{(0)}]^+ = \psi = 0$ on S_0, which completes the proof. □

These invertibility properties for the operator $\mathcal{D}$ lead to the following existence results for Problem (TD).

Theorem 4.3. *Let*

$$S_0, S_1 \in C^{2,\gamma'}, \;\; f \in [C^{1,\beta'}(S_1)]^7, \;\; F \in [C^{0,\beta'}(S_1)]^7,$$
$$f^{(D)} \in [C^{1,\beta'}(S_0)]^7, \;\; 0 < \beta' < \gamma' \le 1.$$

Then the problem (3.2)–(3.5) *has a unique solution in the class of regular vector functions which can be represented by the single layer potentials* (4.1)–(4.2)*, where the triplet*

$$(\varphi, \chi, \psi)^\top \in [C^{1,\beta'}(S_1)]^7 \times [C^{1,\beta'}(S_1)]^7 \times [C^{1,\beta'}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (4.3).

Theorem 4.4. *Let* $p > 1$*,* $s \ge 1$*, and*

$$S_0, S_1 \in C^\infty, \;\; f \in \left[B^{s-\frac{1}{p}}_{p,p}(S_1)\right]^7,$$
$$F \in \left[B^{s-1-\frac{1}{p}}_{p,p}(S_1)\right]^7, \;\; f^{(D)} \in \left[B^{s-\frac{1}{p}}_{p,p}(S_0)\right]^7.$$

Then the problem (3.2)–(3.5) *has a unique solution*

$$(U^{(0)}, U^{(1)}) \in [W^s_p(\Omega_0)]^7 \times [W^s_p(\Omega_1)]^7$$

which can be represented by the single layer potentials (4.1)–(4.2)*, where the triplet*

$$(\varphi, \chi, \psi)^\top \in [B^{s-\frac{1}{p}}_{p,p}(S_1)]^7 \times [B^{s-\frac{1}{p}}_{p,p}(S_1)]^7 \times [B^{s-\frac{1}{p}}_{p,p}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (4.3).

Proof. Existence of solutions directly follows from the representations (4.1)–(4.2) and invertibility of the operator (4.23). Uniqueness for $p = 2$ follows from Theorem 3.1. It remains to show uniqueness of solutions for arbitrary $p > 1$ and $s = 1$.

First we prove that any solution $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$ of the homogeneous equation

$$L^{(\ell)}(\partial,\sigma)U^{(\ell)} = 0 \ \text{ in } \ \Omega_\ell, \ \ \ell = 0,1,$$

can be represented by the single layer potentials:

$$U^{(1)}(x) = V_{S_1}^{(1)}(\varphi^*)(x), \ \ x \in \Omega_1, \tag{4.24}$$

$$U^{(0)}(x) = V_{S_0}^{(0)}(\psi^*)(x) + V_{S_1}^{(0)}(\chi^*)(x), \ \ x \in \Omega_0, \tag{4.25}$$

where $\varphi^*, \chi^* \in [B_{p,p}^{-\frac{1}{p}}(S_1)]^7$ and $\psi^* \in [B_{p,p}^{-\frac{1}{p}}(S_0)]^7$.

We show it for the vector $U^{(0)} \in [W_p^1(\Omega_0)]^7$. By the general integral representation formula we have (see [44, corollary 3.6, formulae (3.77)])

$$U^{(0)} = W_{S_0}^{(0)}([U^{(0)}]^+) - V_{S_0}^{(0)}\big([\mathcal{P}^{(0)}U^{(0)}]^+\big) - \\ - W_{S_1}^{(0)}([U^{(0)}]^-) + V_{S_1}^{(0)}\big([\mathcal{P}^{(0)}U^{(0)}]^-\big) \ \text{ in } \ \Omega_0. \tag{4.26}$$

Furthermore, we establish that the double-layer potentials $W_{S_0}^{(0)}([U^{(0)}]^+)$ and $W_{S_1}^{(0)}([U^{(0)}]^-)$ involved in (4.26) can be represented by the single layer potentials in the interior of S_0 (i.e., in Ω) and in the exterior of S_1 (i.e., in $\mathbb{R}^3 \setminus \overline{\Omega}_1$), respectively. Indeed, denote $\widetilde{U} := W_{S_0}^{(0)}([U^{(0)}]^+)$ in Ω, and consider the vector $U^* := \widetilde{U} - V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}[\widetilde{U}]^+\big) \in [W_p^1(\Omega)]^7$. Clearly, $L^{(1)}(\partial,\sigma)U^* = 0$ in Ω and $[U^*]^+ = 0$ on S_0. Therefore, applying again the general integral representation formula in Ω, we derive

$$U^* = -V_{S_0}^{(0)}\big([\mathcal{P}^{(0)}U^*]^+\big) \in [W_p^1(\Omega)]^7.$$

Whence it follows that

$$\widetilde{U} = V_{S_0}^{(0)}\Big([\mathcal{H}_{S_0}^{(0)}]^{-1}[\widetilde{U}]^+ - [\mathcal{P}^{(0)}U^*]^+\Big) \ \text{ in } \ \Omega.$$

Quite analogously we can show that $W_{S_1}^{(0)}([U^{(0)}]^-)$ is representable by a single layer potential in $\mathbb{R}^3 \setminus \overline{\Omega}_1$. Finally, from (4.26) we conclude that $U^{(0)}$ can be represented in the form (4.25). Similarly we derive the representation (4.24).

Due to invertibility of the operators $\mathcal{H}_{S_j}^{(\ell)}$, $\ell, j = 0,1$, we conclude that any solution pair $(U^{(0)}, U^{(1)}) \in [W_p^1(\Omega_0)]^7 \times [W_p^1(\Omega_1)]^7$ of the homogeneous Problem (TD) can be represented by formulae (4.1) and (4.2). This implies that the homogeneous problem (TD) with $p > 1$ possesses only the trivial solution since the operator $\mathcal{D}$ is invertible by Theorem 4.2. □

Corollary 4.5. *Let*

$$S_0, S_1 \in C^\infty, \ \ f \in [H_2^{\frac{1}{2}}(S_1)]^7, \ \ F \in [H_2^{-\frac{1}{2}}(S_1)]^7, \ \ f^{(D)} \in [H_2^{\frac{1}{2}}(S_0)]^7.$$

Then the problem (3.2)–(3.5) *has a unique solution*

$$(U^{(0)}, U^{(1)}) \in [W_2^1(\Omega_0)]^7 \times [W_2^1(\Omega_1)]^7$$

which can be represented by the single layer potentials (4.1)–(4.2)*, were the triplet*

$$(\varphi,\chi,\psi)^\top\in[H_2^{\frac12}(S_1)]^7\times[H_2^{\frac12}(S_1)]^7\times[H_2^{\frac12}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (4.3)*.*

Remark 4.6. Applying the results in the references [8] and [42] (see also [32]) concerning the properties of the potentials on Lipschitz domains one can prove that the inequality (4.18) remains valid when S_1 is a Lipschitz surface and the operator (4.20) is invertible. This implies that Corollary 4.5 holds true when S_0 and S_1 are Lipschitz surfaces.

5. Existence Results for Problem (TN)

We look for a solution pair $(U^{(0)},U^{(1)})$ of Problem (TN) again in the form (4.1)–(4.2). Taking into consideration the transmission and boundary conditions of Problem (TN) and using the properties of the single layer potentials we arrive at the system of boundary pseudodifferential equations with respect to the triplet of unknown densities (φ,χ,ψ):

$$\begin{aligned}\varphi-\chi-r_{S_1}V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\psi\big)&=f\ \text{ on }\ S_1,\\ \mathcal{A}_{S_1}^{(1)}\varphi-\mathcal{A}_{S_1}^{(0)}\chi-r_{S_1}\mathcal{P}^{(0)}(\partial,n)V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\psi\big)&=F\ \text{ on }\ S_1,\\ r_{S_0}\mathcal{P}^{(0)}(\partial,n)V_{S_1}^{(0)}\big([\mathcal{H}_{S_1}^{(0)}]^{-1}\chi\big)+\mathcal{A}_{S_0}^{(0)}\psi&=F^{(N)}\ \text{ on }\ S_0,\end{aligned}\tag{5.1}$$

where $\mathcal{A}_{S_1}^{(1)}$ and $\mathcal{A}_{S_1}^{(0)}$ are the Steklov–Poincaré operators given by (4.4), and

$$\mathcal{A}_{S_0}^{(0)}:=(-2^{-1}I_7+\mathcal{K}_{S_0}^{(0)})[\mathcal{H}_{S_0}^{(0)}]^{-1}.$$

Denote by $\mathcal{N}$ the matrix integral operator generated by the left hand side expressions in (5.1)

$$\mathcal{N}=[\mathcal{N}_{kj}]_{21\times21}:=$$

$$:=\begin{bmatrix} I_7 & -I_7 & -r_{S_1}V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\big)\\ \mathcal{A}_{S_1}^{(1)} & -\mathcal{A}_{S_1}^{(0)} & -r_{S_1}\mathcal{P}^{(0)}V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\big)\\ 0 & r_{S_0}\mathcal{P}^{(0)}V_{S_1}^{(0)}([\mathcal{H}_{S_1}^{(0)}]^{-1}) & \mathcal{A}_{S_0}^{(0)}\end{bmatrix}_{21\times21}.\tag{5.2}$$

Set

$$\Psi=(\varphi,\chi,\psi)^\top,\quad Q=(f,F,F^{(N)})^\top,$$

and rewrite (5.1) in matrix form

$$\mathcal{N}\Psi=Q.$$

Further, let us introduce the function spaces

$$\begin{gathered}\boldsymbol{Z}^{k,\beta'}:=[C^{k,\beta'}(S_1)]^7\times[C^{k-1,\beta'}(S_1)]^7\times[C^{k-1,\beta'}(S_0)]^7,\\ S_0,S_1\in C^{k+1,\gamma'},\ \ k\geq1,\ \ 0<\beta'<\gamma'\leq1,\\ \boldsymbol{Z}_p^s:=[H_p^s(S_1)]^7\times[H_p^{s-1}(S_1)]^7\times[H_p^{s-1}(S_0)]^7,\end{gathered}$$

$$\boldsymbol{Z}^s_{p,t} := [B^s_{p,t}(S_1)]^7 \times [B^{s-1}_{p,t}(S_1)]^7 \times [B^{s-1}_{p,t}(S_0)]^7,$$
$$s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ 1 \le t \le \infty, \ \ S_0, S_1 \in C^\infty.$$

The operator $\mathcal{N}$ possesses the mapping properties

$$\begin{aligned} \mathcal{N} : \boldsymbol{X}^{k,\beta'} &\longrightarrow \boldsymbol{Z}^{k,\beta'}, \\ : \boldsymbol{X}^s_p &\longrightarrow \boldsymbol{Z}^s_p, \\ : \boldsymbol{X}^s_{p,t} &\longrightarrow \boldsymbol{Z}^s_{p,t}, \end{aligned}$$

where the spaces $\boldsymbol{X}^{k,\beta'}$, $\boldsymbol{X}^s_p$, and $\boldsymbol{X}^s_{p,t}$ are defined in (4.6)–(4.8) respectively. To establish Fredholm properties of these operators let us consider the principal part $\widetilde{\mathcal{N}}$ of the operator (5.2)

$$\widetilde{\mathcal{N}} := \begin{bmatrix} I_7 & -I_7 & 0 \\ \mathcal{A}^{(1)}_{S_1} & -\mathcal{A}^{(0)}_{S_1} & 0 \\ 0 & 0 & \mathcal{A}^{(0)}_{S_0} \end{bmatrix}_{21\times 21}.$$

It is evident that $\widetilde{\mathcal{N}}$ has the same mapping properties as $\mathcal{N}$ and that the difference $\mathcal{N} - \widetilde{\mathcal{N}}$ is a compact operator in the corresponding spaces.

As we have shown in Section 4, the upper 14×14 principal block of the matrix operator $\widetilde{\mathcal{N}}$ and the elliptic pseudodifferential operator $\mathcal{A}^{(0)}_{S_0}$ are invertible in the appropriate function spaces. Consequently, $\widetilde{\mathcal{N}}$ is an invertible operator. Then it follows that the operator $\mathcal{N}$ is Fredholm with zero index. Now let us show that the operator $\mathcal{N}$ has a trivial kernel which implies its invertibility. Indeed, let $\Psi = (\varphi, \chi, \psi)^\top$ be a solution of the homogeneous equation

$$\mathcal{N}\Psi = 0.$$

Construct the single layer potentials:

$$U^{(1)}(x) = V^{(1)}_{S_1}\big([\mathcal{H}^{(1)}_{S_1}]^{-1}\varphi\big)(x), \ \ x \in \Omega_1,$$
$$U^{(0)}(x) = V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)(x) + V^{(0)}_{S_1}\big([\mathcal{H}^{(0)}_{S_1}]^{-1}\chi\big)(x), \ \ x \in \Omega_0.$$

It is easy to verify that the pair $(U^{(0)}, U^{(1)})$ solves the homogeneous Problem (TN) and, consequently, by the uniqueness Theorem 3.2 we conclude that

$$U^{(1)}(x) = 0, \ \ x \in \Omega_1, \quad U^{(0)}(x) = 0, \ \ x \in \Omega_0. \tag{5.3}$$

As in the proof of Theorem 4.2 one can easily show that the relations (5.3) implies $\Psi = 0$.

Now we can formulate the following existence results for Problem (TN).

Theorem 5.1.

(i) *Let*

$$S_0, S_1 \in C^{2,\gamma'}, \ \ f \in [C^{1,\beta'}(S_1)]^7, \ \ F \in [C^{0,\beta'}(S_1)]^7,$$
$$F^{(N)} \in [C^{0,\beta'}(S_0)]^7, \ \ 0 < \beta' < \gamma' \le 1.$$

Then the problem (3.2)–(3.4), (3.6) *possesses a unique solution in the class of regular vector functions which can be represented by single layer potentials* (4.1)–(4.2), *where the triplet*

$$(\varphi, \chi, \psi)^\top \in [C^{1,\beta'}(S_1)]^7 \times [C^{1,\beta'}(S_1)]^7 \times [C^{1,\beta'}(S_0)]^7$$

is uniquely defined by the system of boundary pseudodifferential equations (5.1).

(ii) *Let*

$$S_0, S_1 \in C^\infty, \quad f \in \left[B_{p,p}^{1-\frac{1}{p}}(S_1)\right]^7, \quad F \in \left[B_{p,p}^{-\frac{1}{p}}(S_1)\right]^7,$$
$$F^{(N)} \in \left[B_{p,p}^{-\frac{1}{p}}(S_0)\right]^7, \quad p > 1.$$

Then the problem (3.2)–(3.4), (3.6) *possesses a unique solution*

$$(U^{(0)}, U^{(1)}) \in [W_p^1(\Omega_0)]^7 \times [W_p^1(\Omega_1)]^7$$

which can be represented by the single layer potentials (4.1)–(4.2), *where the triplet*

$$(\varphi, \chi, \psi) \in [B_{p,p}^{1-\frac{1}{p}}(S_1)]^7 \times [B_{p,p}^{1-\frac{1}{p}}(S_1)]^7 \times [B_{p,p}^{1-\frac{1}{p}}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (5.1).

From this theorem, as a particular case, we have the following

Corollary 5.2. *Let*

$$S_0, S_1 \in C^\infty, \quad f \in [H_2^{\frac{1}{2}}(S_1)]^7, \quad F \in [H_2^{-\frac{1}{2}}(S_1)]^7, \quad F^{(N)} \in [H_2^{-\frac{1}{2}}(S_0)]^7.$$

Then the problem (3.2)–(3.4), (3.6) *has a solution*

$$(U^{(0)}, U^{(1)}) \in [W_2^1(\Omega_0)]^7 \times [W_2^1(\Omega_1)]^7$$

which can be represented by the singlelayer potentials (4.1)–(4.2), *where the triplet*

$$(\varphi, \chi, \psi) \in [H_2^{\frac{1}{2}}(S_1)]^7 \times [H_2^{\frac{1}{2}}(S_1)]^7 \times [H_2^{\frac{1}{2}}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (5.1).

Remark 5.3. Applying again the results in the references [8], [42], and [32]) concerning the properties of the potentials on Lipschitz domains one can prove that Corollary 5.2 holds true when S_0 and S_1 are Lipschitz surfaces.

6. Interface Crack Problem (ICP)

6.1. **Formulation of the problem.** Throughout this section, let $\Omega_1 = \Omega^+$ be a bounded region in $\mathbb{R}^3$ with a simply connected boundary $S = \partial\Omega_1 \in C^\infty$ and let $\Omega_0 = \Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}_1$. As in Section 3, we assume that the domains Ω_ℓ are filled with elastic hemitropic materials having different material constants, $\alpha^{(\ell)}$, $\beta^{(\ell)}$, $\gamma^{(\ell)}$, $\delta^{(\ell)}$, $\lambda^{(\ell)}$, $\mu^{(\ell)}$, $\nu^{(\ell)}$, $\varkappa^{(\ell)}$ and $\varepsilon^{(\ell)}$, $\ell = 0, 1$. We preserve the notation employed in Section 3 for differential and integral operators. In what follows, $n(z)$ stands for the outward unit normal vector with respect to the bounded domain Ω_1 at the point $z \in S$. Further, let the interface surface S be divided into two disjoint, simply connected parts S_T (where the transmission conditions are given) and S_C (where the crack conditions are given): $S = \overline{S}_T \cup \overline{S}_C$. We assume that $\partial S_T = \partial S_C$ is a simple, C^∞-smooth curve. We identify S_C as an interface crack surface with smooth boundary ∂S_C.

We will study the following interface crack type mixed transmission Problem (ICP):

Find vector-functions

$$U^{(1)} \in [W^1_p(\Omega_1)]^7, \ \ U^{(0)} \in [W^1_{p,loc}(\Omega_0)]^7, \ \ 1 < p < \infty,$$

satisfying the differential equations,

$$L^{(\ell)}(\partial, \sigma) U^{(\ell)} = 0 \ \text{ in } \ \Omega_\ell, \ \ \ell = 0, 1, \tag{6.1}$$

the transmission conditions on S_T,

$$\{U^{(1)}\}^+ - \{U^{(0)}\}^- = \widetilde{f}, \tag{6.2}$$

$$\big\{\mathcal{P}^{(1)}(\partial, n) U^{(1)}\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial, n) U^{(0)}\big\}^- = \widetilde{F} \ \text{ on } \ S_T, \tag{6.3}$$

and the interface crack conditions on S_C,

$$\big\{\mathcal{P}^{(1)}(\partial, n) U^{(1)}\big\}^+ = F^{(1)}, \ \ \big\{\mathcal{P}^{(0)}(\partial, n) U^{(0)}\big\}^- = F^{(0)} \ \text{ on } \ S_C. \tag{6.4}$$

Moreover, we assume that $U^{(0)}$ is bounded at infinity, whence in view of (3.1) it follows that actually $U^{(0)}$ decays exponentially at infinity and $U^{(0)} \in [W^1_p(\Omega_0)]^7 \cap [C^\infty(\Omega_0)]^7$ (for details see [44]).

In our analysis we replace the conditions (6.4) by the equivalent ones:

$$\big\{\mathcal{P}^{(1)}(\partial, n) U^{(1)}\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial, n) U^{(0)}\big\}^- = F^{(1)} - F^{(0)} \ \text{ on } \ S_C, \tag{6.5}$$

$$\big\{\mathcal{P}^{(1)}(\partial, n) U^{(1)}\big\}^+ + \big\{\mathcal{P}^{(0)}(\partial, n) U^{(0)}\big\}^- = F^{(1)} + F^{(0)} \ \text{ on } \ S_C. \tag{6.6}$$

The boundary data involved in the above formulation belong to the natural spaces:

$$\widetilde{f} \in \big[B^{1-\frac{1}{p}}_{p,p}(S_T)\big]^7, \ \ \widetilde{F} \in [B^{-\frac{1}{p}}_{p,p}(S_T)]^7, \ \ F^{(1)}, F^{(0)} \in [B^{-\frac{1}{p}}_{p,p}(S_C)]^7. \tag{6.7}$$

Denote

$$F := \begin{cases} \widetilde{F} & \text{on } \ S_T, \\ F^{(1)} - F^{(0)} & \text{on } \ S_C. \end{cases} \tag{6.8}$$

Clearly, F represents the difference of generalized traces of the stress vectors,

$$F = \{\mathcal{P}^{(1)}(\partial, n)U^{(1)}\}^{+} - \{\mathcal{P}^{(0)}(\partial, n)U^{(0)}\}^{-} \text{ on } S.$$

Therefore the imbedding

$$F \in [B_{p,p}^{-\frac{1}{p}}(S)]^7 \tag{6.9}$$

is the necessary condition for the interface crack problem (ICP) to be solvable in the space $[W_p^1(\Omega_0)]^7 \times [W_p^1(\Omega_1)]^7$.

Now we reformulate the problem (ICP) (6.1)–(6.7) in the following form:

Find vector-functions $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$, $\ell = 0, 1$, $1 < p < \infty$, satisfying the conditions

$$L^{(\ell)}(\partial, \sigma)U^{(\ell)} = 0 \text{ in } \Omega_\ell, \quad \ell = 0, 1, \tag{6.10}$$

$$\{U^{(1)}\}^{+} - \{U^{(0)}\}^{-} = \widetilde{f} \text{ on } S_T, \tag{6.11}$$

$$\{\mathcal{P}^{(1)}(\partial, n)U^{(1)}\}^{+} - \{\mathcal{P}^{(0)}(\partial, n)U^{(0)}\}^{-} = F \text{ on } S, \tag{6.12}$$

$$\{\mathcal{P}^{(1)}(\partial, n)U^{(1)}\}^{+} + \{\mathcal{P}^{(0)}(\partial, n)U^{(0)}\}^{-} = F^{(1)} + F^{(0)} \text{ on } S_C. \tag{6.13}$$

One can easily prove the following particular uniqueness result using Green's identities for domains Ω_1 and Ω_0 (see the proof of Theorem 3.1).

Theorem 6.1. *The interface crack problem* (6.10)–(6.13) *with* $p = 2$ *may have at most onesolution.*

6.2. **Auxiliary problem.** Let us consider the following basic transmission problem (BTP):

Find vector-functions $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$, $\ell = 0, 1$, $1 < p < \infty$, satisfying the conditions $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$:

$$L^{(\ell)}(\partial, \sigma)U^{(\ell)} = 0 \text{ in } \Omega_\ell, \quad \ell = 0, 1, \tag{6.14}$$

$$\{U^{(1)}\}^{+} - \{U^{(0)}\}^{-} = f \text{ on } S, \tag{6.15}$$

$$\{\mathcal{P}^{(1)}(\partial, n)U^{(1)}\}^{+} - \{\mathcal{P}^{(0)}(\partial, n)U^{(0)}\}^{-} = F \text{ on } S, \tag{6.16}$$

where

$$f \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^7, \quad F \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^7, \ 1 < p < \infty. \tag{6.17}$$

Using Green's formulas it can easily be shown that this problem possesses at most one solution for $p = 2$.

Let us look for a solution pair $(U^{(1)}, U^{(2)})$ in the form of single layer potentials:

$$U^{(\ell)}(x) = V^{(\ell)}\big([\mathcal{H}^{(\ell)}]^{-1} g^{(\ell)}\big)(x), \quad \ell = 0, 1, \tag{6.18}$$

where $V^{(\ell)} = V_S^{(\ell)}$ and $g^{(\ell)} \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^7$ are unknown densities.

The transmission conditions (6.15)–(6.16) lead then to the relations

$$g^{(1)} - g^{(0)} = f \text{ on } S, \tag{6.19}$$

$$\mathcal{A}^{(1)} g^{(1)} - \mathcal{A}^{(0)} g^{(0)} = F \text{ on } S, \tag{6.20}$$

where $\mathcal{A}^{(\ell)}$, $\ell = 0, 1$, are the above introduced Steklov–Poincaré operators (see (4.4)):

$$\mathcal{A}^{(1)} = (-2^{-1}I_7 + \mathcal{K}^{(1)})[\mathcal{H}^{(1)}]^{-1}, \quad \mathcal{A}^{(0)} = (2^{-1}I_7 + \mathcal{K}^{(0)})[\mathcal{H}^{(0)}]^{-1}.$$

From (6.19)–(6.20) we get

$$g^{(1)} = f - g^{(0)} \ \text{ on } \ S, \tag{6.21}$$

$$\big(\mathcal{A}^{(1)} - \mathcal{A}^{(0)}\big)g^{(0)} = F - \mathcal{A}^{(1)}f \ \text{ on } \ S. \tag{6.22}$$

As we have shown in the proof of Lemma 4.1 (Step 2) the operator

$$\mathcal{A}^{(1)} - \mathcal{A}^{(0)} : [B_{p,p}^{-\frac{1}{p}}(S)]^7 \longrightarrow [B_{p,p}^{-\frac{1}{p}}(S)]^7$$

is invertible. Therefore we have from (6.22)

$$g^{(0)} = [\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(1)}f). \tag{6.23}$$

By (6.21) we then get

$$g^{(1)} = [\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}F - [\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}f. \tag{6.24}$$

Substituting (6.23) and (6.24) into (6.18) finally we get the following representation of the solution to the (BTP)

$$U^{(1)} = V^{(1)}\Big([\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(0)}f)\Big) \ \text{ in } \ \Omega_1, \tag{6.25}$$

$$U^{(0)} = V^{(0)}\Big([\mathcal{H}^{(0)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(1)}f)\Big) \ \text{ in } \ \Omega_0. \tag{6.26}$$

Theorem 6.2. *Let $1 < p < \infty$ and conditions* (6.17) *be satisfied. Then the basic transmission problem* (6.14)–(6.17) *is uniquely solvable in the space $[W_p^1(\Omega_1)]^7 \times [W_p^1(\Omega_0)]^7$ and the solution can be represented by formulas* (6.25)–(6.26).

Proof. It is word for word of the proof of Theorem 4.4. □

6.3. **Existence and regularity of solutions to the (ICP).** Let us now consider the (ICP) (6.10)–(6.13). Denote by f a fixed extension of the vector $\widetilde{f}$ from S_T onto the whole of S, preserving the space. Any extension of the same vector can be then represented as a sum $f + \varphi$ with $\varphi \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^7$.

We look for a solution pair $(U^{(1)}, U^{(0)})$ to the (ICP) (6.10)–(6.13) in the form

$$U^{(1)} = V^{(1)}\Big([\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(0)}(f + \varphi))\Big) \ \text{ in } \ \Omega_1, \tag{6.27}$$

$$U^{(0)} = V^{(0)}\Big([\mathcal{H}^{(0)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(1)}(f + \varphi))\Big) \ \text{ in } \ \Omega_0, \tag{6.28}$$

where F is a known vector-function given by (6.8), f is the fixed extension of the vector $\widetilde{f}$ and $\varphi \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^7$ is unknown.

One can easily verify that the differential equations (6.10) and the transmission conditions (6.11) and (6.12) are automatically satisfied, while the boundary condition (6.13) on the crack surface S_C leads to the pseudodifferential equation on S_C for the unknown vector-function φ:

$$r_{S_C}\Big\{\mathcal{A}^{(1)}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}+\mathcal{A}^{(0)}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(1)}\Big\}\,\varphi=\Phi \ \text{ on } \ S_C, \tag{6.29}$$

where

$$\Phi := F^{(1)}-F^{(0)}-r_{S_C}(-2^{-1}I_7+\mathcal{K}^{(1)})[\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}(F-\mathcal{A}^{(0)}f)- \\ - r_{S_C}(2^{-1}I_7+\mathcal{K}^{(0)})[\mathcal{H}^{(0)}]^{-1}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}(F-\mathcal{A}^{(1)}f).$$

Clearly,

$$\Phi \in [B^{-\frac{1}{p}}_{p,p}(S_C)]^7.$$

Denote the principal homogeneous symbol matrices of the pseudodifferential operators $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(0)}$ by $\mathfrak{S}_1=\mathfrak{S}_1(x,\xi_1,\xi_2)$ and $\mathfrak{S}_0=\mathfrak{S}_0(x,\xi_1,\xi_2)$ respectively with $x\in\overline{S}_C$ and $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

Note that, since the principal homogeneous parts of the differential operators $L^{(\ell)}(\partial,\sigma)$ are formally selfadjoint, from (4.19) one can conclude that the principal homogeneous symbol matrices $\mathfrak{S}_1$ and $-\mathfrak{S}_0$ of the operators $\mathcal{A}^{(1)}_{S_1}$ and $-\mathcal{A}^{(0)}_{S_1}$ are positive definite for all $x\in\overline{S}_C$ and $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

For the principal homogeneous symbol matrix of the operator

$$\boldsymbol{K} := -\mathcal{A}^{(1)}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}-\mathcal{A}^{(0)}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(1)} \tag{6.30}$$

we have

$$\mathfrak{S}_{\boldsymbol{K}}=-\mathfrak{S}_1(\mathfrak{S}_1-\mathfrak{S}_0)^{-1}\mathfrak{S}_0-\mathfrak{S}_0(\mathfrak{S}_1-\mathfrak{S}_0)^{-1}\mathfrak{S}_1=2(\mathfrak{S}_1^{-1}-\mathfrak{S}_0^{-1})^{-1}. \tag{6.31}$$

Whence it follows that $\mathfrak{S}_{\boldsymbol{K}}=\mathfrak{S}_{\boldsymbol{K}}(x,\xi_1,\xi_2)$ is positive definite for all $x\in\overline{S}_C$ and $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

Rewrite equation (6.29) in the form

$$r_{S_C}(\boldsymbol{K}\varphi)=-\Phi \ \text{ on } \ S_C,$$

Due to the results in [52] (see also Appendix C in [44]), since $\boldsymbol{K}$ is an elliptic pseudo differential operator of order $+1$ with positive definite principal homogeneous symbol, we conclude that the operator

$$r_{S_C}\boldsymbol{K} : [\widetilde{B}^s_{p,t}(S_C)]^7 \longrightarrow [B^{s-1}_{p,t}(S_C)]^7 \tag{6.32}$$

is Fredholm with zero index for arbitrary $t\in[1,\infty]$, if

$$\frac{1}{p}-1<s-\frac{1}{2}<\frac{1}{p}\,. \tag{6.33}$$

In particular, for $s=1-\frac{1}{p}$ and $t=p$ we get that the operator

$$r_{S_C}\boldsymbol{K} : \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_C)\big]^7 \longrightarrow \big[B^{-\frac{1}{p}}_{p,p}(S_C)\big]^7 \tag{6.34}$$

is Fredholm, if

$$\frac{4}{3}<p<4. \tag{6.35}$$

Moreover, the null space of the operator (6.32) does not depend on t, p and s if (6.33) holds (see, e.g., [5, Theorem 3.5]).

Now we show that the null space of the operator (6.34) with $p=2$,

$$r_{S_C}\boldsymbol{K}:\big[\widetilde{H}_2^{\frac{1}{2}}(S_C)\big]^7\longrightarrow[H_2^{-\frac{1}{2}}(S_C)]^7 \tag{6.36}$$

is trivial.

Let $\psi\in[H_2^{\frac{1}{2}}(S_C)]^7$ be a solution of the homogeneous equation

$$r_{S_C}\boldsymbol{K}\psi=0 \ \text{ on } \ S_C \tag{6.37}$$

and construct the vectors

$$\widetilde{U}^{(1)}(x)=-V^{(1)}\Big([\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}\psi\Big) \ \text{ in } \ \Omega_1,$$

$$\widetilde{U}^{(0)}(x)=-V^{(0)}\Big([\mathcal{H}^{(0)}]^{-1}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(1)}\psi\Big) \ \text{ in } \ \Omega_0.$$

It is easy to check that the pair $(\widetilde{U}^{(1)},\widetilde{U}^{(0)})$ solve the homogeneous problem (ICP) (6.10)–(6.13). Due to the uniqueness Theorem 6.1 it follows that

$$\widetilde{U}^{(1)}=0 \ \text{ in } \ \Omega_0 \ \text{ and } \ \widetilde{U}^{(1)}=0 \ \text{ in } \ \Omega_1.$$

Whence

$$0=\{\widetilde{U}^{(1)}\}^+-\{\widetilde{U}^{(0)}\}^-=-[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}\psi+[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(1)}\psi=\psi.$$

Thus, equation (6.37) possesses only the zero solution and consequently the null space of the operator (6.36) is trivial. Therefore it follows that the operator (6.32) with s and p satisfying the condition (6.33) is invertible.

The same holds true for the operator (6.34) with p satisfying the inequalities (6.35). The above results lead to the following existence and regularity theorems.

Theorem 6.3. *Let* $4/3<p<4$,

$$\widetilde{f}\in\big[B_{p,p}^{1-\frac{1}{p}}(S_T)\big]^7,\quad \widetilde{F}\in\big[B_{p,p}^{-\frac{1}{p}}(S_T)\big]^7,\quad F^{(0)},F^{(1)}\in\big[B_{p,p}^{-\frac{1}{p}}(S_C)\big]^7,$$

and for F given by (6.8) *the inclusion* (6.9) *hold. Then the interface crack problem* (ICP) *possesses a unique solution pair*

$$(U^{(0)},U^{(1)})\in[W_p^1(\Omega_0)]^7\times[W_p^1(\Omega_1)]^7,$$

which is representable in the form (6.27)*–*(6.28)*, where the unknown vector φ is a unique solution to the pseudodifferential equation* (6.29)*.*

Proof. It is quite similar to the proof of Theorem 4.4. The existence of solution follows from the mapping properties of the layer potentials described in Theorems A.1–A.4 (see Appendix A), while the uniqueness of solution is a consequence of the invertibility of the operator (6.34) with p satisfying the inequality (6.35). □

Theorem 6.4. *Let*

$$1<t<\infty,\ \ 1\le r\le\infty,\ \ \frac{4}{3}<p<4,\ \ \frac{1}{t}-\frac{1}{2}<s<\frac{1}{t}+\frac{1}{2}, \tag{6.38}$$

and let a pair $(U^{(0)},U^{(1)})\in[W_p^1(\Omega_0)]^7\times[W_{p,loc}^1(\Omega_1)]^7$ *be a solution to Problem* (ICP)*.*

(i) *If* $\widetilde{f} \in [B^{s}_{t,t}(S_T)]^7$, $\widetilde{F} \in [B^{s-1}_{t,t}(S_T)]^7$, $F^{(0)}, F^{(1)} \in [B^{s-1}_{t,t}(S_C)]^7$, *and* $F \in [B^{s-1}_{t,t}(S)]^7$, *where* F *is defined by* (6.8), *then*

$$(U^{(0)}, U^{(1)}) \in [H^{s+\frac{1}{t}}_t(\Omega_0)]^7 \times [H^{s+\frac{1}{t}}_t(\Omega_1)]^7;$$

(ii) *If* $\widetilde{f} \in [B^{s}_{t,r}(S_T)]^7$, $\widetilde{F} \in [B^{s-1}_{t,r}(S_T)]^7$, $F^{(0)}, F^{(1)} \in [B^{s-1}_{t,r}(S_C)]^7$, *and* $F \in [B^{s-1}_{t,r}(S)]^6$, *where* F *is defined by* (6.8),*then*

$$(U^{(0)}, U^{(1)}) \in [B^{s+\frac{1}{t}}_{t,r}(\Omega_0)]^7 \times [B^{s+\frac{1}{t}}_{t,r}(\Omega_1)]^7; \tag{6.39}$$

(iii) *If*

$$\begin{gathered}\widetilde{f} \in [C^{\beta'}(S_T)]^7, \ \ \widetilde{F} \in \big[B^{\beta'-1}_{\infty,\infty}(S_T)\big]^7, \ \ F \in \big[B^{\beta'-1}_{\infty,\infty}(S)\big]^7, \\ F^{(0)}, F^{(1)} \in \big[B^{\beta'-1}_{\infty,\infty}(S_C)\big]^7, \ \ \beta' > 0,\end{gathered} \tag{6.40}$$

where F *is defined by* (6.8), *then*

$$U^{(\ell)} \in \bigcap_{\sigma'<\nu'} \big[C^{\sigma'}(\overline{\Omega}_\ell)\big]^7, \ \ \ell = 0, 1,$$

where $\nu' = \min\{\beta', 1/2\}$.

Proof. Under the restrictions on the parameters r, t and s stated in the theorem we see that the operator (6.32) is invertible. Therefore the items (i) and (ii) immediately follow from the mapping properties of the single layer potentials and the boundary operators $\mathcal{A}^{(1)} - \mathcal{A}^{(0)}$ and $\mathcal{H}^{(\ell)}$, $\mathcal{A}^{(\ell)}$, $\ell = 0, 1$.

To prove (iii) we use the following embeddings (see, e.g., [54], [55])

$$B^{\alpha'}_{\infty,\infty}(\mathcal{S}) \subset B^{\alpha'-\varepsilon'}_{\infty,1}(\mathcal{S}) \subset B^{\alpha'-\varepsilon'}_{\infty,r}(\mathcal{S}) \subset B^{\alpha'-\varepsilon'}_{t,r}(\mathcal{S}), \tag{6.41}$$

$$\begin{aligned}C^{\beta'}(\mathcal{S}) = B^{\beta'}_{\infty,\infty}(\mathcal{S}) \subset B^{\beta'-\varepsilon'}_{\infty,1}(\mathcal{S}) \subset B^{\beta'-\varepsilon'}_{\infty,r}(\mathcal{S}) \subset \\ \subset B^{\beta'-\varepsilon'}_{t,r}(\mathcal{S}) \subset C^{\beta'-\varepsilon'-\frac{k}{t}}(\mathcal{S}),\end{aligned} \tag{6.42}$$

where $\alpha' \in \mathbb{R}$, ε' is an arbitrary small positive number, $\mathcal{S} \subset \mathbb{R}^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with smooth boundary, $1 \le r \le \infty$, $1 < t < \infty$, $\beta' - \varepsilon' - k/t > 0$, β' and $\beta' - \varepsilon' - k/t$ are not integers. From (6.40) and the embeddings (6.41) the condition (6.39) follows with any $s \le \beta' - \varepsilon'$.

Bearing in mind the conditions (6.38) and taking t sufficiently large and ε' sufficiently small, we may put $s = \beta' - \varepsilon'$ if

$$\frac{1}{t} - \frac{1}{2} < \beta' - \varepsilon' < \frac{1}{t} + \frac{1}{2}, \tag{6.43}$$

and $s \in (1/t - 1/2, 1/t + 1/2)$ if

$$\frac{1}{t} + \frac{1}{2} < \beta' - \varepsilon'. \tag{6.44}$$

By the inclusion (6.39) the vector $U^{(\ell)}$ belongs then to $[B^{s+\frac{1}{t}}_{t,r}(\Omega_\ell)]^7$ with $s+1/t = \beta'-\varepsilon'+1/t$ if (6.43) holds, and with $s+1/t \in (2/t-1/2, 2/t+1/2)$

if (6.44) holds. In the last case we can take $s + 1/t = 2/t + 1/2 - \varepsilon'$. Therefore, we have either $U^{(\ell)} \in [B^{\beta'-\varepsilon'+\frac{1}{t}}_{t,r}(\Omega_\ell)]^7$, or $U^{(\ell)} \in [B^{\frac{1}{2}+\frac{2}{t}-\varepsilon'}_{t,r}(\Omega_\ell)]^7$ in accordance with the inequalities (6.43) and (6.44). The last embedding in (6.42) (with $k = 3$) yields that either $U^{(\ell)} \in [C^{\beta'-\varepsilon'-\frac{2}{t}}(\overline{\Omega}_\ell)]^7$, or $U^{(\ell)} \in [C^{\frac{1}{2}-\varepsilon'-\frac{1}{t}}(\overline{\Omega}_\ell)]^7$ which lead to the inclusion

$$U^{(\ell)} \in \big[C^{\nu'-\varepsilon'-\frac{2}{t}}(\overline{\Omega}_\ell)\big]^7, \quad \ell = 0, 1, \tag{6.45}$$

where $\nu' := \min\{\beta', 1/2\}$. Since t is sufficiently large and ε' is sufficiently small, the embedding (6.45) completes the proof. $\square$

Remark 6.5. More detailed analysis based on the asymptotic expansions of solutions (see [6], [9]) shows that for sufficiently smooth boundary data (e.g., C^∞-smooth data say) the leading asymptotic terms of the solution vectors $U^{(0)}$ and $U^{(1)}$ near the interface crack edge, i.e., near the curve $\partial S_T = \partial S_C$ can be represented as a product of a "good" vector-function and a singular factor of the form $[\ln \varrho(x)]^{q_j}[\varrho(x)]^{\alpha_j+i\beta_j}$, $0 \le q_j \le m_j - 1$. Here $\varrho(x)$ is the distance from a reference point x to the curve $\partial S_T = \partial S_C$. Therefore, near the interface crack edge, the leading dominant singular terms of the corresponding generalized stress vectors $\mathcal{P}^{(\ell)}U^{(\ell)}$ are represented as a product of a "good" vector-function and the factors $[\ln \varrho(x)]^{q_j}[\varrho(x)]^{-1+\alpha_j+i\beta_j}$. Clearly when the numbers β_j are different from zero then we have the oscillating stress singularities.

The exponents $\alpha_j + i\beta_j$ are related to the eigenvalues $\lambda_j = \lambda_j(x)$, $j = \overline{1,7}$, of the matrix (see (6.30), (6.31))

$$\big[\mathfrak{S}_{\boldsymbol{K}}(x, 0, +1)\big]^{-1}\mathfrak{S}_{\boldsymbol{K}}(x, 0, -1)$$

for $x \in \partial S_T = \partial S_C$, and the following relations hold

$$\alpha_j = \frac{1}{2} + \frac{\arg \lambda_j}{2\pi}\,, \quad \beta_j = -\frac{\ln|\lambda_j|}{2\pi}\,, \quad j = \overline{1,7}.$$

In the above expressions the parameter m_j denotes the algebraic multiplicity of the eigenvalue λ_j.

Note that due to the positive definiteness of the matrix $\mathfrak{S}_{\boldsymbol{K}}(x, \xi_1, \xi_2)$ for all $x \in S_1$ and $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ it is easy to show that all eigenvalues λ_j are positive which implies that $\alpha_j = \frac{1}{2}$, $j = \overline{1,7}$.

It is evident that when $|\lambda_j| \neq 1$, then the corresponding $\beta_j \neq 0$ and oscillating stress singularities arise near the interface crack edge. Moreover, the components of the generalized stress vector $\mathcal{P}^{(\ell)}U^{(\ell)}$ behave like $\mathcal{O}\big([\ln \varrho(x)]^{q_0-1}[\varrho(x)]^{-\frac{1}{2}}\big)$, where q_0 denotes the maximal algebraic multiplicity of the eigenvalues. This is a global singularity effect for the first order derivatives of the vectors $U^{(0)}$ and $U^{(1)}$. As we see, the stress singularity exponents for the interface crack problem in the case of hemitropic solids have the form $-\frac{1}{2} + i\beta_j$ where β_j depends on the material parameters of the constituent solids of the composite structure.

7. Appendix A

Here we collect some results concerning mapping and regularity properties of the single and double layer potentials and the boundary pseudodifferential operators generated by them in the Hölder ($C^{m,\kappa}$), Sobolev–Slobodetski (W_p^s), Bessel potential (H_p^s) and Besov ($B_{p,q}^s$) spaces. They can be found in [10], [11], [12], [13], [16], [21], [22], [32], [36], [37], [38], [40], [43], and [44].

We assume (if not otherwise stated) that $\Omega^+ \subset \mathbb{R}^3$ is a bounded domain with boundary $S = \partial\Omega^+$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}^+$,

$$S = \partial\Omega^\pm \in C^{m,\gamma'} \ \text{ with integer } \ m \geq 2 \ \text{ and } \ 0 < \gamma' \leq 1, \qquad \sigma = \sigma_1 + i\sigma_2, \ \ \sigma_1 \in \mathbb{R}, \ \ \sigma_2 > 0. \tag{A.1}$$

Introduce the single and double layer potentials

$$V(x) = V_S(x) := \int\limits_S \Gamma(x-y,\sigma)g(y)\,dS_y, \tag{A.2}$$

$$W(x) = W_S(x) := \int\limits_S \big[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(x-y,\sigma)\big]^\top g(y)\,dS_y, \tag{A.3}$$

where $x \in \mathbb{R}^3 \setminus S$, $\Gamma(x-y,\sigma)$ is the fundamental matrix of the operator $L(\partial,\sigma)$ which is explicitly constructed in [44]. The proofs of the following theorems can be found in [44].

Theorem A.1. *Let S, m, and γ' be as in* (A.1), $0 < \beta' < \gamma'$, *and let $k \leq m-1$ be a nonnegative integer. Then the operators*

$$V : [C^{k,\beta'}(S)]^7 \longrightarrow \big[C^{k+1,\beta'}(\overline{\Omega^\pm})\big]^7, \qquad W : [C^{k,\beta'}(S)]^7 \longrightarrow \big[C^{k,\beta'}(\overline{\Omega^\pm})\big]^7 \tag{A.4}$$

are continuous.

For any $g \in [C^{0,\beta'}(S)]^7$, $h \in [C^{1,\beta'}(S)]^7$, and for all $x \in S$

$$[V(g)(x)]^\pm = V(g)(x) = \mathcal{H}g(x), \tag{A.5}$$

$$\big[\mathcal{P}(\partial_x, n(x))V(g)(x)\big]^\pm = [\mp 2^{-1}I_7 + \mathcal{K}]g(x), \tag{A.6}$$

$$[W(g)(x)]^\pm = [\pm 2^{-1}I_7 + \mathcal{N}]g(x), \tag{A.7}$$

$$\big[\mathcal{P}(\partial_x, n(x))W(h)(x)\big]^+ = [\mathcal{P}(\partial_x, n(x))W(h)(x)]^- = \mathcal{L}h(x), \tag{A.8}$$

where

$$\mathcal{H}g(x) = \mathcal{H}_S g(x) := \int\limits_S \Gamma(x-y,\sigma)g(y)\,dS_y, \tag{A.9}$$

$$\mathcal{K}g(x) = \mathcal{K}_S g(x) := \int\limits_S \big[\mathcal{P}(\partial_x, n(x))\Gamma(x-y,\sigma)\big]g(y)\,dS_y, \tag{A.10}$$

$$\mathcal{N}g(x)=\mathcal{N}_S g(x):=\int\limits_S \big[\mathcal{P}^*(\partial_y,n(y))\Gamma^\top(x-y,\sigma)\big]^\top g(y)\,dS_y, \tag{A.11}$$

$$\mathcal{L}h(x)=\mathcal{L}_S h(x):= \tag{A.12}$$

$$:=\lim_{\Omega^\pm\ni z\to x\in S}\mathcal{P}(\partial_z,n(x))\int\limits_S\big[\mathcal{P}^*(\partial_y,n(y))\Gamma^\top(z-y,\sigma)\big]^\top h(y)\,dS_y. \tag{A.13}$$

Theorem A.2. *Let S be a Lipschitz surface. Then the operators* (A.4) *can be extended to the continuous mappings*

$$V:[H_2^{-\frac12}(S)]^7\longrightarrow[H_2^1(\Omega^\pm)]^7,\quad W:[H_2^{\frac12}(S)]^7\longrightarrow[H_2^1(\Omega^\pm)]^7.$$

The jump relations (A.5)–(A.8) *on S remain valid for the extended operators in the corresponding function spaces.*

Theorem A.3. *Let S, m, γ', β' and k be as in Theorem A.*1*. Then the operators*

$$\begin{aligned}\mathcal{H}&:[C^{k,\beta'}(S)]^7\longrightarrow[C^{k+1,\beta'}(S)]^7,\\ &:[H_2^{-\frac12}(S)]^7\longrightarrow[H_2^{\frac12}(S)]^7,\end{aligned} \tag{A.14}$$

$$\begin{aligned}\mathcal{K}&:[C^{k,\beta'}(S)]^7\longrightarrow[C^{k,\beta'}(S)]^7,\\ &:[H_2^{-\frac12}(S)]^7\longrightarrow[H_2^{-\frac12}(S)]^7,\end{aligned} \tag{A.15}$$

$$\begin{aligned}\mathcal{N}&:[C^{k,\beta'}(S)]^7\longrightarrow[C^{k,\beta'}(S)]^7,\\ &:[H_2^{\frac12}(S)]^7\longrightarrow[H_2^{\frac12}(S)]^7,\end{aligned} \tag{A.16}$$

$$\begin{aligned}\mathcal{L}&:[C^{k,\beta'}(S)]^7\longrightarrow[C^{k-1,\beta'}(S)]^7,\\ &:[H_2^{\frac12}(S)]^7\longrightarrow[H_2^{-\frac12}(S)]^7\end{aligned} \tag{A.17}$$

are continuous. Moreover,

(i) *the principal homogeneous symbol matrices of the singular integral operators $\pm2^{-1}I_7+\mathcal{K}$ and $\pm2^{-1}I_7+\mathcal{N}$ are non-degenerate, while the principal homogeneous symbol matrices of the pseudodifferential operators $-\mathcal{H}$ and $\mathcal{L}$ are positive definite;*

(ii) *the operators $\mathcal{H}$, $\pm2^{-1}I_7+\mathcal{K}$, $\pm2^{-1}I_7+\mathcal{N}$, and $\mathcal{L}$ are elliptic pseudodifferential operators* (*of order* -1, 0, 0, *and* 1, *respectively*) *with zero index;*

(iii) *the following equalities hold in appropriate function spaces:*

$$\mathcal{N}\mathcal{H}=\mathcal{H}\mathcal{K},\quad \mathcal{L}\mathcal{N}=\mathcal{K}\mathcal{L},$$
$$\mathcal{H}\mathcal{L}=-4^{-1}I_7+\mathcal{N}^2,\quad \mathcal{L}\mathcal{H}=-4^{-1}I_7+\mathcal{K}^2.$$

(iv) *The operators* (A.14), (A.15), (A.16), *and* (A.17) *are bounded if S is a Lipschitz surface.*

Theorem A.4. *Let V, W, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{N}$, and $\mathcal{L}$ be as in Theorems A.*1 *and A.*3 *and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The layer potential operators* (A.2), (A.3) *and the boundary integral (pseudodifferential) operators* (A.9)–(A.12) *can be extended to the following continuous operators*

$$
\begin{aligned}
V &: [B^s_{p,p}(S)]^7 \longrightarrow [H_p^{s+1+\frac{1}{p}}(\Omega^\pm)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow \big[B_{p,q}^{s+1+\frac{1}{p}}(\Omega^\pm)\big]^7\Big),\\
W &: [B^s_{p,p}(S)]^7 \longrightarrow [H_p^{s+\frac{1}{p}}(\Omega^\pm)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow \big[B_{p,q}^{s+\frac{1}{p}}(\Omega^\pm)\big]^7\Big),\\
\mathcal{H} &: [H^s_p(S)]^7 \longrightarrow [H^{s+1}_p(S)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow [B^{s+1}_{p,q}(S)]^7\Big), \qquad \text{(A.18)}\\
\mathcal{K} &: [H^s_p(S)]^7 \longrightarrow [H^{s}_p(S)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow [B^{s}_{p,q}(S)]^7\Big), \qquad \text{(A.19)}\\
\mathcal{N} &: [H^s_p(S)]^7 \longrightarrow [H^{s}_p(S)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow [B^{s}_{p,q}(S)]^7\Big), \qquad \text{(A.20)}\\
\mathcal{L} &: [H^{s+1}_p(S)]^7 \longrightarrow [H^{s}_p(S)]^7 \quad \Big([B^{s+1}_{p,q}(S)]^7 \longrightarrow [B^{s}_{p,q}(S)]^7\Big). \qquad \text{(A.21)}
\end{aligned}
$$

The jump relations (A.5)–(A.8) *remain valid for arbitrary $g \in [B^s_{p,q}(S)]^7$ with $s \in \mathbb{R}$ if the limiting values (traces) on S are understood in the sense described in* [51].

The operators (A.18)–(A.21) *are elliptic pseudodifferential operators with zero index. The null-spaces of the operators* (A.18)–(A.21) *are invariant with respect to p, q, and s.*

Theorem A.5. *Let $S \in C^{2,\gamma'}$ and $0 < \beta' < \gamma' \le 1$. Then the operator*

$$\mathcal{H} : [C^{0,\beta'}(S)]^7 \longrightarrow [C^{1,\beta'}(S)]^7$$

is invertible.

Theorem A.6. *Let S be Lipschitz. Then the operator*

$$\mathcal{H} : [H_2^{-\frac{1}{2}}(S)]^7 \longrightarrow [H_2^{\frac{1}{2}}(S)]^7$$

is invertible.

Let us introduce the volume Newtonian potential

$$N_\Omega(\Psi)(x) := \int\limits_\Omega \Gamma(x-y,\sigma)\Psi(y)\,dx,$$

where $\Omega \subset \mathbb{R}^3$ is an arbitrary bounded domain and either $\Psi \in [L_2(\Omega)]^7$ or $\Psi \in [C^{0,\beta'}(\overline{\Omega})]^7$ with $0 < \beta' < 1$. There holds the following proposition (see, e.g., [33], [32]).

Theorem A.7. *Let $S \in C^{1,\gamma'}$ and $0 < \beta' < \gamma' \le 1$. Then operators*

$$
\begin{aligned}
N_\Omega &: [L_2(\Omega)]^7 \longrightarrow [W_2^2(\Omega)]^7,\\
&: [C^{0,\beta'}(\overline{\Omega})]^7 \longrightarrow [C^{2,\beta'}(\Omega)]^7 \cap [C^{1,\beta'}(\overline{\Omega})]^7,
\end{aligned} \qquad \text{(A.22)}
$$

are bounded. The mapping property (A.22) *holds for Lipschitz domains as well. Moreover,*

$$L(\partial,\sigma)N_{\Omega}(\Psi)(x)=\Psi(x),\quad x\in\Omega,$$

for almost all x in Ω if $\Psi\in[L_2(\Omega)]^7$ and for all x in Ω if $\Psi\in[C^{0,\beta'}(\overline{\Omega})]^7$.

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Authors' address:

Department of Mathematics, Georgian Technical University, 77 Kostava St., Tbilisi 0175, Georgia.

E-mail: lgiorgashvili@gmail.com; natrosh@hotmail.com; zaza-ude@hotmail.com.

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Sulkhan Mukhigulashvili and Nino Partsvania

ON ONE ESTIMATE FOR SOLUTIONS OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR HIGHER-ORDER STRONGLY SINGULAR LINEAR DIFFERENTIAL EQUATIONS

Abstract. For higher-order strongly singular differential equations with deviating arguments, the estimates for solutions of two-point conjugated and right-focal boundary value problems are established.

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Key words and phrases. Higher-order differential equation, linear, two-point boundary value problem, deviating argument, strong singularity.

რეზიუმე. მაღალი რიგის ძლიერად სინგულარული გადახრილარგუმენტე-ბიანი დიფერენციალური განტოლებებისათვის დადგენილია ორწერტილოვანი შე-უღლებული და მარჯვნივ ფოკალური სასაზღვრო ამოცანების ამონახსნთა შე-ფასებები.

1. Statement of the Main Results

Consider the differential equation with deviating arguments

$$u^{(n)}(t)=\sum_{j=1}^{m}p_j(t)u^{(j-1)}(\tau_j(t))+q(t)\ \text{ for }\ a<t<b \tag{1.1}$$

with the two-point conjugated and right-focal boundary conditions

$$u^{(i-1)}(a)=0\ \ (i=1,\dots,m),\quad u^{(j-1)}(b)=0\ \ (j=1,\dots,n-m), \tag{1.2}$$

and

$$u^{(i-1)}(a)=0\ (i=1,\dots,m),\quad u^{(j-1)}(b)=0\ \ (j=m+1,\dots,n). \tag{1.3}$$

Here $n\geq 2$, m is the integer part of $n/2$, $-\infty<a<b<+\infty$, $p_j,q\in L_{loc}(]a,b[)$ $(j=1,\dots,m)$, and $\tau_j:]a,b[\to]a,b[$ are measurable functions. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we mean the right (the left) limit of the function $u^{(j-1)}$ at the point a (at the point b).

Following R. P. Agarwal and I. Kiguradze [1], we say that the equation (1.1) is strongly singular if $\int_a^b P(s)ds=+\infty$, where

$$P(t)=(t-a)^{n-1}(b-t)^{n-1}\left[(-1)^{n-m}p_1(t)\right]_+ +\sum_{i=2}^{m}(t-a)^{n-i}(b-t)^{n-i}|p_i(t)|.$$

If the equation (1.1) is strongly singular, then we say that the problem (1.1), (1.2) (the problem (1.1), (1.3)) is also strongly singular.

In the case, where $\tau_j(t)\equiv t$ $(j=1,\dots,m)$, the strongly singular problems (1.1), (1.2) and (1.1), (1.3) are investigated in detail by I. Kiguradze and R. P. Agarwal [1], [2]. In particular, unimprovable in a certain sense conditions are established by them for the unique solvability of those problems in the spaces $\widetilde{C}^{n-1,m}(]a,b[)$ and $\widetilde{C}^{n-1,m}(]a,b])$. For $\tau_j(t)\not\equiv t$ $(j=1,\dots,m)$, the analogous results are obtained in [5], [6]. In the present paper, on the basis of the results of [6], the estimates for solutions of the strongly singular problems (1.1), (1.2) and (1.1), (1.3) are established.

Throughout the paper we use the following notations.

$R_+=[0,+\infty[$;

$[x]_+$ is the positive part of a number x, i.e., $[x]_+=\frac{x+|x|}{2}$;

$L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$ is the space of functions $y:]a,b[\to R$, which are integrable on $[a+\varepsilon,b-\varepsilon]$ $([a+\varepsilon,b])$ for an arbitrarily small $\varepsilon>0$;

$L_{\alpha,\beta}(]a,b[)$ $(L^2_{\alpha,\beta}(]a,b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y:]a,b[\to R$, with the norm

$$\|y\|_{L_{\alpha,\beta}}=\int_a^b(s-a)^\alpha(b-s)^\beta|y(s)|\,ds$$

$$\left(\|y\|_{L^2_{\alpha,\beta}}=\left(\int_a^b(s-a)^\alpha(b-s)^\beta y^2(s)\,ds\right)^{1/2}\right);$$

$L([a,b])=L_{0,0}(]a,b[)$, $L^2([a,b])=L^2_{0,0}(]a,b[)$;

$M(]a,b[)$ is the set of measurable functions $\tau:]a,b[\,\to\,]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ $(\widetilde{L}^2_{\alpha}(]a,b])$ is the Banach space of functions $y\in L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$ such that

$$\mu_1\equiv\max\bigg\{\bigg[\int_a^t (s-a)^\alpha\Big(\int_s^t y(\xi)\,d\xi\Big)^2 ds\bigg]^{1/2}:\ a\le t\le \frac{a+b}{2}\bigg\}+$$

$$+\max\bigg\{\bigg[\int_t^b (b-s)^\beta\Big(\int_t^s y(\xi)\,d\xi\Big)^2 ds\bigg]^{1/2}:\ \frac{a+b}{2}\le t\le b\bigg\}<+\infty,$$

$$\mu_2\equiv\max\bigg\{\bigg[\int_a^t (s-a)^\alpha\Big(\int_s^t y(\xi)\,d\xi\Big)^2 ds\bigg]^{1/2}:\ a\le t\le b\bigg\}<+\infty.$$

Norms in this spaces are defined by the equalities $\|\cdot\|_{\widetilde{L}^2_{\alpha,\beta}}=\mu_1$ $(\|\cdot\|_{\widetilde{L}^2_{\alpha}}=\mu_2)$.

$\widetilde{C}^{n-1,m}(]a,b[)$ $(\widetilde{C}^{n-1,m}(]a,b]))$ is the space of functions $y\in \widetilde{C}^{n-1}_{loc}(]a,b[)$ $(y\in \widetilde{C}^{n-1}_{loc}(]a,b]))$ such that

$$\int_a^b |u^{(m)}(s)|^2\,ds<+\infty. \tag{1.4}$$

When the problem (1.1), (1.2) is discussed, we assume that for $n=2m$ the conditions

$$p_j\in L_{loc}(]a,b[)\ \ (j=1,\dots,m) \tag{1.5}$$

are fulfilled, and for $n=2m+1$, along with (1.5), the condition

$$\limsup_{t\to b}\bigg|(b-t)^{2m-1}\int_{t_1}^t p_1(s)\,ds\bigg|<+\infty\ \ \Big(t_1=\frac{a+b}{2}\Big) \tag{1.6}$$

is fulfilled. The problem (1.1), (1.3) is discussed under the assumptions

$$p_j\in L_{loc}(]a,b])\ \ (j=1,\dots,m). \tag{1.7}$$

A solution of the problem (1.1), (1.2) ((1.1), (1.3)) is sought in the space $\widetilde{C}^{n-1,m}(]a,b[)$ $(\widetilde{C}^{n-1,m}(]a,b]))$.

By $h_j:]a,b[\times]a,b[\,\to R_+$ and $f_j: R\times M(]a,b[)\to C_{loc}(]a,b[\times]a,b[)$ $(j=1,\dots,m)$ we denote, respectively, functions and operators defined by the equalities

$$\begin{gathered} h_1(t,s)=\bigg|\int_s^t(\xi-a)^{n-2m}\big[(-1)^{n-m}p_1(\xi)\big]_+\,d\xi\bigg|,\\ h_j(t,s)=\bigg|\int_s^t(\xi-a)^{n-2m}p_j(\xi)\,d\xi\bigg|\ \ (j=2,\dots,m), \end{gathered} \tag{1.8}$$

and

$$f_j(c,\tau_j)(t,s)=\left|\int\limits_s^t(\xi-a)^{n-2m}|p_j(\xi)|\left|\int\limits_\xi^{\tau_j(\xi)}(\xi_1-c)^{2(m-j)}\,d\xi_1\right|^{1/2}d\xi\right|. \tag{1.9}$$

Suppose also that

$$m!!=\begin{cases}1 & \text{for } m\le 0\\ 1\cdot 3\cdot 5\cdots m & \text{for } m\ge 1\end{cases},$$

if $m=2k+1$.

In [6] (see, Theorems 1.4 and 1.5), the following two theorems are proved.

Theorem 1.1. *Let there exist numbers $t^*\in\,]a,b[\,$, $\ell_{kj}>0$, $\bar{l}_{kj}\ge 0$, and $\gamma_{kj}>0$ $(k=0,1;\ j=1,\dots,m)$ such that along with*

$$B_0\equiv\sum_{j=1}^m\left(\frac{(2m-j)2^{2m-j+1}l_{0j}}{(2m-1)!!(2m-2j+1)!!}+\right.$$
$$\left.+\frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}}\right)<\frac{1}{2}, \tag{1.10}$$

$$B_1\equiv\sum_{j=1}^m\left(\frac{(2m-j)2^{2m-j+1}l_{1j}}{(2m-1)!!(2m-2j+1)!!}+\right.$$
$$\left.+\frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}}\right)<\frac{1}{2}, \tag{1.11}$$

the conditions

$$(t-a)^{2m-j}h_j(t,s)\le l_{0j},\quad (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s)\le\bar{l}_{0j} \quad \textit{for}\quad a<t\le s\le t^*, \tag{1.12}$$

$$(b-t)^{2m-j}h_j(t,s)\le l_{1j},\quad (b-t)^{m-\gamma_{1j}-1/2}f_j(b,\tau_j)(t,s)\le\bar{l}_{1j} \quad \textit{for}\quad t^*\le s\le t<b \tag{1.13}$$

hold. Then for every $q\in\widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ the problem (1.1), (1.2) *is uniquely solvable in the space $\widetilde{C}^{n-1,m}(]a,b[)$.*

Theorem 1.2. *Let there exist numbers $t^*\in\,]a,b[\,$, $\ell_{0j}>0$, $\bar{\ell}_{0j}\ge 0$, and $\gamma_{0j}>0$ $(j=1,\dots,m)$ such that the conditions*

$$(t-a)^{2m-j}h_j(t,s)\le l_{0j},\quad (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s)\le\bar{l}_{0j} \quad \textit{for}\quad a<t\le s\le b, \tag{1.14}$$

and

$$B_3 \equiv \sum_{j=1}^{m} \bigg(\frac{(2m-j)2^{2m-j+1} l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \\ + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}} \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!! \sqrt{2\gamma_{0j}}} \bigg) < 1 \quad (1.15)$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, *the problem* (1.1), (1.3) *is uniquely solvable in the space* $\widetilde{C}^{n-1,m}(]a,b])$.

In the paper, we prove the following two theorems on the estimates of solutions of the problems (1.1), (1.2) and (1.1), (1.3), the existence of which is guaranteed by Theorems 1.1 and 1.2.

Theorem 1.3. *Let all the conditions of Theorem* 1.1 *be satisfied. Then the unique solution* u *of the problem* (1.1), (1.2) *for every* $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a,b[)$ *admits the estimate*

$$\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}, \quad (1.16)$$

where

$$r = \frac{(1+b-a)(2n-2m-1)2^m}{(\nu_n - 2\max\{B_0,B_1\})(2m-1)!!}, \quad \nu_{2m} = 1, \ \ \nu_{2m+1} = \frac{2m+1}{2},$$

and thus the constant $r > 0$ *depends only on the numbers* l_{kj}, $\bar{l}_{kj}$, γ_{kj} $(k = 1, 2; \ j = 1, \dots, m)$, *and* a, b, t^*, n.

Theorem 1.4. *Let all the conditions of Theorem* 1.2 *be satisfied. Then the unique solution* u *of the problem* (1.1), (1.3) *for every* $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ *admits the estimate*

$$\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2}}, \quad (1.17)$$

where

$$r = \frac{2^{m-1}(2n-2m-1)}{(\nu_n - B_3)(2m-1)!!}, \quad \nu_{2m} = 1, \ \ \nu_{2m+1} = \frac{2m+1}{2},$$

end thus the constant $r > 0$ *depends only on the numbers* l_{0j}, $\bar{l}_{0j}$, γ_{0j} $(j = 1, \dots, m)$, *and* a, b, n.

2. Auxiliary Propositions

To prove Theorems 1.3 and 1.4, we need Lemmas 2.1–2.6 below.

Lemma 2.1. *Let* $\in \widetilde{C}^{m-1}_{loc}(]t_0,t_1[)$ *and*

$$u^{(j-1)}(t_0) = 0 \ \ (j = 1, \dots, m), \quad \int_{t_0}^{t_1} |u^{(m)}(s)|^2\, ds < +\infty. \quad (2.1)$$

Then

$$\int_{t_0}^{t} \frac{(u^{(j-1)}(s))^2}{(s-t_0)^{2m-2j+2}}\, ds \leq$$

$$\leq \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!}\Big)^2 \int_{t_0}^{t} |u^{(m)}(s)|^2\, ds \ \ \textit{for}\ \ t_0 \leq t \leq t_1. \tag{2.2}$$

Lemma 2.2. *Let* $u \in \widetilde{C}_{loc}^{m-1}(]t_0, t_1[)$, *and*

$$u^{(j-1)}(t_1) = 0 \ \ (j = 1, \dots, m), \qquad \int_{t_0}^{t_1} |u^{(m)}(s)|^2\, ds < +\infty. \tag{2.3}$$

Then

$$\int_{t}^{t_1} \frac{(u^{(j-1)}(s))^2}{(t_1-s)^{2m-2j+2}}\, ds \leq$$

$$\leq \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!}\Big)^2 \int_{t}^{t_1} |u^{(m)}(s)|^2\, ds \ \ \textit{for}\ \ t_0 \leq t \leq t_1. \tag{2.4}$$

Let $t_0, t_1 \in]a, b[$, $u \in \widetilde{C}_{loc}^{m-1}(]t_0, t_1[)$, and $\tau_j \in M(]a, b[)$ $(j = 1, \dots, m)$. Then we define the functions $\mu_j : [a, (a+b)/2] \times [(a+b)/2, b] \times [a, b] \to [a, b]$, $\rho_k : [t_0, t_1] \to R_+$ $(k = 0, 1)$, $\lambda_j : [a, b] \times]a, (a+b)/2] \times [(a+b)/2, b[\times]a, b[\to R_+$ by the equalities

$$\mu_j(t_0, t_1, t) = \begin{cases} \tau_j(t) & \text{for } \ \tau_j(t) \in [t_0, t_1] \\ t_0 & \text{for } \ \tau_j(t) < t_0 \\ t_1 & \text{for } \ \tau_j(t) > t_1 \end{cases},$$

$$\rho_k(t) = \Big| \int_{t}^{t_k} |u^{(m)}(s)|^2\, ds \Big|, \tag{2.5}$$

$$\lambda_j(c, t_0, t_1, t) = \Big| \int_{t}^{\mu_j(t_0, t_1, t)} (s-c)^{2(m-j)}\, ds \Big|^{1/2}.$$

Moreover, we define the functions $\alpha_j : R_+^3 \times [0, 1[\to R_+$ and $\beta_j \in R_+ \times [0, 1[\to R_+$ $(j = 1, \dots, m)$ as follows

$$\alpha_j(x, y, z, \gamma) = x + \frac{2^{m-j} y z^{\gamma}}{(2m-2j-1)!!}\,,$$
$$\beta_j(y, \gamma) = \frac{2^{2m-j-1}}{(2m-2j-1)!!(2m-3)!!}\, \frac{y^{\gamma}}{\sqrt{2\gamma}}\,. \tag{2.6}$$

Lemma 2.3. *Let $a_0 \in]a,b[$, $t_0 \in]a,a_0[$, $t_1 \in]a_0,b[$, and a function $u \in \widetilde{C}^{m-1}_{loc}(]t_0,t_1[)$ be such that the conditions* (2.1) *hold. Moreover, let constants $l_{0j} > 0$, $\bar{l}_{0j} \geq 0$, $\gamma_{0j} > 0$, and functions $\overline{p}_j \in L_{loc}(]t_0,t_1[)$, $\tau_j \in M(]a,b[)$ be such that the inequalities*

$$(t-t_0)^{2m-1}\int_t^{a_0}\big[\overline{p}_1(s)\big]_+\,ds \leq l_{01}, \tag{2.7}$$

$$(t-t_0)^{2m-j}\bigg|\int_t^{a_0}\overline{p}_j(s)\,ds\bigg| \leq l_{0j} \quad (j=2,\dots,m), \tag{2.8}$$

$$(t-t_0)^{m-\frac{1}{2}-\gamma_{0j}}\bigg|\int_t^{a_0}\overline{p}_j(s)\lambda_j(t_0,t_0,t_1,s)\,ds\bigg| \leq \bar{l}_{0j} \quad (j=1,\dots,m), \tag{2.9}$$

hold for $t_0 < t \leq a_0$. Then

$$\begin{aligned}\int_t^{a_0}&\overline{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))\,ds \leq \\ &\leq \alpha_j(l_{0j},\bar{l}_{0j},a_0-a,\gamma_{0j})\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(t)+\bar{l}_{0j}\beta_j(a_0-a,\gamma_{0j})\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(a_0)+ \\ &+l_{0j}\,\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\,\rho_0(a_0) \ \text{ for } \ t_0 < t \leq a_0,\end{aligned} \tag{2.10}$$

where $\tau^* = \sup\big\{\mu_j(t_0,t_1,t):\ t_0 \leq t \leq a_0,\ j=1,\dots,m\big\} \leq t_1$.

Lemma 2.4. *Let $b_0 \in]a,b[$, $t_1 \in]b_0,b[$, $t_0 \in]a,b_0[$, and a function $u \in \widetilde{C}^{m-1}_{loc}(]t_0,t_1[)$ be such that the conditions* (2.3) *hold. Moreover, let constants $l_{1j} > 0$, $\bar{l}_{1j} \geq 0$, $\gamma_{1j} > 0$, and functions $\overline{p}_j \in L_{loc}(]t_0,t_1[)$, $\tau_j \in M(]a,b[)$ be such that the inequalities*

$$(t_1-t)^{2m-1}\int_{b_0}^{t}\big[\overline{p}_1(s)\big]_+\,ds \leq l_{11}, \tag{2.11}$$

$$(t_1-t)^{2m-j}\bigg|\int_{b_0}^{t}\overline{p}_j(s)\,ds\bigg| \leq l_{1j} \quad (j=2,\dots,m), \tag{2.12}$$

$$(t_1-t)^{m-\frac{1}{2}-\gamma_{1j}}\bigg|\int_{b_0}^{t}\overline{p}_j(s)\lambda_j(t_1,t_0,t_1,s)\,ds\bigg| \leq \bar{l}_{1j} \quad (j=1,\dots,m) \tag{2.13}$$

hold for $b_0 < t \le t_1$. *Then*

$$\int\limits_{b_0}^{t} \overline{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))\,ds \le$$

$$\le \alpha_j(l_{1j},\bar{l}_{1j},b-b_0,\gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t)+\bar{l}_{1j}\beta_j(b-b_0,\gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(b_0)+$$
$$+\,l_{1j}\,\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\,\rho_1(b_0) \;\; for \;\; b_0 \le t < t_1, \quad (2.14)$$

where $\tau_* = \inf\big\{\mu_j(t_0,t_1,t):\; b_0 \le t \le t_1,\; j=1,\dots,m\big\} \ge t_0$.

Lemma 2.5. *If* $u \in C^{n-1}_{loc}(]a,b[)$, *then for any* $s,t \in]a,b[$ *the equality*

$$(-1)^{n-m}\int\limits_{s}^{t}(\xi-a)^{n-2m}u^{(n)}(\xi)u(\xi)\,d\xi =$$
$$= w_n(t)-w_n(s)+\nu_n\int\limits_{s}^{t}|u^{(m)}(\xi)|^2\,d\xi \quad (2.15)$$

is valid, where

$$\nu_{2m}=1,\quad \nu_{2m+1}=\frac{2m+1}{2},\quad w_{2m}(t)=\sum_{j=1}^{m}(-1)^{m+j-1}u^{(2m-j)}(t)u(t),$$
$$w_{2m+1}(t)=\sum_{j=1}^{m}(-1)^{m+j}\Big[(t-a)u^{(2m+1-j)}(t)-ju^{(2m-j)}(t)\Big]u^{(j-1)}(t)-$$
$$-\frac{t-a}{2}\,|u^{(m)}(t)|^2.$$

Lemma 2.6. *Let*

$$w(t)=\sum_{i=1}^{n-m}\sum_{k=i}^{n-m}c_{ik}(t)u^{(n-k)}(t)u^{(i-1)}(t),$$

where $\widetilde{C}^{n-1,m}(]a,b[)$, *and each* $c_{ik}:[a,b]\to R$ *is an* $(n-k-i+1)$*-times continuously differentiable function. If, moreover,* $u^{(i-1)}(a)=0$ $(i=1,\dots,m)$,

$$\limsup_{t\to a}\frac{|c_{ii}(t)|}{(t-a)^{n-2m}} < +\infty \;\; (i=1,\dots,n-m),$$

then $\liminf\limits_{t\to a}|w(t)| = 0$, *and if* $u^{(i-1)}(b) = 0$ $(i = 1,\dots,n-m)$, *then* $\liminf\limits_{t\to b}|w(t)|=0$.

Lemmas 2.1, 2.2 are proved in [1], Lemmas 2.3, 2.4 are proved in [6]. The proof of Lemma 2.6 can be found in [4]. As for Lemma 2.5, it is a particular case of Lemma 4.1 from [3].

3. Proofs

Proof of Theorem 1.3. Let u be a solution of the problem (1.1), (1.2). Then in view of Theorem 1.1, the inclusion $u \in \widetilde{C}^{n\,m-1}(]a,b[)$ holds, i.e.,

$$\rho = \int\limits_a^b |u^{(m)}(s)|^2\,ds < +\infty. \tag{3.1}$$

Multiplying the equation (1.1) by $(-1)^{n-m}(t-a)^{n-2m}u(t)$ and then integrating from t_0 to t_1, by Lemma 2.5 we obtain

$$w_n(t)-w_n(s)+\nu_n\int\limits_s^t |u^{(m)}(\xi)|^2\,d\xi=(-1)^{n-m}\int\limits_s^t(s-a)^{n-2m}q(s)u(s)\,ds+$$
$$+(-1)^{n-m}\sum_{j=1}^m\int\limits_s^t(\xi-a)^{n-2m}p_j(\xi)u^{(j-1)}(\tau_j(\xi))u(\xi)\,d\xi \tag{3.2}$$

for $a<s\leq t<b$. Hence by Lemma 2.6 it is evident that

$$\liminf_{s\to a}|w_n(s)|=0, \quad \liminf_{t\to b}|w_n(t)|=0. \tag{3.3}$$

Moreover, due to the conditions (1.10) and (1.11), a number $\nu\in]0,1[$ can be chosen so that the inequalities

$$\begin{aligned} B_0 &\equiv \sum_{j=1}^m\left(l_{0j}\,\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar l_{0j}\beta_j(t^*-a,\gamma_{0j})\right)< \\ &\qquad <(\nu_n-\nu)/2, \\ B_1 &\equiv \sum_{j=1}^m\left(l_{1j}\,\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar l_{1j}\beta_j(b-t^*,\gamma_{1j})\right)< \\ &\qquad <(\nu_n-\nu)/2, \end{aligned} \tag{3.4}$$

would be satisfied, and then

$$0<\nu<\nu_n-2\max\{B_0,B_1\}. \tag{3.5}$$

It is obvious that the maximum of ν depends only on the numbers l_{kj}, $\bar l_{kj}$, γ_{kj} ($k=1,2$; $j=1,\dots,m$), and a, b, t^*, n. Now, if we put $c=(a+b)/2$, then by virtue of Lemmas 2.1, 2.2, and Young's inequality we get

$$\left|\int\limits_s^t(\psi-a)^{n-2m}q(\psi)u(\psi)\,d\psi\right|\leq$$
$$\leq\left|\int\limits_s^c(\psi-a)^{n-2m}q(\psi)u(\psi)\,d\psi\right|+\left|\int\limits_c^t(\psi-a)^{n-2m}q(\psi)u(\psi)\,d\psi\right|=$$

$$= \Big| \int_s^c \Big[(n-2m)u(\psi) + (\psi-a)^{n-2m} u'(\psi) \Big] \Big(\int_\psi^c q(\xi)\, d\xi \Big)\, d\psi \Big| +$$

$$+ \Big| \int_c^t \Big[(n-2m)u(\psi) + (\psi-a)^{n-2m} u'(\psi) \Big] \Big(\int_c^\psi q(\xi)\, d\xi \Big)\, d\psi \Big| \le$$

$$\le \bigg[(n-2m) \Big(\int_s^c \frac{u^2(\psi)}{(\psi-a)^{2m}}\, d\psi \Big)^{1/2} + \Big(\int_s^c \frac{u'^2(\psi)}{(\psi-a)^{2m-2}}\, d\psi \Big)^{1/2} \bigg] \times$$

$$\times \bigg(\int_s^c (\psi-a)^{2n-2m-2} \Big(\int_\psi^c q(\xi)\, d\xi \Big)^2 d\psi \bigg)^{1/2} +$$

$$+ (1+b-a) \bigg[(n-2m) \Big(\int_c^t \frac{u^2(\psi)}{(b-\psi)^{2m}}\, d\psi \Big)^{1/2} + \Big(\int_c^t \frac{u'^2(\psi)}{(b-\psi)^{2m-2}}\, d\psi \Big)^{1/2} \bigg] \times$$

$$\times \bigg(\int_c^t (b-\psi)^{2m-2} \Big(\int_c^\psi q(\xi)\, d\xi \Big)^2 d\psi \bigg)^{1/2} \le$$

$$\le \frac{(1+b-a)(2n-2m-1)2^{m-1}}{(2m-1)!!} \, \|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \times$$

$$\times \bigg[\Big(\int_a^c |u^{(m)}(s)|^2\, ds \Big)^{1/2} + \Big(\int_c^b |u^{(m)}(s)|^2\, ds \Big)^{1/2} \bigg] \le \frac{\nu}{2} \int_a^b |u^{(m)}(s)|^2\, ds +$$

$$+ \frac{1}{2\nu} \Big(\frac{(1+b-a)(2n-2m-1)2^m}{(2m-1)!!} \Big)^2 \|q\|^2_{\widetilde{L}^2_{2n-2m-2,2m-2}} \tag{3.6}$$

for $a < s \le t^* \le t < b$. Due to Lemmas 2.3 and 2.4 with $a_0 = t^*$, $t_0 = a$, $b_0 = t^*$, $t_1 = b$, $\overline{p}_j(t) = (t-a)^{n-2m}(-1)^{n-m} p_j(t)$, and the equalities $\rho_0(a) = \rho_1(b) = 0$, $\mu_j(a, b, t) = \tau_j(t)$, we have

$$(-1)^{n-m} \int_s^t (\xi-a)^{n-2m} p_j(\xi) u^{(j-1)}(\tau_j(\xi)) u(\xi)\, d\xi \le$$

$$\le \bar{l}_{0j} \beta_j(t^*-a, \gamma_{0j}) \rho_0^{1/2}(b) \rho_0^{1/2}(t^*) +$$

$$+ l_{0j} \, \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \, \rho_0(t^*) + \bar{l}_{1j} \beta_j(b-t^*, \gamma_{1j}) \rho_1^{1/2}(a) \rho_1^{1/2}(t^*) +$$

$$+ l_{1j} \, \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \, \rho_1(t^*) \tag{3.7}$$

for $a < s \le t^* \le t < b$. Thus according to (3.3)–(3.7), and the inequalities $\rho_0^{1/2}(b)\rho_0^{1/2}(t^*) \le \rho$, $\rho_1^{1/2}(a)\rho_1^{1/2}(t^*) \le \rho$, we have the estimate

$$\nu_n\rho \le (\nu_n - \nu)\rho + \frac{\nu}{2}\,\rho + \\ + \frac{1}{2\nu}\Big(\frac{(1+b-a)(2n-2m-1)2^m}{(2m-1)!!}\Big)^2 \|q\|^2_{\widetilde{L}^2_{2n-2m-2,2m-2}}. \tag{3.8}$$

From (3.5) and (3.8) it immediately follows that

$$\|u^{(m)}\|_{L^2} \le r_\nu \|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}} \quad \text{for } \ 0 < \nu < \nu_n - 2\max\{B_0, B_1\}, \tag{3.9}$$

where $r_\nu = [(1+b-a)(2n-2m-1)2^m]/[\nu(2m-1)!!]$. Thus from (3.9) we obtain

$$\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2,2m-2}}, \tag{3.10}$$

where

$$r = \frac{(1+b-a)(2n-2m-1)2^m}{(\nu_n - 2\max\{B_0,\, B_1\})(2m-1)!!}\,.$$

Hence, by the definition of the numbers ν_n, B_0, B_1, it is clear that r depends only on the numbers l_{kj}, $\bar{l}_{kj}$, γ_{kj} $(k = 1,2;\, j = 1,\dots,m)$, and a, b, t^*, n. □

The proof of Theorem 1.4 is analogous to that of Theorem 1.3. The only difference is that instead of Theorem 1.1, Theorem 1.2 is applied, and we put $t = c = b$.

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Authors' addresses:

Sulkhan Mukhigulashvili

1. Mathematical Institute of the Academy of Sciences of the Czech Republic, Branch in Brno, 22 Žižkova, Brno 616 62, Czech Republic;

2. Ilia State University, Faculty of Physics and Mathematics, 32 I. Chavchavadze Ave., Tbilisi 0179, Georgia.

E-mail: mukhig@ipm.cz

Nino Partsvania

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. International Black Sea University, 2 David Agmashenebeli Alley 13km, Tbilisi 0131, Georgia.

E-mail: ninopa@rmi.ge

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Belgacem Rebiai

INVARIANT DOMAINS AND GLOBAL EXISTENCE FOR REACTION-DIFFUSION SYSTEMS WITH A TRIDIAGONAL MATRIX OF DIFFUSION COEFFICIENTS

Abstract. The aim of this study is to prove the global existence of solutions for reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients and nonhomogeneous boundary conditions. Towards this end, we make use of the appropriate techniques which are based on the invariant domains and on Lyapunov functional methods. The nonlinear reaction term has been supposed to be of polynomial growth. This result is a continuation of that due to Kouachi and Rebiai [13].

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რეზიუმე. ნაშრომის მიზანია დამტკიცდეს დიფუზიის კოეფიციენტების ტრიდიაგონალური მატრიციანი რეაქციულ-დიფუზიური სისტემების გლობალური ამონახსნების არსებობა არაერთგვაროვან სასაზღვრო პირობებში. ამ მიზნით გამოიყენება შესაბამისი ტექნიკა, რომელიც დაფუძნებულია ინვარიანტულ არეებზე და ლიაპუნოვის ფუნქციონალის მეთოდებზე. არაწრფივი რეაქციული წევრის შესახებ გაკეთებულია დაშვება, რომ მისი ზრდის რიგი პოლინომიალურია. ეს შედეგი წარმოდგენს კუაჩისა და რებიაის [13] შედეგის გავრცელებას.

1. Introduction

We consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a_{11}\Delta u - a_{12}\Delta v = f(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - a_{21}\Delta u - a_{22}\Delta v - a_{23}\Delta w = g(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.2}$$

$$\frac{\partial w}{\partial t} - a_{32}\Delta v - a_{33}\Delta w = h(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.3}$$

with the boundary conditions

$$\lambda u + (1-\lambda)\frac{\partial u}{\partial \eta} = \beta_1, \quad \lambda v + (1-\lambda)\frac{\partial v}{\partial \eta} = \beta_2, \quad \lambda w + (1-\lambda)\frac{\partial w}{\partial \eta} = \beta_3, \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.4}$$

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \text{ in } \Omega, \tag{1.5}$$

where

(i) $0 < \lambda < 1$ and $\beta_i \in \mathbb{R}$, $i = 1,2,3$, for nonhomogeneous Robin boundary conditions.

(ii) $\lambda = \beta_i = 0$, $i = 1,2,3$, for homogeneous Neumann boundary conditions.

(iii) $1 - \lambda = \beta_i = 0$, $i = 1,2,3$, for homogeneous Dirichlet boundary conditions.

Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^N$ with boundary $\partial\Omega$ and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. The diffusion terms a_{ij} ($i,j = 1,2,3$ and $(i,j) \neq (1,3),(3,1)$) are supposed to be positive constants such that

$$a_{12}a_{21}(a_{22} - a_{33}) = a_{23}a_{32}(a_{11} - a_{22})$$

and

$$a_{33}(a_{12} + a_{21})^2 + a_{11}(a_{23} + a_{32})^2 < 4a_{11}a_{22}a_{33}$$

which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

is positive definite. The eigenvalues λ_1, λ_2 and λ_3 ($\lambda_1 < \lambda_2 = a_{22} < \lambda_3$) of A are positive. If we put

$$\underline{a} = \min\{a_{11}, a_{33}\} \text{ and } \overline{a} = \max\{a_{11}, a_{33}\},$$

then the positivity of the a_{ij} implies that

$$\lambda_1 < \underline{a} < \lambda_2 < \overline{a} < \lambda_3.$$

The initial data are assumed to be in the domain

$$\Sigma = \begin{cases} \big\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ \mu_i u_0 + \nu_i w_0 \le v_0, \ i = 1, 2, 3\big\} \\ \qquad \text{if } \ \mu_i\beta_1 + \nu_i\beta_3 \le \beta_2, \ \ i = 1, 2, 3, \\ \big\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ \mu_i u_0 + \nu_i w_0 \le v_0 \le \mu_1 u_0 + \nu_1 w_0, \ i = 2, 3\big\} \\ \qquad \text{if } \ \mu_i\beta_1 + \nu_i\beta_3 \le \beta_2 \le \mu_1\beta_1 + \nu_1\beta_3, \ \ i = 2, 3, \\ \big\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ \mu_i u_0 + \nu_i w_0 \le v_0 \le \mu_2 u_0 + \nu_2 w_0, \ i = 1, 3\big\} \\ \qquad \text{if } \ \mu_i\beta_1 + \nu_i\beta_3 \le \beta_2 \le \mu_2\beta_1 + \nu_2\beta_3, \ \ i = 1, 3, \\ \big\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ \mu_3 u_0 + \nu_3 w_0 \le v_0 \le \mu_i u_0 + \nu_i w_0, \ i = 1, 2\big\} \\ \qquad \text{if } \ \mu_3\beta_1 + \nu_3\beta_3 \le v_0 \le \mu_i\beta_1 + \nu_i\beta_3, \ \ i = 1, 2, \end{cases}$$

where $\mu_1 = a_{21}/(a_{11} - \lambda_1) > 0 > \mu_2 = a_{21}/(a_{11} - \lambda_2) > \mu_3 = a_{21}/(a_{11} - \lambda_3)$, $\nu_1 = a_{23}/(a_{33} - \lambda_1) > \nu_2 = a_{23}/(a_{33} - \lambda_2) > 0 > \nu_3 = a_{23}/(a_{33} - \lambda_3)$, if we assume without loss of generality that $a_{11} < a_{33}$.

Since we use the same methods to treat all the cases, we will tackle only with the first one. We suppose that the functions f, g and h are continuously differentiable, polynomially bounded on Σ,

$$\big(f(r_1, r_2, r_3), g(r_1, r_2, r_3), h(r_1, r_2, r_3)\big) \ \text{ is in } \Sigma \text{ for all } (r_1, r_2, r_3) \text{ in } \partial\Sigma$$

(we say that (f, g, h) points into Σ on $\partial\Sigma$), i.e.,

$$\mu_i f(r_1, r_2, r_3) + \nu_i h(r_1, r_2, r_3) \le g(r_1, r_2, r_3), \tag{1.6}$$

for all r_1, r_2 and r_3 such that $\mu_j r_1 + \nu_j r_3 \le r_2 = \mu_i r_1 + \nu_i r_3, \ j = 1, 2, 3$ $(j \ne i)$, $i = 1, 2, 3$, and for positive constants E and D, we have

$$(Ef + Dg + h)(u, v, w) \le C_1(u + v + w + 1) \tag{1.7}$$

for all (u, v, w) in Σ, where C_1 is a positive constant.

In the two-component case, where $a_{12} = 0$, Kouachi and Youkana [14] generalized the method of Haraux and Youkana [4] with the reaction terms $f(u, v) = -\lambda F(u, v)$ and $g(u, v) = +\mu F(u, v)$ with $F(u, v) \ge 0$, requiring the condition

$$\lim_{s \to +\infty} \Big[\frac{\ln(1 + F(r, s))}{s}\Big] < \alpha^* \ \text{ for any } \ r \ge 0,$$

with

$$\alpha^* = \frac{2a_{11}a_{22}}{n(a_{11} - a_{22})^2\|u_0\|_\infty} \min\Big\{\frac{\lambda}{\mu}, \frac{a_{11} - a_{22}}{a_{21}}\Big\},$$

where the positive diffusion coefficients a_{11}, a_{22} satisfy $a_{11} > a_{22}$ and a_{21}, λ, μ are positive constants. This condition reflects a weak exponential growth of the function F. Kanel and Kirane [6] proved the global existence in the case where $g(u, v) = -f(u, v) = uv^n$ and n is an odd integer, under the embarrassing condition $|a_{12} - a_{21}| < C_p$, where C_p contains a constant from Solonnikov's estimate [19]. Later, in [7] they improved their results to obtain the global existence under the restrictions

H$_1$. $a_{22} < a_{11} + a_{21}$,

H$_2$. $a_{12} < \varepsilon_0 = \dfrac{a_{11}a_{22}(a_{11} + a_{21} - a_{22})}{a_{11}a_{22} + a_{21}(a_{11} + a_{21} - a_{22})}$ if $a_{11} \le a_{22} < a_{11} + a_{21}$,

H$_3$. $a_{12} < \min\left\{\dfrac{1}{2}(a_{11} + a_{21}), \varepsilon_0\right\}$ if $a_{22} < a_{11}$,

and $|F(v)| \le C_F(1 + |v|^{1-\varepsilon})$, $vF(v) \ge 0$ for all $v \in \mathbb{R}$, where ε and C_F are positive constants with $\varepsilon<1$ and $g(u,v)=-f(u,v)=uF(v)$.

Kouachi [12] has proved the global existence for solutions of two-component reaction-diffusion systems with a general full matrix of diffusion coefficients and nonhomogeneous boundary conditions. Recently, we proved the global existence for solutions of three-component reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients and nonhomogeneous boundary conditions where the positive diffusion coefficients a_{11}, a_{33} are equal (see Kouachi and Rebiai [13]).

The present investigation is a continuation work of that obtained in [13]. In this study we will treat the case where $a_{11} \ne a_{33}$.

We note that the case of strongly coupled systems which are not triangular in the diffusion part is quite more difficult. As a consequence of the blow-up of the solutions found in [17], we can indeed prove that there is the blow-up of the solutions in finite time for such nontriangular systems even though the initial data are regular, the solutions are positive and the nonlinear terms are negative, a structure that ensured the global existence in the diagonal case. For this purpose, we construct the invariant domains in which we can demonstrate that for any initial data in those domains, problem (1.1)–(1.5) is equivalent to the problem for which the global existence follows from the usual techniques based on Lyapunov functionals (see Kirane and Kouachi [8], Kouachi and Youkana [14] and Kouachi [12]).

Many chemical and biological operations are described by means of reaction diffusion systems with a tridiagonal matrix of diffusion coefficients. The components $u(t,x)$, $v(t,x)$ and $w(t,x)$ can be represented either by chemical concentrations or biological population densities (see, e.g., Cussler [1] and [2]). For example, in chemistry, an n-species reaction-diffusion system with cross-diffusion can be described by the following system of partial differential equations

$$\frac{\partial c_i}{\partial t} - \operatorname{div}(\nabla D_{ii} c_i) - \sum_{j\ne i} \operatorname{div}(\nabla D_{ij} c_j) = R_i(c_1, \dots, c_n), \quad i,j = 1,2,\dots,n,$$

where $R_i(c_1, \dots, c_n)$ are the reactive terms, D_{ii} are the main-diffusion coefficients and the cross-diffusion term $\operatorname{div}(\nabla D_{ij} c_j)$ links the gradient of species c_j to the flux of species c_i. If $D_{ij} \ge 0$, then the ith species diffuses from larger to smaller concentrations of the jth species, analogous to the case of ordinary self-diffusion. If $D_{ij} < 0$, then the ith species diffuses in the opposite direction, against the gradient ∇c_j.

Throughout this work, we denote by $\|\cdot\|_p$, $p \in [1, +\infty[$ the norm in $L^p(\Omega)$ and $\|\cdot\|_\infty$ the norm in $C(\overline{\Omega})$ or $L^\infty(\Omega)$.

2. The Local Existence and Invariant Domains

The study of local existence and uniqueness of solutions (u,v,w) of (1.1)–(1.5) follows from the basic existence theory for parabolic semilinear equations (see, e.g., [3], [5] and [16]). As a consequence, for any initial data in $C(\overline{\Omega})$ or $L^\infty(\Omega)$ there exists $T^* \in]0,+\infty]$ such that (1.1)–(1.5) has a unique classical solution on $[0,T^*[\times\Omega$. Furthermore, if $T^* < +\infty$, then

$$\lim_{t\uparrow T^*}\big(\|u(t)\|_\infty + \|v(t)\|_\infty + \|w(t)\|_\infty\big) = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty + \|w(t)\|_\infty \le C \text{ for all } t \in [0,T^*[\,,$$

then $T^* = +\infty$.

Since the initial conditions are in Σ, then under the assumptions (1.6), the next proposition says that the classical solution of (1.1)–(1.5) on $[0,T^*[\times\Omega$ remains in Σ for all t in $[0,T^*[$.

Proposition 1. *Suppose that (f,g,h) points into Σ on $\partial\Sigma$. Then for any (u_0,v_0,w_0) in Σ the solution (u,v,w) of the problem (1.1)–(1.5) remains in Σ for all t in $[0,T^*[$.*

Proof. Let $(x_{i1},x_{i2},x_{i3})^t$, $i=1,2,3$, be the eigenvectors of the matrix A^t associate with its eigenvalues λ_i, $i=1,2,3$ ($\lambda_1<\lambda_2<\lambda_3$). Multiplying equations (1.1), (1.2) and (1.3) of the given reaction-diffusion system by x_{i1}, x_{i2} and x_{i3}, respectively, and summing the resulting equations, we get

$$\frac{\partial}{\partial t}z_1 - \lambda_1\Delta z_1 = F_1(z_1,z_2,z_3) \text{ in }]0,T^*[\times\Omega, \tag{2.1}$$

$$\frac{\partial}{\partial t}z_2 - \lambda_2\Delta z_2 = F_2(z_1,z_2,z_3) \text{ in }]0,T^*[\times\Omega, \tag{2.2}$$

$$\frac{\partial}{\partial t}z_3 - \lambda_3\Delta z_3 = F_3(z_1,z_2,z_3) \text{ in }]0,T^*[\times\Omega, \tag{2.3}$$

with the boundary conditions

$$\lambda z_i + (1-\lambda)\frac{\partial z_i}{\partial\eta} = \rho_i, \quad i=1,2,3, \text{ on }]0,T^*[\times\partial\Omega, \tag{2.4}$$

and the initial data

$$z_i(0,x) = z_i^0(x), \quad i=1,2,3, \text{ in } \Omega, \tag{2.5}$$

where

$$z_i = x_{i1}u + x_{i2}v + x_{i3}w, \quad i=1,2,3, \text{ in }]0,T^*[\times\Omega, \tag{2.6}$$
$$\rho_i = x_{i1}\beta_1 + x_{i2}\beta_2 + x_{i3}\beta_3, \quad i=1,2,3,$$

and

$$F_i(z_1,z_2,z_3) = x_{i1}f + x_{i2}g + x_{i3}h, \quad i=1,2,3, \tag{2.7}$$

for all (u,v,w) in Σ.

We note that the condition of the parabolicity of the system (1.1)–(1.3) implies one of (2.1)–(2.3). Since λ_1, λ_2 and λ_3 are the eigenvalues of the matrix A^t, the problem (1.1)–(1.5) is equivalent to the problem (2.1)–(2.5), and to prove that Σ is an invariant domain for the system (1.1)–(1.3) it suffices to prove that the domain

$$\left\{(z_1^0, z_2^0, z_3^0) \in \mathbb{R}^3 : \ z_i^0 \geq 0, \ i = 1,2,3\right\} = (\mathbb{R}^+)^3 \tag{2.8}$$

is invariant for the system (2.1)–(2.3) and there exist some constants x_{ij}, $i, j = 1, 2, 3$, such that

$$\Sigma = \left\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ z_i^0 = x_{i1}u_0 + x_{i2}v_0 + x_{i3}w_0 \geq 0, \ i = 1,2,3\right\}. \tag{2.9}$$

Since $(x_{i1}, x_{i2}, x_{i3})^t$ is an eigenvector of the matrix A^t associated to the eigenvalue λ_i, $i = 1, 2, 3$, we have

$$\begin{cases} (a_{11} - \lambda_i)x_{i1} + a_{21}x_{i2} = 0, \\ a_{23}x_{i2} + (a_{33} - \lambda_i)x_{i3} = 0, \end{cases} \qquad i = 1,2,3.$$

If we assume, without loss of generality, that $a_{11} < a_{33}$ and choose $x_{12} = x_{22} = x_{32} = 1$, then we have $x_{i1}u_0 + x_{i2}v_0 + x_{i3}w_0 \geq 0, \ i = 1,2,3 \iff \mu_i u_0 + \nu_i w_0 \leq v_0, \ i = 1,2,3$. Thus (2.9) is proved and (2.6) can be written as

$$z_i = -\mu_i u + v - \nu_i w, \ \ i = 1,2,3. \tag{2.6a}$$

Now, to prove that the domain $(\mathbb{R}^+)^3$ is invariant for the system (2.1)–(2.3), it suffices to show that $F_i(z_1, z_2, z_3) \geq 0$ for all (z_1, z_2, z_3) such that $z_i = 0$ and $z_j \geq 0$, $j = 1,2,3$ $(j \neq i)$, $i = 1,2,3$, thanks to the invariant domain method (see Smoller [18]). Using the expressions (2.7), we get

$$F_i = -\mu_i f + g - \nu_i h, \ \ i = 1,2,3, \tag{2.7a}$$

for all (u, v, w) in Σ. Since from (1.6) we have $F_i(z_1, z_2, z_3) \geq 0$ for all (z_1, z_2, z_3) such that $z_i = 0$ and $z_j \geq 0$, $j = 1,2,3$ $(j \neq i)$, $i = 1,2,3$, we obtain $z_i(t, x) \geq 0$, $i = 1,2,3$, for all $(t, x) \in [0, T^*[\times \Omega$. As a consequence, Σ is an invariant domain for the system (1.1)–(1.3). □

In addition, the system (1.1)–(1.3) with the boundary conditions (1.4) and initial data in Σ is equivalent to the system (2.1)–(2.3) with the boundary conditions (2.4) and positive initial data (2.5).

Once the invariant domains are constructed and since ρ_i, $i = 1,2,3$, given by $\rho_i = -\mu_i\beta_1 + \beta_2 - \nu_i\beta_3$, $i = 1,2,3$, are positive, we can apply the Lyapunov technique and establish the global existence of unique solutions for (1.1)–(1.5).

3. Global Existence

As the determinant of the linear algebraic system (2.6), with respect to variables u, v and w, is different from zero, to prove the global existence of solutions of the problem (1.1)–(1.5) one needs to prove it for the problem (2.1)–(2.5). To this end, it is well known that (see Henry [5]) it suffices to derive a uniform estimate of $\|F_i(z_1, z_2, z_3)\|_p$, $i = 1,2,3$, on $[0, T]$, $T < T^*$, for some $p > N/2$.

Let θ and σ be two positive constants such that

$$\theta > A_{12}, \tag{3.1}$$

$$(\theta^2 - A_{12}^2)(\sigma^2 - A_{23}^2) > (A_{13} - A_{12}A_{23})^2, \tag{3.2}$$

where $A_{ij} = \frac{\lambda_i+\lambda_j}{2\sqrt{\lambda_i\lambda_j}}$, $i, j = 1, 2, 3$ $(i < j)$, and let

$$\theta_q = \theta^{q^2} \text{ and } \sigma_p = \sigma^{p^2} \text{ for } q = 0, 1, \ldots, p \text{ and } p = 0, 1, \ldots, n, \tag{3.3}$$

with n as a positive integer. The main result of this section is

Theorem 1. *Let* (z_1, z_2, z_3) *be any positive solution of* (2.1)–(2.5) *on* $[0, T^*[\times\Omega$*; let the functional*

$$t \longmapsto L(t) = \int_\Omega H_n\big(z_1(t,x), z_2(t,x), z_3(t,x)\big)\, dx, \tag{3.4}$$

where

$$H_n(z_1, z_2, z_3) = \sum_{p=0}^{n}\sum_{q=0}^{p} C_n^p C_p^q \theta_q\sigma_p z_1^q z_2^{p-q} z_3^{n-p}, \tag{3.5}$$

with n *being a positive integer and* $C_n^p = \frac{n!}{(n-p)!p!}$.

Then, the functional L *is uniformly bounded on* $[0, T]$, $T < T^*$.

For the proof of Theorem 1 we need some preparatory Lemmas.

Lemma 1. *Let* H_n *be the homogeneous polynomial defined by* (3.5). *Then*

$$\frac{\partial H_n}{\partial z_1} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-1)-p}, \tag{3.6}$$

$$\frac{\partial H_n}{\partial z_2} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-1)-p}, \tag{3.7}$$

$$\frac{\partial H_n}{\partial z_3} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q\sigma_p z_1^q z_2^{p-q} z_3^{(n-1)-p}. \tag{3.8}$$

Proof. Differentiating H_n with respect to z_1 and using the fact that

$$qC_p^q = pC_{p-1}^{q-1} \text{ and } pC_n^p = nC_{n-1}^{p-1} \tag{3.9}$$

for $q = 1, 2, \ldots, p$, $p = 1, 2, \ldots, n$, we get

$$\frac{\partial H_n}{\partial z_1} = n\sum_{p=1}^{n}\sum_{q=1}^{p} C_{n-1}^{p-1} C_{p-1}^{q-1} \theta_q\sigma_p z_1^{q-1} z_2^{p-q} z_3^{n-p}.$$

Replacing in the sums the indices $q-1$ by q and $p-1$ by p, we deduce (3.6). For the formula (3.7), differentiating H_n with respect to z_2, taking into account

$$C_p^q = C_p^{p-q}, \quad q = 0, 1, \ldots, p-1 \text{ and } p = 1, 2, \ldots, n, \tag{3.10}$$

using (3.9) and replacing the index $p-1$ by p, we get (3.7).

Finally, we have

$$\frac{\partial H_n}{\partial z_3} = \sum_{p=0}^{n-1}\sum_{q=0}^{p}(n-p)C_n^p C_p^q \theta_q \sigma_p z_1^q z_2^{p-q} z_3^{n-p-1}.$$

Since $(n-p)C_n^p = (n-p)C_n^{n-p} = nC_{n-1}^{n-p-1} = nC_{n-1}^p$, we get (3.8). $\square$

Lemma 2. *The second partial derivatives of H_n are given by*

$$\frac{\partial^2 H_n}{\partial z_1^2} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+2}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.11}$$

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+1}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.12}$$

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.13}$$

$$\frac{\partial^2 H_n}{\partial z_2^2} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.14}$$

$$\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.15}$$

$$\frac{\partial^2 H_n}{\partial z_3^2} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q}\sigma_{p} z_1^q z_2^{p-q} z_3^{(n-2)-p}. \tag{3.16}$$

Proof. Differentiating $\frac{\partial H_n}{\partial z_1}$ given by (3.6) with respect to z_1, we obtain

$$\frac{\partial^2 H_n}{\partial z_1^2} = n\sum_{p=1}^{n-1}\sum_{q=1}^{p} qC_{n-1}^p C_p^q \theta_{q+1}\sigma_{q+1} z_1^{q-1} z_2^{p-q} z_3^{(n-1)-p}.$$

Using (3.9), we get (3.11).

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n\sum_{p=1}^{n-1}\sum_{q=0}^{p-1}(p-q)C_{n-1}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q-1} z_3^{(n-1)-p}.$$

Applying (3.10) and then (3.9), we get (3.12).

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n\sum_{p=0}^{n-2}\sum_{q=0}^{p}((n-1)-p)C_{n-1}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-2)-p}.$$

Applying successively (3.10), (3.9) and (3.10) for the second time, we deduce (3.13).

$$\frac{\partial^2 H_n}{\partial z_2^2} = n \sum_{p=1}^{n-1} \sum_{q=0}^{p-1} (p-q) C_{n-1}^p C_p^q \theta_q \sigma_{p+1} z_1^q z_2^{p-q-1} z_3^{(n-1)-p}.$$

The application of (3.10) and then (3.9) yields (3.14).

$$\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n \sum_{p=0}^{n-2} \sum_{q=0}^{p} ((n-1)-p) C_{n-1}^p C_p^q \theta_q \sigma_p z_1^q z_2^{p-q} z_3^{(n-2)-p}.$$

Applying (3.10) and then (3.9), we get (3.15). Finally, we get (3.16) by differentiating $\frac{\partial H_n}{\partial z_3}$ with respect to z_3 and applying successively (3.10), (3.9) and (3.10) for the second time. □

Proof of Theorem 1. Differentiating L with respect to t, we find that

$$\begin{aligned} L'(t) &= \int_\Omega \Big(\frac{\partial H_n}{\partial z_1} \frac{\partial z_1}{\partial t} + \frac{\partial H_n}{\partial z_2} \frac{\partial z_2}{\partial t} + \frac{\partial H_n}{\partial z_3} \frac{\partial z_3}{\partial t} \Big) \, dx = \\ &= \int_\Omega \Big(\lambda_1 \frac{\partial H_n}{\partial z_1} \Delta z_1 + \lambda_2 \frac{\partial H_n}{\partial z_2} \Delta z_2 + \lambda_3 \frac{\partial H_n}{\partial z_3} \Delta z_3 \Big) \, dx + \\ &\quad + \int_\Omega \Big(\frac{\partial H_n}{\partial z_1} F_1 + \frac{\partial H_n}{\partial z_2} F_2 + \frac{\partial H_n}{\partial z_3} F_3 \Big) \, dx =: I + J, \end{aligned}$$

Using Green's formula in I, we get $I = I_1 + I_2$, where

$$I_1 = \int_{\partial\Omega} \Big(\lambda_1 \frac{\partial H_n}{\partial z_1} \frac{\partial z_1}{\partial \eta} + \lambda_2 \frac{\partial H_n}{\partial z_2} \frac{\partial z_2}{\partial \eta} + \lambda_3 \frac{\partial H_n}{\partial z_3} \frac{\partial z_3}{\partial \eta} \Big) \, ds,$$

where ds denotes the $(n-1)$-dimensional surface element, and

$$\begin{aligned} I_2 = - \int_\Omega \Big[& \lambda_1 \frac{\partial^2 H_n}{\partial z_1^2} |\nabla z_1|^2 + (\lambda_1 + \lambda_2) \frac{\partial^2 H_n}{\partial z_1 \partial z_2} \nabla z_1 \nabla z_2 + \\ & + (\lambda_1 + \lambda_3) \frac{\partial^2 H_n}{\partial z_1 \partial z_3} \nabla z_1 \nabla z_3 + \lambda_2 \frac{\partial^2 H_n}{\partial z_2^2} |\nabla z_2|^2 + \\ & + (\lambda_2 + \lambda_3) \frac{\partial^2 H_n}{\partial z_2 \partial z_3} \nabla z_2 \nabla z_3 + \lambda_3 \frac{\partial^2 H_n}{\partial z_3^2} |\nabla z_3|^2 \Big] \, dx. \end{aligned}$$

We prove that there exists a positive constant C_2 independent of $t \in [0, T^*[$ such that

$$I_1 \le C_2 \ \text{ for all } \ t \in [0, T^*[\,, \tag{3.17}$$

and that

$$I_2 \le 0. \tag{3.18}$$

To see this, we follow the same reasoning as in [11].

(i) If $0 < \lambda < 1$, using the boundary conditions (2.4), we get

$$I_1 = \int_{\partial\Omega} \Big(\lambda_1 \frac{\partial H_n}{\partial z_1} (\gamma_1 - \alpha z_1) + \lambda_2 \frac{\partial H_n}{\partial z_2} (\gamma_2 - \alpha z_2) + \lambda_3 \frac{\partial H_n}{\partial z_3} (\gamma_3 - \alpha z_3) \Big) \, ds,$$

where $\alpha = \frac{\lambda}{1-\lambda}$ and $\gamma_i = \frac{\rho_i}{1-\lambda}$, $i = 1, 2, 3$. Since

$$H(z_1, z_2, z_3) = \lambda_1 \frac{\partial H_n}{\partial z_1}(\gamma_1 - \alpha z_1) + \lambda_2 \frac{\partial H_n}{\partial z_2}(\gamma_2 - \alpha z_2) + \\ + \lambda_3 \frac{\partial H_n}{\partial z_3}(\gamma_3 - \alpha z_3) = P_{n-1}(z_1, z_2, z_3) - Q_n(z_1, z_2, z_3),$$

where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n, and since the solution is positive, we obtain

$$\limsup_{(|z_1|+|z_2|+|z_3|)\to+\infty} H(z_1, z_2, z_3) = -\infty, \tag{3.19}$$

which proves that H is uniformly bounded on $(\mathbb{R}^+)^3$, and consequently (3.17).

(ii) If $\lambda = 0$, then $I_1 = 0$ on $[0, T^*[$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0, T^*[\times \Omega$ implies $\partial z_1/\partial\eta \leq 0$, $\partial z_2/\partial\eta \leq 0$ and $\partial z_3/\partial\eta \leq 0$ on $[0, T^*[\times\partial\Omega$. Consequently, one again gets (3.17) with $C_2 = 0$.

We now prove (3.18). Applying Lemma 2, we obtain

$$I_2 = -n(n-1)\int_\Omega \sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \left[(B_{pq}z)\cdot z\right] dx,$$

where

$$B_{pq} = \begin{pmatrix} \lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} \\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} \\ \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} & \lambda_3\theta_q\sigma_p \end{pmatrix}$$

for $q = 0, 1, \ldots, p$, $p = 0, 1, \ldots, n-2$ and $z = (\nabla z_1, \nabla z_2, \nabla z_3)^t$.

The quadratic forms (with respect to $\nabla z_1, \nabla z_2$ and ∇z_3) associated with the matrices B_{pq}, $q = 0, 1, \ldots, p$, $p = 0, 1, \ldots, n-2$, are positive, since their main determinants Δ_1, Δ_2 and Δ_3 are positive too, according to the Sylvester criterion. To see this, we have

1) $\Delta_1 = \lambda_1\theta_{q+2}\sigma_{p+2} > 0$ for $q = 0, 1, \ldots, p$ $p = 0, 1, \ldots, n-2$.

2) $$\Delta_2 = \begin{vmatrix} \lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} \\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} \end{vmatrix} = \lambda_1\lambda_2\theta_{q+1}^2\sigma_{p+2}^2(\theta^2 - A_{12}^2),$$
for $q = 0, 1, \ldots, p$ and $p = 0, 1, \ldots, n-2$.
Using (3.1), we get $\Delta_2 > 0$.

$$3)\ \Delta_3 = \begin{vmatrix} \lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\,\theta_{q+1}\sigma_{p+2} & \frac{\lambda_1+\lambda_3}{2}\,\theta_{q+1}\sigma_{p+1} \\ \frac{\lambda_1+\lambda_2}{2}\,\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} & \frac{\lambda_2+\lambda_3}{2}\,\theta_q\sigma_{p+1} \\ \frac{\lambda_1+\lambda_3}{2}\,\theta_{q+1}\sigma_{p+1} & \frac{\lambda_2+\lambda_3}{2}\,\theta_q\sigma_{p+1} & \lambda_3\theta_q\sigma_p \end{vmatrix} =$$

$$= \lambda_1\lambda_2\lambda_3\theta_{q+1}^2\theta_q\sigma_{p+2}\sigma_{p+1}^2\big[(\theta^2 - A_{12}^2)(\sigma^2 - A_{23}^2) - (A_{13} - A_{12}A_{23})^2\big],$$

for $q = 0, 1, \dots, p$ and $p = 0, 1, \dots, n-2$.

Using (3.2), we get $\Delta_3 > 0$. Consequently we have (3.18).

Substitution of the expressions of the partial derivatives given by Lemma 1 in the second integral yields

$$J = \int_\Omega \Big[n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q z_1^q z_2^{p-q} z_3^{(n-1)-p}\Big]\times$$
$$\times\,(\theta_{q+1}\sigma_{p+1}F_1 + \theta_q\sigma_{p+1}F_2 + \theta_q\sigma_p F_3)\,dx.$$

Using the expressions (2.7a), we obtain

$$\theta_{q+1}\sigma_{p+1}F_1 + \theta_q\sigma_{p+1}F_2 + \theta_q\sigma_p F_3 = -(\mu_1\theta_{q+1}\sigma_{p+1} + \mu_2\theta_q\sigma_{p+1} + \mu_3\theta_q\sigma_p)f +$$
$$+ (\theta_{q+1}\sigma_{p+1} + \theta_q\sigma_{p+1} + \theta_q\sigma_p)g - (\nu_1\theta_{q+1}\sigma_{p+1} + \nu_2\theta_q\sigma_{p+1} + \nu_3\theta_q\sigma_p)h =$$
$$= -\theta_{q+1}\sigma_{p+1}\Big(\nu_1 + \nu_2\frac{\theta_q}{\theta_{q+1}} + \nu_3\,\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}\Big)\times$$
$$\times\left(\frac{\mu_1 + \mu_2\,\frac{\theta_q}{\theta_{q+1}} + \mu_3\,\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}}{\nu_1 + \nu_2\,\frac{\theta_q}{\theta_{q+1}} + \nu_3\,\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}}\,f - \frac{1 + \frac{\theta_q}{\theta_{q+1}} + \frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}}{\nu_1 + \nu_2\,\frac{\theta_q}{\theta_{q+1}} + \nu_3\,\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}}\,g + h\right).$$

Since $\frac{\theta_q}{\theta_{q+1}}$ and $\frac{\sigma_p}{\sigma_{p+1}}$ are sufficiently large if we choose θ and σ sufficiently large, by using the condition (1.7) and the relation (2.6a) successively, for an appropriate constant C_3, we get

$$J \le C_3\int_\Omega\Big[\sum_{p=0}^{n-1}\sum_{q=0}^{p}(z_1 + z_2 + z_3 + 1)C_{n-1}^p C_p^q z_1^q z_2^{p-q} z_3^{(n-1)-p}\Big]\,dx.$$

To prove that the functional L is uniformly bounded on the interval $[0, T]$, we first write

$$\sum_{p=0}^{n-1}\sum_{q=0}^{p}(z_1 + z_2 + z_3 + 1)C_{n-1}^p C_p^q z_1^q z_2^{p-q} z_3^{(n-1)-p} =$$
$$= R_n(z_1, z_2, z_3) + S_{n-1}(z_1, z_2, z_3),$$

where $R_n(z_1, z_2, z_3)$ and $S_{n-1}(z_1, z_2, z_3)$ are two homogeneous polynomials of degrees n and $n-1$, respectively. First, since the polynomials H_n and R_n are of degree n, there exists a positive constant C_4 such that $\int_\Omega R_n(z_1, z_2, z_3)\,dx \le C_4\int_\Omega H_n(z_1, z_2, z_3)\,dx$. Applying Hölder's inequality to the integral $\int_\Omega S_{n-1}(z_1, z_2, z_3)\, dx$, one gets

$$\int_\Omega S_{n-1}(z_1, z_2, z_3)\, dx \le (meas\ \Omega)^{\frac{1}{n}} \Big(\int_\Omega \big(S_{n-1}(z_1, z_2, z_3)\big)^{\frac{n}{n-1}}\, dx\Big)^{\frac{n-1}{n}}.$$

Since for all $z_1 \ge 0$ and $z_2, z_3 > 0$,

$$\frac{(S_{n-1}(z_1, z_2, z_3))^{\frac{n}{n-1}}}{H_n(z_1, z_2, z_3)} = \frac{(S_{n-1}(\xi_1, \xi_2, 1))^{\frac{n}{n-1}}}{H_n(\xi_1, \xi_2, 1)},$$

where $\xi_1 = z_1/z_2$, $\xi_2 = z_2/z_3$ and

$$\lim_{\substack{\xi_1\to+\infty\\ \xi_2\to+\infty}} \frac{(S_{n-1}(\xi_1, \xi_2, 1))^{\frac{n}{n-1}}}{H_n(\xi_1, \xi_2, 1)} < +\infty,$$

one asserts that there exists a positive constant C_5 such that

$$\frac{(S_{n-1}(z_1, z_2, z_3))^{\frac{n}{n-1}}}{H_n(z_1, z_2, z_3)} \le C_5 \ \text{ for all }\ z_1, z_2, z_3 \ge 0.$$

Due to (3.19), there exist η_i, $i = 1, 2, 3$, such that for all $z_i > \eta_i$ the functional L satisfies the differential inequality $L'(t) \le C_6 L(t) + C_7 L^{\frac{n-1}{n}}(t)$, which for $Z = L^{\frac{1}{n}}$ can be written as $nZ' \le C_6 Z + C_7$. A simple integration gives a uniform bound of the functional L on the interval $[0, T]$.

On the other hand, if z_i is in the compact interval $[0, \eta_i]$, then the continuous function $(z_1, z_2, z_3) \longmapsto H_n(z_1, z_2, z_3)$ is bounded. Thus, the functional L is uniformly bounded on $[0, T]$. This completes the proof of Theorem 1. □

Corollary 1. *Suppose that the functions f, g and h are continuously differentiable on Σ, point into Σ on $\partial\Sigma$ and satisfy the condition* (1.7)*. Then all uniformly bounded solutions on Ω of* (1.1)–(1.5) *with initial data in Σ are in $L^\infty(0, T; L^p(\Omega))$ for all $p \ge 1$.*

Proof. The proof of this Corollary is an immediate consequence of Theorem 1, the trivial inequality $\int_\Omega (z_1+z_2+z_3)^p\, dx \le L(t)$ on $[0, T^*[$, and (2.6a). □

Proposition 2. *Under the hypothesis of Corollary* 1*, if the functions f, g and h are polynomially bounded on Σ, then all uniformly bounded solutions on Ω of* (1.1)–(1.4) *with the initial data in Σ are global in time.*

Proof. As it has been mentioned above, it suffices to derive a uniform estimate of $\|F_1(z_1, z_2, z_3)\|_p$, $\|F_2(z_1, z_2, z_3)\|_p$ and $\|F_3(z_1, z_2, z_3)\|_p$ on $[0, T]$, $T < T^*$ for some $p > \frac{N}{2}$. Since the reaction terms $f(u, v, w)$, $g(u, v, w)$ and $h(u, v, w)$ are polynomially bounded on Σ, by using the relations (2.6a) and (2.7a) we get that such are $F_1(z_1, z_2, z_3)$, $F_2(z_1, z_2, z_3)$ and $F_3(z_1, z_2, z_3)$, and the proof becomes an immediate consequence of Corollary 1. □

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Author's address:

Department of Mathematics, University of Tebessa, 12002, Algeria.
e-mail: `brebiai@gmail.com`

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Akihito Shibuya

ASYMPTOTIC ANALYSIS OF POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS IN THE FRAMEWORK OF REGULAR VARIATION

Dedicated to the 80th birthday anniversary
of Professor Takaŝi Kusano

Abstract. This paper is devoted to the asymptotic analysis of positive solutions of a class of second order functional differential equations in the framework of regular variation. It is shown that precise asymptotic behavior of intermediate positive solutions of the equations under consideration can be established by means of Karamata's integration theorem combined with fixed point techniques.

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რეზიუმე. ნაშრომი ეძღვნება მეორე რიგის არაწრფივი ფუნქციონალურ დიფერენციალური განტოლებვბის დადებითი ამონახსნების ასიმპტოტურ ანალიზს რეგულარული ვარიაციის ჩარჩოებში. ნაჩვენებია, რომ განხილული განტოლებების შუალედური დადებითი ამონახსნების ასიმპტოტური ყიფაქცევა შეიძლება დადგენილ იქნას კარამატას ინტეგრირების თეორემისა და უძრავი წერტილის ტექნიკის შერწყმით.

1. Introduction

This paper is devoted to the study of the existence and asymptotic behavior of positive solutions of second order Emden–Fowler type functional differential equations of the form

$$x''(t) + q(t)|x(g(t))|^{\gamma} \operatorname{sgn} x(g(t)) = 0, \tag{A}$$

where

(a) γ is a positive constant less than 1,

(b) $q : [a, \infty) \to (0, \infty)$ is a continuous function, $a > 0$,

(c) $g : [a, \infty) \to (0, \infty)$ is a continuous increasing function such that

$$g(t) < t \quad \text{and} \quad \lim_{t\to\infty} g(t) = \infty.$$

This equation (A) is called *sublinear*. Equation (A) with $\gamma > 1$ is said to be *superlinear*.

By a *proper solution* of equation (A) we mean a function $x(t)$ which is defined in a neighborhood of infinity and is nontrivial in the sense that

$$\sup\big\{|x(t)| : \ t \geqq T\big\} > 0 \ \text{ for any sufficiently large } \ T > a.$$

A proper solution of (A) is said to be *oscillatory* if it has an infinite sequence of zeros clustering at infinity and *nonoscillatory* otherwise. Thus a nonoscillatory solution is eventually positive or negative.

We are interested in the existence and asymptotic behavior of possible nonoscillatory solutions of (A). If $x(t)$ is a solution of (A), then so is $-x(t)$, and hence in studying nonoscillatory solutions it suffices to restrict our consideration to *positive solutions*. It is known that any positive solution $x(t)$ falls into one of the following three types:

(I) $\lim_{t\to\infty} x(t) = const > 0$,

(II) $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} \dfrac{x(t)}{t} = 0$,

(III) $\lim_{t\to\infty} \dfrac{x(t)}{t} = const > 0$.

Our primary concern in this paper will be with type (II)-solutions, which are referred to as *intermediate solutions of* (A), because the other two types of solutions are fully understood as the following statements show:

(i) (A) has solutions of type (I) if and only if $\int_a^\infty tq(t)\,dt < \infty$;

(ii) (A) has solutions of type (III) if and only if $\int_a^\infty g(t)^{\gamma} q(t)\,dt < \infty$.

It seems to be very difficult to obtain detailed information about the existence of intermediate solutions of (A) having precise asymptotic behavior at infinity in the case of general positive continuous $q(t)$, and hence we limit ourselves to the case where the coefficient $q(t)$ is a regularly varying function (in the sense of Karamata) and focus our attention on regularly varying solutions of (A). Analyzing equation (A) in the framework of regular variation was motivated by a recent interesting paper [2] in which complete analysis has been made of positive regularly varying solutions of type (II) of the sublinear Emden–Folwer equation

$$x'' + q(t)|x|^{\gamma}\operatorname{sgn} x = 0,$$

under the assumption that $q(t)$ is regularly varying.

It is natural to obtain the desired solutions of (A) by solving the integral equation

$$x(t) = x_0 + \int_{T_0}^{t}\int_{s}^{\infty} q(r)x(g(r))^{\gamma}\, dr\, ds, \quad t \geqq T_0, \tag{B}$$

where $x_0 > 0$ and $T_0 > a$. Note that any type (II)-solution of (A) satisfies (B) for some x_0 and T_0. In view of the difficulty in the analysis of (B) for general retarded argument $g(t)$ we confine our attention to the class of $g(t)$ such that

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1. \tag{1.1}$$

Associated with (B) is the following integral asymptotic relation

$$x(t) \sim \int_{T_0}^{t}\int_{s}^{\infty} q(r)x(g(r))^{\gamma}\, dr\, ds, \quad t \to \infty, \tag{C}$$

which is regarded as an approximation at infinity of (B). Here and throughout, the symbol $\sim$ is used to mean the asymptotic equivalence

$$f(t) \sim g(t), \quad t \to \infty \iff \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

It is shown that if $q(t)$ is regularly varying and $g(t)$ satisfies (1.1), then one can acquire full knowledge of the structure of all possible regularly varying solutions of (C), and that the results for (C) thus obtained play a central role in establishing the existence of intermediate solutions with accurate asymptotic behavior at infinity for equation (A).

Our main results are presented in Section 3 consisting of three subsections. The first subsection is devoted to the analysis of relation (C) with regularly varying $q(t)$ by means of regular variation under condition (1.1), and three types of its regularly varying solutions are shown to exist. These three types of solutions are effectively used in the second subsection to construct three kinds of intermediate solutions for equation (A) with the help of fixed point techniques. In the third subsection two kinds of intermediate solutions thus constructed will be verified to be regularly varying. The definition and some basic properties of regularly varying functions will be summarized in Section 2 of preliminary nature.

2. Regularly Varying Functions

We state here the definition and some basic properties of regularly varying functions which will be needed in developing our main results in the next section.

Definition 2.1. A measurable function $f : [0,\infty) \to (0,\infty)$ is called *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \text{ for all } \lambda > 0.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV to denote $\mathrm{RV}(0)$, and call members of SV *slowly varying functions.* Any function $f(t) \in \mathrm{RV}(\rho)$ is written as $f(t) = t^{\rho} g(t)$ with $g(t) \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. One of the most important properties of regularly varying functions is the following *representation theorem.*

Definition 2.2. $f(t) \in \mathrm{RV}(\rho)$ *if and only if* $f(t)$ *is represented in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s}\, ds \right\}, \quad t \geqq t_0,$$

for some $t_0 > 0$ *and for some measurable functions* $c(t)$ *and* $\delta(t)$ *such that*

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \text{ and } \lim_{t\to\infty} \delta(t) = \rho.$$

If $c(t) \equiv c_0$, then $f(t)$ is referred to as a *normalized* regularly varying function of index ρ, and is denoted by $f(t) \in \text{n-RV}(\rho)$.

Typical examples of slowly varying functions are: all functions tending to some positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbb{R}, \quad \text{and} \quad \exp\left\{ \prod_{n=1}^{N} (\log_n t)^{\beta_n} \right\}, \quad \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function $L(t) = \exp\left\{(\log t)^{\frac{1}{3}} \cos\, (\log t)^{\frac{1}{3}}\right\}$ is a slowly varying function which is oscillating in the sense that $\limsup_{t\to\infty} L(t) = \infty$ and $\liminf_{t\to\infty} L(t) = 0$.

The following result concerns operations which preserve slow variation.

Proposition 2.1. *Let* $L(t)$, $L_1(t)$, $L_2(t)$ *be slowly varying. Then,* $L(t)^{\alpha}$ *for any* $\alpha \in \mathbb{R}$, $L_1(t) + L_2(t)$, $L_1(t) L_2(t)$ *and* $L_1(L_2(t))$ (*if* $L_2(t) \to \infty$) *are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following

Proposition 2.2. *Let $f(t) \in \mathrm{SV}$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{\varepsilon} f(t) = \infty, \quad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0.$$

A simple criterion for determining the regularity of differentiable positive functions follows.

Proposition 2.3. *A differentiable positive function $f(t)$ is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t\frac{f'(t)}{f(t)} = \rho.$$

The following result which is called Karamata's integration theorem is useful in handling slowly and regularly varying functions analytically.

Proposition 2.4. *Let $L(t) \in \mathrm{SV}$. Then,*

(i) *if $\alpha > -1$,*

$$\int_a^t s^{\alpha} L(s)\, ds \sim \frac{1}{\alpha+1}\, t^{\alpha+1} L(t), \quad t \to \infty.$$

(ii) *if $\alpha < -1$,*

$$\int_t^{\infty} s^{\alpha} L(s)\, ds \sim -\frac{1}{\alpha+1}\, t^{\alpha+1} L(t), \quad t \to \infty.$$

(iii) *if $\alpha = -1$,*

$$l(t) = \int_a^t \frac{L(s)}{s}\, ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

and

$$m(t) = \int_t^{\infty} \frac{L(s)}{s}\, ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

Definition 2.3. A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it is expressed in the form $f(t) = t^{\rho} L(t)$ with $L(t) \in \mathrm{SV}$ satisfying $\lim_{t\to\infty} L(t) = \mathit{const} > 0$. Otherwise $f(t)$ is called a *nontrivial* regularly varying function of index ρ. The symbol tr-RV(ρ) (or ntr-RV(ρ)) denotes the set of all trivial RV(ρ)-functions (or the set of all nontrivial RV(ρ)-functions)

For the most complete exposition of the theory of regular variation and its applications the reader is referred to the book of Bingham, Goldie and Teugels [1]. See also Seneta [7]. A comprehensive survey of results up to 2000 on the asymptotic analysis of ordinary differential equations by means of regular variation can be found in the monograph of Marić [6].

3. Existence of Intermediate Solutions of Equation (A)

Intermediate solutions of (A), that is, positive solutions $x(t)$ such that

$$\lim_{t\to\infty} x(t) = \infty \ \text{ and } \ \lim_{t\to\infty} \frac{x(t)}{t} = 0, \tag{3.1}$$

are constructed as solutions of the integral equation (B) under the assumption that $q(t) \in \mathrm{RV}(\sigma)$ ($\sigma \in \mathbb{R}$) and $g(t)$ satisfy (1.1). For this purpose an essential role is played by the fact that regularly varying solutions of the integral asymptotic relation (C) satisfying (3.1) can be thoroughly analyzed in the framework of regular variation. Throughout this section, the use is made of the following expression for $q(t)$

$$q(t) = t^{\sigma} l(t), \ \ l(t) \in \mathrm{SV}. \tag{3.2}$$

3.1. **Regularly varying solutions of asymptotic relation (C).** Let $x(t) = t^{\rho}\xi(t)$, $\xi(t) \in \mathrm{SV}$, be a regularly varying solution of (C) satisfying (3.1). We see that ρ must satisfy $\rho \in [0,1]$, and that $\xi(t) \to \infty$, $t \to \infty$, if $\rho = 0$ and $\xi(t) \to 0$, $t \to \infty$, if $\rho = 1$, which means that $x(t)$ must be in one of the following three classes of regularly varying functions:

$$\mathrm{ntr}-\mathrm{SV}, \ \ \mathrm{RV}(\rho) \ \text{ with } \ \rho \in (0,1), \ \ \mathrm{ntr}-\mathrm{RV}(1). \tag{3.3}$$

One can establish the existence of these three kinds of regularly varying solutions of (C) as the following theorems demonstrate.

Theorem 3.1. *Relation* (C) *has nontrivial slowly varying solutions if and only if* $\sigma = -2$ *and*

$$\int_a^{\infty} t q(t)\, dt = \infty, \tag{3.4}$$

in which case any such solution $x(t)$ *has one and the same asymptotic behavior*

$$x(t) \sim \left[(1-\gamma)\int_a^t s q(s)\, ds\right]^{\frac{1}{1-\gamma}}, \ \ t \to \infty. \tag{3.5}$$

Theorem 3.2. *Relation* (C) *has regularly varying solutions of index* $\rho \in (0,1)$ *if and only if* $\sigma \in (-2, -\gamma-1)$, *in which case* ρ *is given by*

$$\rho = \frac{\sigma+2}{1-\gamma} \tag{3.6}$$

and any such solution $x(t)$ *has one and the same asymptotic behavior*

$$x(t) \sim \left[\frac{t^2 q(t)}{\rho(1-\rho)}\right]^{\frac{1}{1-\gamma}}, \ \ t \to \infty. \tag{3.7}$$

Theorem 3.3. *Relation* (C) *has nontrivial regularly varying solutions of index* 1 *if and only if* $\sigma = -\gamma - 1$ *and*

$$\int_a^\infty t^\gamma q(t)\,dt < \infty, \tag{3.8}$$

in which case any such solution $x(t)$ *has one and the same asymptotic behavior*

$$x(t) \sim t\biggl[(1-\gamma)\int_t^\infty s^\gamma q(s)\,ds\biggr]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.9}$$

Lemma 3.1. *If* $f(t)$ *is regularly varying and* $g(t)$ *satisfies* (1.1)*, then* $f(g(t)) \sim f(t)$ *as* $t \to \infty$.

Proof. Suppose that $f(t) \in \mathrm{RV}(\rho)$. Then by Proposition 2.1 it is expressed as

$$f(t) = c(t)\exp\biggl\{\int_{t_0}^t \frac{\delta(s)}{s}\,ds\biggr\}, \quad t \geqq t_0,$$

for some constant $t_0 > 0$ and some functions $c(t)$ and $\delta(t)$ such that $c(t) \to c_0 > 0$ and $\delta(t) \to \rho$ as $t \to \infty$. Then, we have

$$\frac{f(g(t))}{f(t)} = \frac{c(g(t))}{c(t)}\,\exp\biggl\{-\int_{g(t)}^t \frac{\delta(s)}{s}\,ds\biggr\}, \quad t \geqq t_0. \tag{3.10}$$

Noting that $|\delta(t)| \leqq k$, $t \geqq t_0$, for some constant $k > 0$, we see because of (1.1) that

$$\biggl|\int_{g(t)}^t \frac{\delta(s)}{s}\,ds\biggr| \leqq k\biggl|\int_{g(t)}^t \frac{ds}{s}\biggr| \leqq k\log\biggl|\frac{t}{g(t)}\biggr| \longrightarrow 0, \quad t \to \infty,$$

which, combined with (3.10), implies that $f(g(t))/f(t) \to 1$ or $f(g(t)) \sim f(t)$ as $t \to \infty$. This completes the proof. □

Proof of the "only if" parts of Theorems 3.1, 3.2 *and* 3.3. Let $x(t) = t^\rho \xi(t)$, $\xi(t) \in \mathrm{SV}$, be a solution of (C) satisfying (3.1). Using (3.2) and Lemma 3.1, we have

$$\int_t^\infty q(s)x(g(s))^\gamma\,ds \sim \int_t^\infty q(s)x(s)^\gamma\,ds = \int_t^\infty s^{\sigma+\rho\gamma} l(s)\xi(s)^\gamma\,ds, \quad t \to \infty. \tag{3.11}$$

The convergence of the last integral in (3.11) implies $\sigma + \rho\gamma \leqq -1$.

(i) We first consider the case where $\sigma + \rho\gamma = -1$. Then, since

$$\int_t^\infty s^{-1} l(s)\xi(s)^\gamma\,ds \in \mathrm{SV},$$

we have by Karamata's integration theorem ((i) of Proposition 2.5)

$$\int_{T_0}^{t}\int_{s}^{\infty} r^{-1}l(r)\xi(r)^{\gamma}\,dr\,ds \sim t\int_{t}^{\infty} s^{-1}l(s)\xi(s)^{\gamma}\,ds,$$

and hence by (C)

$$x(t) \sim t\int_{t}^{\infty} s^{-1}l(s)\xi(s)^{\gamma}\,ds \in \mathrm{RV}(1),\ \ t\to\infty. \tag{3.12}$$

This means that $\rho = 1$, so that $\sigma = -\gamma - 1$. From (3.12) we see that

$$\xi(t) \sim \int_{t}^{\infty} s^{-1}l(s)\xi(s)^{\gamma}\,ds,\ \ t\to\infty. \tag{3.13}$$

Let $\eta(t)$ denote the right-hand side of (3.13). Then, we obtain the following differential asymptotic relation for $\eta(t)$:

$$-\eta(t)^{-\gamma}\eta'(t) \sim t^{-1}l(t) = t^{\gamma}q(t),\ \ t\to\infty. \tag{3.14}$$

Since the left-hand side of (3.14) is integrable on $[T_0,\infty)$, so is $t^{\gamma}q(t)$, which shows that (3.8) is satisfied, and integrating (3.14) from t to ∞, we obtain

$$\xi(t) \sim \eta(t) \sim \left[(1-\gamma)\int_{t}^{\infty} s^{\gamma}q(s)\,ds\right]^{\frac{1}{1-\gamma}},\ \ t\to\infty,$$

which, in view of (3.13), leads to

$$x(t) \sim t\left[(1-\gamma)\int_{t}^{\infty} s^{\gamma}q(s)\,ds\right]^{\frac{1}{1-\gamma}},\ \ t\to\infty,$$

implying that $x(t)$ satisfies (3.9).

(ii) Next, we consider the case where $\sigma + \rho\gamma < -1$. Then, applying Karamata's integration theorem ((ii) of Proposition 2.5) to (3.11), we have

$$\int_{t}^{\infty} q(s)x(s)^{\gamma}\,ds \sim \frac{t^{\sigma+\rho\gamma+1}l(t)\xi(t)^{\gamma}}{-(\sigma+\rho\gamma+1)},\ \ t\to\infty. \tag{3.15}$$

We distinguish the three cases:

(a) $\sigma + \rho\gamma + 2 > 0$,

(b) $\sigma + \rho\gamma + 2 = 0$,

(c) $\sigma + \rho\gamma + 2 < 0$.

If (a) holds, then applying Karamata's integration theorem to (3.15), we find that

$$x(t) \sim \int_{T_0}^{t} \int_{s}^{\infty} q(r)x(r)^{\gamma}\, dr\, ds \sim$$

$$\sim \frac{t^{\sigma+\rho\gamma+2} l(t)\xi(t)^{\gamma}}{[-(\sigma+\rho\gamma+1)](\sigma+\rho\gamma+2)}, \quad t \to \infty, \tag{3.16}$$

which shows that $x(t) \in \mathrm{RV}(\sigma+\rho\gamma+2)$, where $\sigma+\rho\gamma+2 \in (0,1)$. This means that $\rho = \sigma+\rho\gamma+2$ or $\rho = (\sigma+2)/(1-\gamma)$, that is, ρ is given by (3.6). From $\rho \in (0,1)$ the range of σ is determined to be $\sigma \in (-2, -\gamma-1)$. Note that (3.16) is rewritten as

$$x(t) \sim \frac{t^{\sigma+2} l(t) x(t)^{\gamma}}{\rho(1-\rho)} = \frac{t^2 q(t) x(t)^{\gamma}}{\rho(1-\rho)},$$

from which it follows that

$$x(t) \sim \left[\frac{t^2 q(t)}{\rho(1-\rho)}\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty.$$

This shows that $x(t)$ satisfies (3.7).

If (b) holds, then (3.15) takes the form $\int_t^{\infty} q(s)x(s)^{\gamma}\, ds \sim t^{-1} l(t)\xi(t)^{\gamma}$ and we have

$$x(t) \sim \int_{T_0}^{t} \int_{s}^{\infty} q(r)x(r)^{\gamma}\, dr\, ds \sim \int_{T_0}^{t} s^{-1} l(s)\xi(s)^{\gamma}\, ds \in \mathrm{SV}, \quad t \to \infty, \tag{3.17}$$

which implies that $\rho = 0$, so that $x(t) = \xi(t)$ and $\sigma = -2$. Denoting the right-hand side of (3.17) by $y(t)$, we obtain from (3.17)

$$y(t)^{-\gamma} y'(t) \sim t^{-1} l(t) = t q(t), \quad t \to \infty. \tag{3.18}$$

Noting that the left-hand side of (3.18) and hence $tq(t)$ is not integrable on $[T_0, \infty)$ because $y(t) \to \infty$ as $t \to \infty$, we see that (3.4) holds and integrating (3.18) on $[T_0, t]$ yields

$$x(t) \sim y(t) \sim \left[(1-\gamma)\int_{T_0}^{t} s q(s)\, ds\right]^{\frac{1}{1-\gamma}} \sim \left[(1-\gamma)\int_{a}^{t} s q(s)\, ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

showing that $x(t)$ satisfies (3.5).

Finally, we note that case (c) is impossible. In fact, if (c) would hold, then the last integral in (3.15) would be integrable over $[T_0, \infty)$, which would imply that $x(t)$ tends to a constant as $t \to \infty$, that is, $x(t) \in \mathrm{ntr}-\mathrm{SV}$, an impossibility.

Let us now suppose that relation (C) admits a regularly varying solution $x(t)$ belonging to one of the three classes in (3.3). If $x(t) \in \mathrm{ntr}-\mathrm{SV}$ and $x(t) \to \infty$, $t \to \infty$, then from the above observations it is clear that $x(t)$

must fall into case (b) of (ii), which means that $\sigma = -2$ and (3.4) holds and that the asymptotic behavior of $x(t)$ is given by (3.5). Next, let (C) have a solution $x(t) \in \mathrm{RV}(\rho)$ with $\rho \in (0,1)$. Then, only case (a) of (ii) is admissible, showing that $\sigma \in (-2, -\gamma-1)$ and $x(t)$ must satisfy (3.7) with ρ defined by (3.6). Finally, if $x(t) \in \mathrm{ntr}-\mathrm{RV}(1)$ and its slowly varying part $\xi(t)$ tends to 0 as $t \to \infty$, then case (i) necessarily fits $x(t)$, so that $\sigma = -\gamma - 1$, (3.8) holds and the asymptotic behavior of $x(t)$ is governed by the formula (3.9). □

Proof of the "if" parts of Theorems 3.1*,* 3.2 *and* 3.3. Let $X(t)$ denote any one of the functions $X_i(t)$, $i = 1,2,3$, defined on $[a, \infty)$ as follows:

$$X_1(t) = \Bigg[(1-\gamma)\int_a^t sq(s)\,ds\Bigg]^{\frac{1}{1-\gamma}} \in \mathrm{SV}, \tag{3.19}$$

$$\text{if } \ \sigma = -2 \ \text{ and (3.4) holds,}$$

$$X_2(t) = \Big[\frac{t^2 q(t)}{\rho(1-\rho)}\Big]^{\frac{1}{1-\gamma}} \in \mathrm{RV}(\rho), \tag{3.20}$$

$$\text{if } \ \sigma \in (-2, -\gamma-1), \ \text{ where } \ \rho = \frac{\sigma+2}{1-\gamma} \in (0,1),$$

$$X_3(t) = t\Bigg[(1-\gamma)\int_t^\infty s^\gamma q(s)\,ds\Bigg]^{\frac{1}{1-\gamma}} \in \mathrm{RV}(1), \tag{3.21}$$

$$\text{if } \ \sigma = -\gamma - 1 \ \text{ and (3.8) holds.}$$

It suffices to verify that $X(t)$ satisfies the asymptotic relation

$$X(t) \sim \int_T^t \int_s^\infty q(r) X(g(r))^\gamma\,dr\,ds \sim \int_T^t \int_s^\infty q(r) X(r)^\gamma\,dr\,ds, \quad t \to \infty, \tag{3.22}$$

for any $T > a$ such that $g(t) \geqq a$ for $t \geqq T$, where the last relation follows from Lemma 3.1 ensuring that $X(g(t)) \sim X(t)$ as $t \to \infty$.

Suppose that $\sigma = -2$ and (3.4) holds. Then, $X_1(t)$ satisfies

$$\int_t^\infty q(s) X_1(s)^\gamma\,ds \sim t^{-1} l(t) \Bigg[(1-\gamma)\int_a^t s^{-1} l(s)\,ds\Bigg]^{\frac{\gamma}{1-\gamma}}$$

and hence

$$\int_T^t \int_s^\infty q(r) X_1(r)^\gamma\,dr\,ds \sim \int_T^t s^{-1} l(s) \Bigg[(1-\gamma)\int_a^s r^{-1} l(r)\,dr\Bigg]^{\frac{\gamma}{1-\gamma}} ds \sim$$

$$\sim \Bigg[(1-\gamma)\int_a^t s^{-1} l(s)\,ds\Bigg]^{\frac{1}{1-\gamma}} = \Bigg[(1-\gamma)\int_a^t sq(s)\,ds\Bigg]^{\frac{1}{1-\gamma}} = X_1(t), \quad t \to \infty.$$

Suppose next that $\sigma \in (-2, -\gamma-1)$. Rewriting $X_2(t)$ as $X_2(t) = t^{\rho}(l(t)/\rho(1-\rho))^{\frac{1}{1-\gamma}}$ and applying Karamata's integration theorem twice, we see that

$$\int_t^{\infty} q(s)X_2(s)^{\gamma}\,ds = \frac{\int_t^{\infty} s^{\rho-2}l(s)^{\frac{1}{1-\gamma}}\,ds}{(\rho(1-\rho))^{\frac{\gamma}{1-\gamma}}} \sim \frac{t^{\rho-1}l(t)^{\frac{1}{1-\gamma}}}{(\rho(1-\rho))^{\frac{\gamma}{1-\gamma}}(1-\rho)},$$

and

$$\int_T^t \int_s^{\infty} q(r)X_2(r)\,dr\,ds \sim \frac{t^{\rho}l(t)^{\frac{1}{1-\gamma}}}{(\rho(1-\rho))^{\frac{\gamma}{1-\gamma}}(1-\rho)\rho} = X_2(t), \quad t \to \infty.$$

Suppose finally that $\sigma = -\gamma - 1$ and (3.8) holds. Then, using

$$\int_t^{\infty} q(s)X_3(s)^{\gamma}\,ds = \left[(1-\gamma)\int_t^{\infty} s^{\gamma}q(s)\,ds\right]^{\frac{1}{1-\gamma}},$$

we conclude via Karamata's integration theorem that

$$\int_T^t \int_s^{\infty} q(r)X_3(r)^{\gamma}\,dr\,ds \sim t\left[(1-\gamma)\int_t^{\infty} s^{\gamma}q(s)\,ds\right]^{\frac{1}{1-\gamma}} = X_3(t), \quad t \to \infty.$$

This completes the proof of Theorems 3.1, 3.2 and 3.3. □

3.2. **Construction of Intermediate Solutions of Equation (A).** The purpose of this subsection is to prove the existence of three kinds of intermediate solutions for equation (A) with regularly varying coefficient $q(t)$ and retarded argument $g(t)$ satisfying (1.1), and furthermore to verify that two kinds of them are really regularly varying solutions. Our discussions here essentially depend on the results on regularly varying solutions of the asymptotic relation (C) developed in the first subsection. We use the following notation.

Notation 3.1. Let $f(t)$ and $g(t)$ be positive functions defined on $[t_0, \infty)$. We write $f(t) \asymp g(t)$, $t \to \infty$, to denote that there exist positive constants m and M such that $mg(t) \leqq f(t) \leqq Mg(t)$ for $t \geqq t_0$. Clearly, $f(t) \sim g(t)$, $t \to \infty$, implies $f(t) \asymp g(t)$, $t \to \infty$, but not conversely. If $f(t) \asymp g(t)$, $t \to \infty$, and $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} f(t) = 0$. Our main results follow.

Theorem 3.4. *Suppose that* $q(t) \in \mathrm{RV}(-2)$ *satisfies* (3.4) *and* $g(t)$ *satisfies* (1.1). *Then equation* (A) *possesses an intermediate solution* $x(t)$ *such that*

$$x(t) \asymp \left[(1-\gamma)\int_a^t sq(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.23}$$

Theorem 3.5. *Suppose that* $q(t) \in \mathrm{RV}(\sigma)$ *with* $\sigma \in (-2, -\gamma - 1)$ *and* $g(t)$ *satisfies* (1.1). *Then equation* (A) *possesses an intermediate solution* $x(t)$ *such that*

$$x(t) \asymp \Big[\frac{t^2 q(t)}{\rho(1-\rho)}\Big]^{\frac{1}{1-\gamma}}, \quad t \to \infty, \tag{3.24}$$

where ρ *is given by* (3.6).

Theorem 3.6. *Suppose that* $q(t) \in \mathrm{RV}(-\gamma - 1)$ *satisfies* (3.8) *and* $g(t)$ *satisfies* (1.1). *Then, equation* (A) *possesses an intermediate solution* $x(t)$ *such that*

$$x(t) \asymp t\bigg[(1-\gamma)\int_t^\infty s^\gamma q(s)\,ds\bigg]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.25}$$

Proof of Theorems 3.4, 3.5 *and* 3.6. Under the assumptions of these theorems one can define the functions $X_i(t)$, $i = 1, 2, 3$, by (3.19), (3.20) or (3.21). Let $X(t)$ denote one of $X_i(t)$, $i = 1, 2, 3$, depending on the indicated values of σ. Since $X(t)$ satisfies (3.22), there exists $T_0 > a$ such that $g(t) \geqq a$ for $t \geqq T_0$ and

$$\int_{T_0}^t \int_s^\infty q(r) X(g(r))^\gamma\,dr\,ds \leqq 2X(t), \quad t \geqq T_0. \tag{3.26}$$

We may assume that $X(t)$ is increasing for $t \geqq g(T_0)$. Using (3.21) again, one can choose $T_1 > T_0$ such that

$$\int_{T_0}^t \int_s^\infty q(r) X(g(r))^\gamma\,dr\,ds \geqq \frac{1}{2} X(t), \quad t \geqq T_1. \tag{3.27}$$

Furthermore, choose positive constants $k < 1$ and $K > 1$ satisfying

$$k^{1-\gamma} \leqq \frac{1}{2}, \quad K^{1-\gamma} \geqq 4, \quad kX(T_1) \leqq \frac{1}{2} KX(g(T_0)), \tag{3.28}$$

and define the set $\mathcal{X}$ and the mapping $\mathcal{F} : \mathcal{X} \to C[g(T_0), \infty)$ as follows:

$$\mathcal{X} = \Big\{x(t) \in C[g(T_0), \infty) : \ kX(t) \leqq x(t) \leqq KX(t), \ t \geqq g(T_0)\Big\}, \tag{3.29}$$

$$\begin{cases} \mathcal{F}x(t) = x_0 + \displaystyle\int_{T_0}^t \int_s^\infty q(r) x(g(t))^\gamma\,dr\,ds, & t \geqq T_0, \\ \mathcal{F}x(t) = x_0, & g(T_0) \leqq t \leqq T_0, \end{cases} \tag{3.30}$$

where x_0 is a constant such that

$$kX(T_1) \leqq x_0 \leqq \frac{1}{2} KX(g(T_0)). \tag{3.31}$$

It can be shown that $\mathcal{F}$ is a continuous self-map of $\mathcal{X}$ which sends $\mathcal{X}$ into a relatively compact subset of $C[g(T_0), \infty)$.

(i) $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$. This follows from the following calculations in which (3.26)–(3.31) are used:

$$\mathcal{F}x(t) \geqq x_0 \geqq kX(T_1) \geqq kX(t) \quad \text{for } g(T_0) \leqq t \leqq T_1,$$

$$\mathcal{F}x(t) \geqq \int_{T_0}^{t} \int_{s}^{\infty} q(r)(kX(g(r)))^{\gamma}\, dr\, ds \geqq \frac{1}{2} k^{\gamma} X(t) \geqq kX(t) \quad \text{for } t \geqq T_1,$$

$$\mathcal{F}x(t) \leqq \frac{1}{2} KX(g(T_0)) \leqq \frac{1}{2} KX(t) \leqq KX(t) \quad \text{for } g(T_0) \leqq t \leqq T_0,$$

$$\mathcal{F}x(t) \leqq \frac{1}{2} KX(T_0) + \int_{T_0}^{t} \int_{s}^{\infty} q(r)(KX(g(r)))^{\gamma}\, dr\, ds$$

$$\leqq \frac{1}{2} KX(t) + 2K^{\gamma} X(t) \leqq \frac{1}{2} KX(t) + \frac{1}{2} KX(t) = KX(t) \quad \text{for } t \geqq T_0.$$

(ii) $\mathcal{F}(\mathcal{X})$ is relatively compact. The set $\mathcal{F}(\mathcal{X})$ is locally uniformly bounded on $[g(T_0), \infty)$, since it is a subset of $\mathcal{X}$. The inequality $0 \leqq (\mathcal{F}x)'(t) \leqq K^{\gamma} \int_t^{\infty} q(s)X(g(s))^{\gamma}\, ds$, $t \geqq T_0$, holding for all $x(t) \in \mathcal{X}$ guarantees that $\mathcal{F}(\mathcal{X})$ is locally equicontinuous on $[T_0, \infty)$ and hence on $[g(T_0), \infty)$. The desired relative compactness then follows from Arzela–Ascoli's lemma.

(iii) $\mathcal{F}$ is continuous. Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging as $n \to \infty$ to $x(t) \in \mathcal{X}$ uniformly on every compact subinterval of $[g(T_0), \infty)$. Naturally, we need only to study the convergence on $[T_0, \infty)$. Our aim is to prove that $\mathcal{F}x_n(t) \to \mathcal{F}x(t)$ as $n \to \infty$ uniformly on compact subintervals of $[T_0, \infty)$. But this follows immediately from the Lebesgue dominated convergence theorem applied to the inner integral of the right-hand side of the inequality

$$\left|\mathcal{F}x_n(t) - \mathcal{F}x(t)\right| \leqq \int_{T_0}^{t} \int_{s}^{\infty} q(r)\left|x_n(g(r))^{\gamma} - x(g(r))^{\gamma}\right| dr\, ds, \quad t \geqq T_0.$$

Therefore, all the hypotheses of the Schauder–Tychonoff fixed point theorem are fulfilled and so there exists $x(t) \in \mathcal{X}$ such that $x(t) = \mathcal{F}x(t)$ for $t \geqq g(T_0)$, which implies in particular that

$$x(t) = x_0 + \int_{T_0}^{t} \int_{s}^{\infty} q(r)x(g(r))^{\gamma}\, dr\, ds, \quad t \geqq T_0.$$

This implies that $x(t)$ is a solution of (A) on $[T_0, \infty)$. Since $x(t) \in \mathcal{X}$, i.e., $x(t) \asymp X(t)$, $t \to \infty$, $x(t)$ is an intermediate solution of (A). This completes the simultaneous proof of Theorems 3.4, 3.5 and 3.6. □

3.3. **Regularity of Intermediate Solutions.** It is shown that the two kinds of intermediate solutions of (A) obtained in Theorems 3.4 and 3.6 are actually regularly varying of indices 0 and 1, respectively. Combining this fact with Theorems 3.1 and 3.3 on the asymptotic relation (C), one can characterize completely the situation in which the sublinear equation (A) with regularly varying $q(t)$ possesses nontrivial regularly varying solutions of indices 0 and 1.

Theorem 3.7. *Let* $q(t) \in \mathrm{RV}(\sigma)$ *and suppose that* $g(t)$ *satisfies* (1.1). *Equation* (A) *possesses nontrivial slowly varying solutions if and only if* $\sigma = -2$ *and* (3.4) *holds, in which case the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula* (3.5).

Proof. (The "if" part) Suppose that $\sigma = -2$ and (3.4) holds. Then $q(t) = t^{-2}l(t)$ and (3.4) is expressed as $\int_a^\infty s^{-1}l(s)\,ds = \infty$. Let $x(t)$ be an intermediate solution of (A) constructed in Theorem 3.4 as a solution of the integral equation (B). It is known that

$$x(t) \asymp X_1(t) = \left[(1-\gamma)\int_a^t s^{-1}l(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.32}$$

Using (B), (3.32) and one of the properties of $X_1(t)$ mentioned in the proof of the "if" part of Theorem 3.1, we find that

$$x'(t) = \int_t^\infty q(s)x(g(s))^\gamma\,ds \asymp \int_t^\infty q(s)X_1(g(s))^\gamma\,ds \sim$$

$$\sim \int_t^\infty q(s)X_1(s)^\gamma\,ds \sim t^{-1}l(t)\left[(1-\gamma)\int_a^t s^{-1}l(s)\,ds\right]^{\frac{\gamma}{1-\gamma}}, \quad t \to \infty. \tag{3.33}$$

We combine (3.32) and (3.33) to obtain

$$t\frac{x'(t)}{x(t)} \asymp \frac{l(t)}{(1-\gamma)\int_a^t s^{-1}l(s)\,ds}, \quad t \to \infty,$$

from which, noting that the right-hand side of the above tends to 0 as $t \to \infty$ by (iii) of Proposition 2.5, we conclude that $\lim_{t\to\infty} tx'(t)/x(t) = 0$. From Proposition 2.4 it follows that $x(t)$ is a nontrivial slowly varying function.

(The "only if" part) If $x(t)$ is a nontrivial slowly varying solution of (A), then it clearly satisfies relation (C) and hence from the "only if" part of Theorem 3.1 it follows that $\sigma = -2$ and (3.4) holds and, moreover, that the asymptotic behavior of $x(t)$ is given by (3.5). This completes the proof of Theorem 3.7. □

Theorem 3.8. *Let* $q(t) \in \mathrm{RV}(\sigma)$ *and suppose that* $g(t)$ *satisfies* (1.1). *Equation* (A) *possesses nontrivial regularly varying solutions of index* 1 *if and only if* $\sigma = -\gamma - 1$ *and* (3.8) *holds, in which case the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula* (3.9).

Proof. (The "if" part) Suppose that $\sigma = -\gamma - 1$ and (3.8) holds. Then, $q(t) = t^{-\gamma-1}l(t)$ and (3.8) is expressed as $\int_a^\infty s^{-1}l(s)\,ds < \infty$. Let $x(t)$ be an intermediate solution of (A) obtained in Theorem 3.4 as a solution of the integral equation (B). It satisfies

$$x(t) \asymp X_3(t) = t\left[(1-\gamma)\int_t^\infty s^{-1}l(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

which implies that

$$\begin{aligned} -x''(t) &= q(t)x(g(t))^\gamma \asymp q(t)X_3(g(t))^\gamma \sim \\ &\sim q(t)X_3(t)^\gamma = t^{-\gamma-1}l(t)\left[(1-\gamma)\int_t^\infty s^{-1}l(s)\,ds\right]^{\frac{\gamma}{1-\gamma}}, \quad t \to \infty. \end{aligned} \tag{3.34}$$

On the other hand, taking the proof of the "if" part of Theorem 3.3, we see that $x'(t)$ satisfies

$$\begin{aligned} x'(t) &= \int_t^\infty q(s)x(g(s))^\gamma\,ds \asymp \int_t^\infty q(s)X_3(g(s))^\gamma\,ds \sim \\ &\sim \int_t^\infty q(s)X_3(s)^\gamma\,ds = \left[(1-\gamma)\int_t^\infty s^{-1}l(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \end{aligned} \tag{3.35}$$

Using (3.34) and (3.35), we obtain

$$-t\frac{x''(t)}{x'(t)} \asymp \frac{l(t)}{(1-\gamma)\int_t^\infty s^{-1}l(s)\,ds} \to 0, \quad t \to \infty,$$

where (iii) of Proposition 2.5 has been used. This means by Proposition 2.4 that $x'(t)$ is slowly varying, and from (i) of Proposition 2.5 we conclude that

$$x(t) \sim \int_{T_0}^t x'(s)\,ds \sim tx'(t) \in \mathrm{RV}(1), \quad t \to \infty,$$

which implies that $x(t)$ is a nontrivial regularly varying solution of index 1.

(The "only if" part) Let $x(t)$ be a nontrivial RV(1)-solution of (A). Then, since it satisfies relation (C), from the "only if" part of Theorem 3.3 it follows that $\sigma = -\gamma - 1$ and (3.8) holds and, moreover, that the asymptotic behavior of $x(t)$ is given by (3.9). This completes the proof of Theorem 3.8. □

Remark 3.1. It is impossible for us to prove that the solution obtained in Theorem 3.5 is regularly varying of index $\rho \in (0,1)$. A more powerful criterion than Proposition 2.4 seems to be necessary.

Example 3.1. Consider equation (A) with $g(t)$ satisfying (1.1). Suppose that $q(t)$ satisfies

$$q(t) \sim \frac{c}{t^2 \log t (\log\log t)^{\gamma}}, \quad t \to \infty,$$

for some positive constant $c > 0$. It is clear that $q(t) \in \mathrm{RV}(-2)$ and (3.4) is satisfied, and that

$$\left[(1-\gamma)\int_a^t sq(s)\,ds\right]^{\frac{1}{1-\gamma}} \sim c^{\frac{1}{1-\gamma}} \log\log t, \quad t \to \infty.$$

By Theorem 3.7, we see that equation (A) possesses nontrivial SV-solutions $x(t)$, all of which have one and the same asymptotic behavior $x(t) \sim c^{\frac{1}{1-\gamma}} \log\log t$, $t \to \infty$, for any retarded argument $g(t)$. If, in particular,

$$q(t) = \frac{1}{t^2 \log t (\log\log g(t))^{\gamma}} \Big(1 + \frac{1}{\log t}\Big),$$

then equation (A) has an exact solution $x_0(t) = \log\log t \in \mathrm{ntr} - \mathrm{SV}$.

Example 3.2. Consider equation (A) with $g(t)$ satisfying (1.1). Suppose that $q(t)$ satisfies

$$q(t) \sim \frac{c}{t^{\gamma+1} \log t (\log\log t)^{2-\gamma}} \in \mathrm{RV}(-\gamma-1), \quad t \to \infty,$$

for some constant $c > 0$. As is easily checked, (3.8) is satisfied and

$$\left[(1-\gamma)\int_t^{\infty} s^{\gamma} q(s)\,ds\right]^{\frac{1}{1-\gamma}} \sim \frac{c^{\frac{1}{1-\gamma}}}{\log\log t}, \quad t \to \infty,$$

and hence by Theorem 3.8, equation (A) possesses nontrivial RV(1)-solutions $x(t)$, all of which have one and the same asymptotic behavior $x(t) \sim \frac{c^{\frac{1}{1-\gamma}} t}{\log\log t}$, $t \to \infty$, for any retarded argument $g(t)$. If, in particular,

$$q(t) = \frac{(\log\log g(t))^{\gamma}}{tg(t)^{\gamma} \log t (\log\log t)^2} \Big(1 - \frac{1}{\log t} - \frac{2}{\log t \cdot \log\log t}\Big),$$

then equation (A) has an exact solution $x_1(t) = t/\log\log t$.

Example 3.3. Consider the equation

$$x''(t) + t^{-\frac{3}{2}} \big(2 + \sin(\log\log t)\big)^2 x(g(t))^{\frac{1}{3}} = 0, \tag{3.36}$$

which is a special case of (A) in which

$$\gamma = \frac{1}{3} \ \text{ and } \ q(t) = t^{-\frac{3}{2}} \Big(2 + \sin(\log\log t)\Big)^2 \in \mathrm{RV}\Big(-\frac{3}{2}\Big).$$

Since $\sigma = -\frac{3}{2}$ satisfies $-2 < \sigma < -\gamma - 1 = -\frac{4}{3}$, Theorem 3.5 is applicable to (3.35) and ensures the existence of its intermediate solution $x(t)$ such that

$$x(t) \asymp \Big(\frac{16}{3}\Big)^{\frac{3}{2}} t^{\frac{3}{4}} \big(2 + \sin(\log\log t)\big)^3, \quad t \to \infty.$$

It is impossible to decide whether or not this solution is regularly varying of index $\frac{3}{4}$.

Remark 3.2. A question naturally arises: what will happen if condition (1.1) on $g(t)$ is not required? The problem of investigating the accurate asymptotic behavior of positive solutions of (A) for general retarded argument is much more difficult to handle as the following example indicates. It is to be noted that very little is known about regularly varying solutions of functional differential equations, linear or nonlinear, with general deviating arguments. See e.g. the papers [3]–[5].

Example 3.4. Consider the equation

$$x''(t) + q(t)x(\log t)^{\gamma} = 0, \quad 0 < \gamma < 1, \tag{3.37}$$

where $q(t)$ is given by

$$q(t) = \frac{(\log\log\log t)^{\gamma}}{t(\log t)^{\gamma+1}(\log\log t)^2}\Big(1 - \frac{1}{\log t} - \frac{2}{\log t \cdot \log\log t}\Big) \in \mathrm{RV}(-1).$$

As is easily checked, equation (3.37) has a nontrivial RV(1)-solution $x(t) = t/\log\log t$ in marked contrast to Theorem 3.6 or Theorem 3.8.

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Authors' addresses:

Department of Mathematics, Graduate School of Science and Technology, Kumamoto University, 2 – 39 – 1 Kurokami, Kumamoto, 860 – 8555, Japan.

e-mail: akihito.shibuya@gmail.com

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Short Communications

Malkhaz Ashordia

ON A TWO-POINT SINGULAR BOUNDARY VALUE PROBLEM FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The two-point boundary value problem is considered for the system of nonlinear generalized ordinary differential equations with singularities on a non-closed interval. Singularity is understood in a sense of the vector-function corresponding to the system which belongs to the local Carathéodory class with respect to the matrix-function corresponding to the system.

The general sufficient conditions are established for the unique solvability of this problem. Relying on these results, the effective conditions are established for the unique solvability of the problem.

რეზიუმე. სინგულარობებიან განზოგადოებულ ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემისათვის განხილულია ორწერტილოვანი სასაზღვრო ამოცანა არაჩაკეტილ ინტერვალზე. სინგულარობა გაიგება იმ აზრით, რომ სისტემის შესაბამისი ვექტორული ფუნქცია მიეკუთვნება ლოკალურ კარათეოდორის კლასს სისტემის შესაბამისი მატრიცული ფუნქციის მიმართ.

მიღებულია ამ ამოცანის ცალსახად ამოხსნადობის ზოგადი საკმარისი პირობები. ამ შედეგებზე დაყრდნობით დადგენილია ცალსახად ამოხსნადობის ეფექტური საკმარისი პირობები.

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1. Statement of the Problem and Basic Notation

In the present paper, for a system of linear generalized ordinary differential equations with singularities

$$dx_i = f_i(t, x_1, \dots, x_n) da_i(t) \ \text{ for } \ t \in [a, b] \ \ (i = 1, \dots, n) \tag{1.1}$$

we consider the two-point boundary value problem

$$x_i(a+) = 0 \ \ (i = 1, \dots, n_0), \quad x_i(b-) = 0 \ \ (i = n_0 + 1, \dots, n), \tag{1.2}$$

where $-\infty < a < b < +\infty$, $n_0 \in \{1, \dots, n\}$, $x_1, \dots, x_n$ are the components of a desired solution x, $a_i : [a, b] \to \mathbb{R}$ $(i = 1, \dots, n)$ are nondecreasing functions, and $f_i :]a, b[\times \mathbb{R}^n \to \mathbb{R}$ is a function belonging to the local Carathéodory class $\mathrm{Car}_{loc}(]a, b[\times \mathbb{R}^n, \mathbb{R}; a_i)$ corresponding to the function a_i for every $i \in \{1, \dots, n\}$.

We investigate the question of solvability of the problem (1.1), (1.2), when the system (1.1) has singularities. Singularity is understood in a sense that the components of the vector-function f may have non-integrable components at the boundary points a and b, in general. We present a general theorem for the solvability of this problem. On the basis of this theorem we obtain the effective criteria for the solvability of the problem.

Analogous and related questions are investigated in [13]–[18] (see also references therein) for the singular two-point and multipoint boundary value problems for linear and nonlinear systems of ordinary differential equations, and in [1]–[7] (see also references therein) for regular two-point and multipoint boundary value problems for systems of linear and nonlinear generalized differential equations. As for the two-point and multipoint singular boundary value problems for generalized differential systems, they are little studied and, despite some results given in [8–10] for two-point and multipoint singular boundary value problem, their theory is rather far from completion even in the linear case. Therefore, the problem under consideration is actual.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has been motivated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see e.g. [1]–[12], [19]–[22] and references therein).

Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$; $R_+ = [0, +\infty[$; $[a, b]$, $]a, b[$ and $]a, b]$, $[a, b[$ are, respectively, closed, open and half-open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{il})_{i,l=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i,l=1}^{n,m} |x_{il}|.$$

$\mathbb{R}_+^{n\times n} = \big\{(x_{il})_{i,l=1}^{n,m} : \ x_{il} \geq 0 \ (i = 1, \dots, n; \ l = 1, \dots, m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X = (x_{il})_{i,l=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{il}|)_{i,l=1}^{n,m}$.

$\mathbb{R}^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.

$\bigvee\limits_{c}^{d}(X)$, where $a < c < d < b$, is the variation of the matrix-function $X :]a, b[\to \mathbb{R}^{n\times m}$ on the closed interval $[c, d]$, i.e., the sum of total variations of the latter components x_{il} $(i = 1, \dots, n;\ l = 1, \dots, m)$ on this interval; if $d < c$, then $\bigvee\limits_{c}^{d}(X) = -\bigvee\limits_{d}^{c}(X)$; $V(X)(t) = (v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(c_0) = 0$, $v(x_{il})(t) = \bigvee\limits_{c_0}^{t}(x_{il})$ for $a < t < b$, and $c_0 = (a + b)/2$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X :]a, b[\to \mathbb{R}^{n\times m}$ at the point $t \in]a, b[$ (we assume $X(t) = X(a+)$ for $t \le a$ and $X(t) = X(b-)$ for $t \ge b$, if necessary).

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a, b], \mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a, b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_{a}^{b}(X) < +\infty$).

$\mathrm{BV}_{loc}(]a, b[\,, \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X :]a, b[\to \mathbb{R}^{n\times m}$ such that $\bigvee\limits_{c}^{d}(X) < +\infty$ for every $a < c < d < b$.

If $X \in \mathrm{BV}_{loc}(]a, b[\,, \mathbb{R}^{n\times n})$, $\det(I_n + (-1)^j d_jX(t)) \ne 0$ for $t \in]a, b[$ $(j = 1, 2)$, and $Y \in \mathrm{BV}_{loc}(]a, b[\,, \mathbb{R}^{n\times m})$, then $\mathcal{A}(X, Y)(t) \equiv \mathcal{B}(X, Y)(c_0, t)$, where $\mathcal{B}$ is the operator defined as follows:

$$\mathcal{B}(X, Y)(t, t) = O_{n\times m} \ \text{ for } \ t \in]a, b[\,,$$

$$\mathcal{B}(X, Y)(s, t) = Y(t) - Y(s) + \sum_{s<\tau\le t} d_1X(\tau)\cdot\big(I_n - d_1X(\tau)\big)^{-1} d_1Y(\tau) - \\ - \sum_{s\le\tau<t} d_2X(\tau)\cdot\big(I_n + d_2X(\tau)\big)^{-1} d_2Y(\tau) \ \text{ for } \ a < s < t < b$$

and

$$\mathcal{B}(X, Y)(s, t) = -\mathcal{B}(X, Y)(t, s) \ \text{ for } \ a < t < s < b.$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $\alpha : [a, b] \to \mathbb{R}$ is a nondecreasing function, then $D_\alpha = \{t \in [a, b] : d_1\alpha(t) + d_2\alpha(t) \ne 0\}$.

If $\alpha \in \mathrm{BV}([a, b], \mathbb{R})$ has no more than a finite number of points of discontinuity, and $m \in \{1, 2\}$, then $D_{\alpha m} = \{t_{\alpha m1}, \dots, t_{\alpha m n_{\alpha m}}\}$ $(t_{\alpha m1} < \dots < t_{\alpha m n_{\alpha m}})$ is the set of all points from $[a, b]$ for which $d_m\alpha(t) \ne 0$, and $\mu_{\alpha m} = \max\{d_m\alpha(t) : \ t \in D_{\alpha m}\}$ $(m = 1, 2)$.

If $\beta \in \mathrm{BV}([a, b], \mathbb{R})$, then

$$\nu_{\alpha m\beta j} = \max\Big\{d_j\beta(t_{\alpha ml}) + \sum_{t_{\alpha m\, l+1-m}<\tau<t_{\alpha m\, l+2-m}} d_j\beta(\tau) : \ l = 1, \dots, n_{\alpha m}\Big\}$$

$(j, m = 1, 2)$; here $t_{\alpha 20} = a - 1$, $t_{\alpha 1 n_{\alpha 1}+1} = b + 1$.

$s_1, s_2, s_c : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \ \text{ and } \ s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \ \text{ for } \ a < t \le b,$$

and

$$s_c(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$ corresponding to the function $s_0(g)$; if $a = b$, then we assume $\int_a^b x(t)\,dg(t) = 0$; thus, $\int_s^t x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [19], [20], [22]). Moreover, we put

$$\int_{s+}^t x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_{s+\varepsilon}^t x(\tau)\,dg(\tau)$$

and

$$\int_s^{t-} x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_s^{t-\varepsilon} x(\tau)\,dg(\tau).$$

$L^p([a,b],\mathbb{R};g)$ $(1 \le p < +\infty)$ is the space of all functions $x : [a,b] \to \mathbb{R}$ measurable and integrable with respect to the measure $\mu(g_c(g))$ for which

$$\sum_{a<\tau\le b} |x(t)|^\mu d_1g(\tau) + \sum_{a\le\tau<b} |x(t)|^\mu d_2g(t) < +\infty,$$

with the norm

$$\|x\|_{p,g} = \left(\int_a^b |x(t)|^p\,dg(t)\right)^{\frac{1}{p}}.$$

$L^{+\infty}([a,b],\mathbb{R};g)$ is the space of all $\mu(s_0(g))$-measurable and $\mu(s_0(g))$-essentially bounded functions $x : [a,b] \to \mathbb{R}$ such that $\sup\{|x(t)| : t \in D_\alpha\} < +\infty$, with the norm

$$\|x\|_{+\infty,g} = \inf\Big\{r > 0 : |x(t)| \le r \ \text{ for } \ \mu(s_0(g))\text{-almost all } \ t \in [a,b] \ \text{ and for } \ t \in D_\alpha\Big\}.$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_s^t x(\tau)\, dg_1(\tau) - \int_s^t x(\tau)\, dg_2(\tau) \text{ for } s \leq t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b], D; G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b], R; g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\, dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m} \text{ for } a \leq s \leq t \leq b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ (j = 0,1,2).$$

The inequalities between the vectors and between the matrices are understood componentwise.

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}$, then $\mathrm{Car}([a,b]\times D_1, D_2; g)$ is the Carathéodory class, i.e., the set of all mappings $f : [a,b]\times D_1 \to D_2$ such that:

(i) the function $f(\cdot\,, x) : [a,b] \to D_2$ is $\mu(g)$-measurable for every $x \in D_1$;

(ii) the function $f(t,\cdot) : D_1 \to D_2$ is continuous for $\mu(g)$-almost all $t \in [a,b]$, and

$$\sup\big\{|f(\cdot\,,x)| : \ x \in D_0\big\} \in L([a,b], R; g)$$

for every compact $D_0 \subset D_1$.

$\mathrm{Car}_{loc}(]a,b[\times D_1, D_2; g)$ is the set of all mappings $f :]a,b[\times D_1 \to D_2$ the restriction of which on every closed interval $[c,d]$ of $]a,b[$ belongs to $\mathrm{Car}([c,d]\times D_1, D_2; g)$. Analogously are defined the sets $\mathrm{Car}_{loc}(]a,b]\times D_1, D_2; G)$ and $\mathrm{Car}_{loc}([a,b[\times D_1, D_2; G)$.

We assume that $a_i : [a,b] \to \mathbb{R}$ $(i = 1,\dots,n)$ are nondecreasing functions and $f_i \in \mathrm{Car}(]a,b[\times\mathbb{R}^n, \mathbb{R}^n; a_i)$ $(i = 1,\dots,n)$. A vector-function $x = (x_i)_{i=1}^n$ is said to be a solution of the system (1.1) if $x_i \in \mathrm{BV}_{loc}(]a,b], \mathbb{R})$ $(i = 1,\dots,n_0)$, $x_i \in \mathrm{BV}_{loc}([a,b[, \mathbb{R})$ $(i = n_0+1,\dots,n)$ and

$$x_i(t) = x_i(s) + \sum_{l=1}^n \int_s^t f_l(\tau, x_1(\tau),\dots,x_n(\tau))\, da_{il}(\tau)$$

for $a<s\leq t\leq b$ if $i\in\{1,\dots,n_0\}$ and for $a\leq s<t<b$ if $i\in\{n_0+1,\dots,n\}$.

Under the solution of the problem (1.1), (1.2) we mean a solution $x(t) = (x_i(t))_{i=1}^n$ of the system (1.1) such that the one-sided limits $x_i(a+)$ $(i = 1,\dots,n_0)$ and $x_i(b-)$ $(i = n_0+1,\dots,n)$ exist and the equalities (1.2) are fulfilled. We assume $x_i(a) = 0$ $(i = 1,\dots,n_0)$ and $x_i(b) = 0$ $(i = n_0 + 1,\dots,n)$, if necessary.

A vector-function $x=(x_i)_{i=1}^n$, $x\in \mathrm{BV}(]a,b[\,,\mathbb{R})$, is said to be a solution of the system of generalized differential inequalities

$$dx_i(t)\leq \sum_{l=1}^n x_l(t)\,db_{il}(t)\ (\geq)\ \text{ for } t\in\,]a,b[\ \ (i=1,\dots,n),$$

where $b_{il}:[a,b]\to\mathbb{R}$ $(i,l=1,\dots,n)$ are nondecreasing functions, if

$$x_i(t)-x_i(s)\leq \sum_{l=1}^n \int\limits_s^t x_l(\tau)\,db_{il}(\tau)\ (\geq)\ \text{ for } a<s\leq t<b\ \ (i=1,\dots,n).$$

Without loss of generality, we assume that $a_i(a)=O_{n\times n}$ $(i=1,\dots,n)$. Moreover, we assume

$$\det\big(I_n+(-1)^j d_j a_i(t)\big)\neq 0\ \text{ for } t\in\,]a,b[\ \ (j=1,2;i=1,\dots,n). \qquad (1.3)$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding system (see [22, Theorem III.1.4]).

If $s\in\,]a,b[$ and $\alpha\in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R})$ are such that

$$1+(-1)^j d_j\beta(t)\neq 0\ \text{ for } (-1)^j(t-s)<0\ \ (j=1,2),$$

then by $\gamma_\beta(\cdot\,,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t)=\gamma(t)d\beta(t),\ \ \gamma(s)=1.$$

It is known (see [11], [12]) that

$$\gamma_\alpha(t,s)=\begin{cases}\exp\big(s_0(\beta)(t)-s_0(\beta)(s)\big)\times & \\ \qquad\times\prod\limits_{s<\tau\leq t}(1-d_1\alpha(\tau))^{-1}\prod\limits_{s\leq\tau<t}(1+d_2\beta(\tau)) & \text{for } t>s,\\ \exp\big(s_0(\beta(t)-s_0(\beta(s))\big)\times & \\ \qquad\times\prod\limits_{t<\tau\leq s}(1-d_1\beta(\tau))\prod\limits_{t\leq\tau<s}(1+d_2\beta(\tau))^{-1} & \text{for } t<s,\\ 1 & \text{for } t=s.\end{cases} \qquad (1.4)$$

It is evident that if the last inequalities are fulfilled on the whole interval $[a,b]$, then $\gamma_\alpha^{-1}(t)$ exists for every $t\in[a,b]$.

Definition 1.1. Let $a_i:[a,b]\to\mathbb{R}$ $(i=1,\dots,n)$ be nondecreasing functions and $n_0\in\{1,\dots,n\}$. We say that the matrix-function $C=(c_{il})_{i,l=1}^n\in \mathrm{BV}([a,b],\mathbb{R}_+^{n\times n})$ belongs to the set $\mathcal{U}(a+,b-;a_1,\dots,a_n;n_0)$, if the system

$$\operatorname{sgn}\Big(n_0+\frac12-i\Big)\,dx_i(t)\leq$$
$$\leq\sum_{l=1}^n c_{il}(t)x_l(t)\,da_i(t)\ \text{ for } t\in[a,b]\ \ (i=1,\dots,n) \qquad (1.5)$$

has no nontrivial nonnegative solution satisfying the condition (1.2).

Definition 1.2. We say that a vector-function $g : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$, $g(t,x) = (g_i(t,x_1,\dots,x_n))_{i=1}^n$, is nondecreasing outside of the diagonal elements (or quasi-nondecreasing) with respect to nondecreasing vector-function $\alpha = (\alpha_i)_{i=1}^n$ if from the condition

$$x_1 \le y_1, \dots, x_{i-1} \le y_{i-1}, x_{i+1} \le y_{i+1}, \dots, x_n \le y_n$$

follows

$$g_i(t,x_1,\dots,x_{i-1},x_i,\dots,x_n) \le g_i(t,y_1,\dots,y_{i-1},x_i,y_{i+1},\dots,y_n)$$

for $\mu(a_i)$-almost all t $(i=1,\dots,n)$.

Definition 1.3. Let $a_i : [a,b] \to \mathbb{R}$ $(i=1,\dots,n)$ be nondecreasing functions and $n_0 \in \{1,\dots,n\}$. We say that a vector-function $g(t,x) = (g_i(t,x_1,\dots,x_n))_{i=1}^n, g_i \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R};a_i)$ $(i=1,\dots,n)$, belongs to the set $\mathcal{U}_0(a+,b-;a_1,\dots,a_n;n_0)$ if it is nonnegative, quasi-nondecreasing and there exists a positive number $r \in \mathbb{R}_+$ such that

$$0 \le x(t) \le r \quad \text{for} \quad t \in [a,b]$$

for every nonnegative solution $x = (x_i)_{i=1}^n$ of the system

$$\operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big)\, dx_i(t) \le \\ \le g_i(t,x_1,\dots,x_n(t))da_i(t) \quad \text{for} \quad t \in [a,b] \quad (i=1,\dots,n) \tag{1.6}$$

under the boundary condition (1.2).

The similar definition of the sets $\mathcal{U}_0$ and $\mathcal{U}$ has been introduced by I. Kiguradze for ordinary differential equations (see [13]–[15]).

Theorem 1.1. *Let the functions* $f_i \in \mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n,\mathbb{R}^n;a_i)$ $(i = 1,\dots,n)$ *be such that*

$$f_i(t,x_1,\dots,x_n)\operatorname{sgn}\Big(\Big(n_0+\frac{1}{2}-i\Big)x_i\Big) \le -b_i(t)|x_i| + g_i(t,|x_1|,\dots,|x_n|)$$
$$\textit{for } \mu(s_c(a_i))\textit{-almost all } t\in[a,b] \textit{ and for every } t \in D_{a_i}, \\ (x_k)_{k=1}^n \in \mathbb{R}^n \quad (i=1,\dots,n),$$

$$f_i(t,x_1,\dots,x_n)d_2a_i(t)\operatorname{sgn}\big(x_i + f_i(t,x_1,\dots,x_n)d_2a_i(t)\big) \le \\ \le -b_i(t)|x_i| + g_i\big(t,|x_1|,\dots,|x_n|\big) \\ \textit{for } t\in[a,b] \textit{ and } (x_k)_{k=1}^n \in \mathbb{R}^n \quad (i=1,\dots,n_0)$$

and

$$f_i(t,x_1,\dots,x_n)d_1a_i(t)\operatorname{sgn}\big(x_i - f_i(t,x_1,\dots,x_n)d_1a_i(t)\big) \ge \\ \ge b_i(t)|x_i| - g_i(t,|x_1|,\dots,|x_n|) \\ \textit{for } t\in[a,b] \textit{ and } (x_k)_{k=1}^n \in \mathbb{R}^n \quad (i=n_0+1,\dots,n),$$

where $g_i \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}_+;a_i)$ $(i=1,\dots,n)$, *the functions* $b_i \in L_{loc}(]a,b],\mathbb{R};a_i)$ *for* $(i=1,\dots,n_0)$ *and* $b_i \in L_{loc}([a,b[\,,\mathbb{R};a_i)$ *for* $(i=n_0+1,\dots,n)$ *are nonnegative. Let, moreover,*

$$\begin{aligned} &g=(g_i)_{i=1}^n \in \mathcal{U}_0(a+,b-;a_1,\dots,a_n;n_0),\\ &\lim_{t\to a+} b_i(t)d_2a_i(t)<1 \ \ (i=1,\dots,n_0),\\ &\lim_{t\to b-} b_i(t)d_1a_i(t)<1 \ \ (i=n_0+1,\dots,n) \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} &\lim_{t\to a+}\ \lim_{k\to\infty}\sup \gamma_{\alpha_i}(t,a+1/k)=0 \ \ (i=1,\dots,n_0),\\ &\lim_{t\to b-}\ \lim_{k\to\infty}\sup \gamma_{\alpha_i}(t,b-1/k)=0 \ \ (i=n_0+1,\dots,n), \end{aligned} \tag{1.8}$$

where $\alpha_i(t)\equiv\int\limits_{c_0}^{t} b_i(\tau)\,da_i(\tau)$ $(i=1,\dots,n)$, $c_0=(a+b)/2$, *and* γ_{α_i} $(i=1,\dots,n)$ *are the functions defined according to* (1.4). *Then the problem* (1.1), (1.2) *is solvable.*

Theorem 1.2. *Let the functions* $f_i \in \mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}^n,\mathbb{R}^n;a_i)$ $(i=1,\dots,n)$ *be such that*

$$\begin{aligned} &f_i(t,x_1,\dots,x_n)\,\mathrm{sgn}\Big(\Big(n_0+\frac12-i\Big)x_i\Big)\le\\ &\qquad\le -b_i(t)|x_i|+\sum_{l=1}^n \eta_{il}(t)|x_l|+q_i\Big(t,\sum_{l=1}^n |x_l|\Big)\\ &\qquad\text{for } \mu(s_c(a_i))\text{-almost all } t\in[a,b] \text{ and for every } t\in D_{a_i},\\ &\qquad\qquad (x_k)_{k=1}^n\in\mathbb{R}^n \ \ (i=1,\dots,n), \end{aligned} \tag{1.9}$$

$$\begin{aligned} &f_i(t,x_1,\dots,x_n)d_2a_i(t)\,\mathrm{sgn}\big(x_i+f_i(t,x_1,\dots,x_n)d_2a_i(t)\big)\le\\ &\le -b_i(t)|x_i|+\sum_{l=1}^n\eta_{il}(t)|x_l|+q_i\Big(t,\sum_{l=1}^n|x_l|\Big) \text{ for } t\in[a,b] \ \ (i=1,\dots,n_0) \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &f_i(t,x_1,\dots,x_n)d_1a_i(t)\,\mathrm{sgn}\big(x_i-f_i(t,x_1,\dots,x_n)d_1a_i(t)\big)\ge\\ &\ge b_i(t)|x_i|-\sum_{l=1}^n\eta_{il}(t)|x_l|-q_i\Big(t,\sum_{l=1}^n|x_l|\Big) \text{ for } t\in[a,b] \ \ (i=n_0+1,\dots,n) \end{aligned} \tag{1.11}$$

where $\eta_{il}\in L([a,b],\mathbb{R};a_i)$ $(i,l=1,\dots,n)$, *the functions* $b_i\in L_{loc}(]a,b],\mathbb{R};a_i)$ $(i=1,\dots,n_0)$ *and* $b_i\in L_{loc}([a,b[,\mathbb{R};a_i)$ $(i=n_0+1,\dots,n)$ *are nonnegative, and* $q_i\in\mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+;a_i)$ $(i=1,\dots,n)$ *are nondecreasing functions in the second variable. Let, moreover, the conditions* (1.7), (1.8),

$$C=(c_{il})_{i,l=1}^n\in\mathcal{U}(a+,b-;a_1,\dots,a_n;n_0),$$

and

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int\limits_a^b q_i(t,\rho)\,da_i(t)=0\ \ (i=1,\dots,n) \tag{1.12}$$

be valid, where $\alpha_i(t)\equiv\int\limits_{c_0}^t b_i(\tau)\,da_i(\tau)$ $(i=1,\dots,n)$, $c_0=(a+b)/2$, $c_{il}(t)\equiv\int\limits_a^t \eta_{il}(\tau)\,da_i(\tau)$ $(i,l=1,\dots,n)$, *and* γ_{α_i} $(i=1,\dots,n)$ *are the functions defined according to* (1.4). *Then the problem* (1.1), (1.2) *is solvable.*

Corollary 1.1. *Let the functions* $f_i\in \mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n,\mathbb{R}^n;a_i)$ $(i=1,\dots,n)$ *be such that the conditions* (1.9)–(1.12) *hold, where the functions* a_i $(i=1,\dots,n)$ *have not more than a finite number of points of discontinuity, the functions* $b_i\in L_{loc}(]a,b],\mathbb{R};a_i)$ $(i=1,\dots,n_0)$ *and* $b_i\in L_{loc}([a,b[\,,\mathbb{R};a_i)$ $(i=n_0+1,\dots,n)$ *are nonnegative,* $q_i\in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+;a_i)$ $(i=1,\dots,n)$ *are nondecreasing functions in the second variable,* $\alpha_i(t)\equiv\int\limits_{c_0}^t b_i(\tau)\,da_i(\tau)$ $(i=1,\dots,n)$, $c_0=(a+b)/2$, γ_{α_i} $(i=1,\dots,n)$ *are the functions defined according to* (1.4),

$$\int\limits_a^t \eta_{il}(\tau)\,da_i(\tau)\equiv\int\limits_c^t h_{il}(\tau)d\beta_l(\tau)\ \ (i,l=1,\dots,n),$$

β_l $(l=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$, $h_{ii}\in L^\mu([a,b],\mathbb{R};\beta_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\beta_l)$ $(i\neq l;\ i,l=1,\dots,n)$, $1\le\mu\le+\infty$. *Let, moreover,*

$$r(\mathcal{H})<1, \tag{1.13}$$

where the $3n\times 3n$*-matrix* $\mathcal{H}=(\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$ *is defined by*

$$\mathcal{H}_{j+1\,m+1}=\Big(\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\beta_i)}\Big)_{i,k=1}^n\ \ (j,m=0,1,2),$$

$$\xi_{ij}=\big(s_j(\beta_i)(b)-s_j(\beta_i)(a)\big)^{\frac{1}{\nu}}\ \ (j=0,1,2,;\ \ i=1,\dots,n);$$

$$\lambda_{k0i0}=\begin{cases}\Big(\dfrac{4}{\pi^2}\Big)^{\frac{1}{\nu}}\xi_{k0}^2 & \text{if } s_0(\beta_i)(t)\equiv s_0(\beta_k)(t),\\ \xi_{k0}\xi_{i0} & \text{if } s_0(\beta_i)(t)\not\equiv s_0(\beta_k)(t)\ \ (i,k=1,\dots,n);\end{cases}$$

$$\lambda_{kmij}=\xi_{km}\xi_{ij}\ \ \text{if } m^2+j^2>0,\ \ mj=0\ \ (j,m=0,1,2;\ \ i,k=1,\dots,n),$$

$$\lambda_{kmij}=\Big(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\Big)^{\frac{1}{\nu}}\ (j,m=1,2;\ i,k=1,\dots,n),$$

and $\frac{1}{\mu}+\frac{2}{\nu}=1$. *Then the problem* (1.1), (1.2) *is solvable.*

Remark 1.1. The $3n \times 3n$-matrix $\mathcal{H}$, appearing in Corollary 1.1 can be replaced by the $n \times n$-matrix

$$\left(\max \left\{ \sum_{j=0}^{2} \lambda_{kmij} \|h_{ik}\|_{\mu, S_m(\alpha_k)} : \ m = 0, 1, 2 \right\} \right)_{i,k=1}^{n}.$$

Remark 1.2. If $a_i(t) \equiv a_0(t)\,(i = 1, \dots, n)$, where the function a_0 has not more than a finite number of points of discontinuity, then we can assume that $h_{il}(t) \equiv \eta_{il}(t)$ and $\beta_l(t) \equiv a_0(t)\,(i, l = 1, \dots, n)$.

By Remark 1.1, Corollary 1.1 has the following form for $a_i(t) \equiv a_0(t)$, $b_i(t) \equiv b_0(t)$, $\eta_{il}(t) \equiv \eta_{il} = const$, $q_i(t,x) \equiv q(t,x)$ $(i, l = 1, \dots, n)$ and $\mu = +\infty$ since, by the choice of $h_{il}(t) \equiv \eta_{il}(t) = \eta_{il}$ $(i, l = 1, \dots, n)$, we have $\beta_l(t) \equiv a_0(t)$ $(l = 1, \dots, n)$ in this case.

Corollary 1.2. *Let the functions* $f_i \in \mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n, \mathbb{R}^n; a_0)$ *be such that the conditions* (1.12),

$$f_i(t, x_1, \dots, x_n) \operatorname{sgn}\left(\left(n_0 + \frac{1}{2} - i\right) x_i\right) \le$$
$$\le -b_0(t)|x_i| + \sum_{l=1}^{n} \eta_{il}|x_l| + q_i\left(t, \sum_{l=1}^{n} |x_l|\right)$$
for $\mu(s_c(a_0))$*-almost all* $t \in [a,b]$ *and for every* $t \in D_a$, $(x_k)_{k=1}^n \in \mathbb{R}^n$ $(i = 1, \dots, n)$,

$$f_i(t, x_1, \dots, x_n) d_2 a_i(t) \operatorname{sgn}\big(x_i + f_i(t, x_1, \dots, x_n) d_2 a_i(t)\big) \le$$
$$\le -b_0(t)|x_i| + \sum_{l=1}^{n} \eta_{il}|x_l| + q_i\left(t, \sum_{l=1}^{n} |x_l|\right) \ \text{for } t \in [a,b] \ (i = 1, \dots, n_0),$$
$$f_i(t, x_1, \dots, x_n) d_1 a_i(t) \operatorname{sgn}\big(x_i - f_i(t, x_1, \dots, x_n) d_1 a_i(t)\big) \ge$$
$$\ge b_0(t)|x_i| - \sum_{l=1}^{n} \eta_{il}|x_l| - q_i\left(t, \sum_{l=1}^{n} |x_l|\right) \ \text{for } t \in [a,b] \ (i = n_0 + 1, \dots, n)$$

and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b q(t, \rho)\, da_0(t) = 0$$

hold, where a_0 *is a nondecreasing function on* $[a,b]$ *having no more than a finite number of points of discontinuity,* $b_0 \in L([a,b], \mathbb{R}_+; a_0)$, $q \in \mathrm{Car}([a,b] \times \mathbb{R}_+, \mathbb{R}_+; a_0)$ *is a nondecreasing function in the second variable, the function* $\alpha(t) \equiv \int_{c_0}^{t} b(\tau)\, da(\tau)$, $c_0 = (a+b)/2$, *satisfies the conditions* (1.7) *and* (1.8), γ_α *is the function defined according to* (1.4), $\eta_{ii} \in \mathbb{R}$, $\eta_{il} \in \mathbb{R}_+$ $(i \ne l;$ $i, l = 1, \dots, n)$. *Let, moreover,*

$$\rho_0\, r(\mathcal{H}) < 1,$$

where

$$\mathcal{H} = (\eta_{ik})_{i,k=1}^{n}, \quad \rho_0 = \max\bigg\{\sum_{j=0}^{2} \lambda_{mj} : \ m = 0,1,2\bigg\},$$

$$\lambda_{00} = \frac{2}{\pi}\,\big(s_0(a_0)(b) - s_0(a_0)(a)\big),$$

$$\lambda_{0j} = \lambda_{j0} = \big(s_0(a_0)(b) - s_0(a_0)(a)\big)^{\frac{1}{2}}\big(s_j(a_0(b) - s_j(a_0(a)\big)^{\frac{1}{2}} \ \ (j = 1,2),$$

$$\lambda_{mj} = \frac{1}{2}\,\big(\mu_{\alpha m}\nu_{\alpha m \alpha j}\big)^{\frac{1}{2}} \sin^{-1}\,\frac{\pi}{4n_{\alpha m}+2} \ \ (m,j = 1,2).$$

Then the problem $(1.1), (1.2)$ *is solvable.*

Theorem 1.3. *Let the functions* $f_i \in \mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n,\mathbb{R}^n;a_i)$ $(i = 1,\dots,n)$ *be such that the conditions* (1.7)–(1.12),

$$d_2\beta_i(a) \le 0 \ \ \textit{and} \ \ 0 \le d_1\beta_i(t) < |\eta_i|^{-1} \ \ \textit{for} \ \ a < t \le b \ \ (i = 1,\dots,n_0),$$

$$d_1\beta_i(b) \le 0 \ \ \textit{and} \ \ 0 \le d_2\beta_i(t) < |\eta_i|^{-1} \ \ \textit{for} \ \ a \le t < b \ \ (i = n_0+1,\dots,n)$$

and

$$\int_c^t \eta_{il}(\tau)\,da_(\tau) = h_{il}\beta_i(t) + \beta_{il}(t) \ \ \textit{for} \ \ t \in [q,b] \ \ (i,l = 1,\dots,n)$$

are fulfilled, where $\eta_{il} \in L([a,b],\mathbb{R};a_i)$ $(i,l = 1,\dots,n)$, *the functions* $b_i \in L_{loc}(]a,b],\mathbb{R};a_i)$ $(i = 1,\dots,n_0)$ *and* $b_i \in L_{loc}([a,b[,\mathbb{R};a_i)$ $(i = n_0+1,\dots,n)$ *are nonnegative, and* $q_i \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+;a_i)$ $(i = 1,\dots,n)$ *are nondecreasing functions in the second variable,* $\alpha_i(t) \equiv \int_{c_0}^{t} b_i(\tau)\,da_i(\tau)$ $(i = 1,\dots,n)$, $c_0 = (a+b)/2$, *and* γ_{α_i} $(i = 1,\dots,n)$ *are the functions defined according to* (1.4), $h_{ii} < 0$, $h_{il} \ge 0$, $\eta_i < 0$ $(i \ne l;\ i,l = 1,\dots,n)$, β_{ii} $(i = 1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$; $\beta_{il}, \beta_i \in \mathrm{BV}([a,b],\mathbb{R})$ $(i \ne l;\ i,l = 1,\dots,n)$ *are the functions nondecreasing on the interval* $]a,b]$ *for* $i \in \{1,\dots,n_0\}$ *and on the interval* $[a,b[$ *for* $i \in \{n_0+1,\dots,n\}$. *Let, moreover, the condition* (1.16) *hold, where* $\mathcal{H} = (\xi_{il})_{i,l=1}^{n}$,

$$\xi_{ii} = \lambda_i, \ \ \xi_{il} = \frac{h_{il}}{|h_{ii}|} \ \ (i \ne l; \ \ i,l = 1,\dots,n),$$

$$\lambda_i = V\big(\mathcal{A}(\zeta_i,\gamma_i)\big)(b) - V\big(\mathcal{A}(\zeta_i,\gamma_i)\big)(a+) \ \ \textit{for} \ \ i \in \{1,\dots,n_0\},$$

$$\lambda_i = V\big(\mathcal{A}(\zeta_i,\gamma_i)\big)(b-) - V\big(\mathcal{A}(\zeta_i,\gamma_i)\big)(a) \ \ \textit{for} \ \ i \in \{n_0+1,\dots,n\};$$

$$\zeta_i(t) \equiv \sum_{k=l}^{n} \beta_{il}(t) \ \ (i = 1,\dots,n);$$

and

$$\gamma_i(t) \equiv \big(\beta_i(t) - \beta_i(a+)\big)h_{ii} \ \ \textit{for} \ \ a < t \le b \ \ (i = 1,\dots,n_0),$$

$$\gamma_i(t) \equiv \big(\beta_i(b-) - \beta_i(t)\big)h_{ii} \ \ \textit{for} \ \ a \le t < b \ \ (i = n_0+1,\dots,n).$$

Then the problem (1.1), (1.2) *is solvable.*

Remark 1.3. If

$$\lambda_i < 1 \ \ (i = 1, \dots, n), \tag{1.14}$$

then, in Theorem 1.2, we can assume that

$$\xi_{ii} = 0, \ \ \xi_{il} = \frac{h_{il}}{(1-\lambda_i)|h_{ii}|} \ \ (i \neq l; \ i, l = 1, \dots, n). \tag{1.15}$$

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Author's address:

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. Sukhumi State University, 12 Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: ashord@rmi.ge

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Malkhaz Ashordia, Goderdzi Ekhvaia, and Nestan Kekelia

ON THE CONTI–OPIAL TYPE EXISTENCE AND UNIQUENESS THEOREMS FOR GENERAL NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF IMPULSIVE EQUATIONS WITH FINITE AND FIXED POINTS OF IMPULSES ACTIONS

Abstract. The general nonlocal boundary value problem is considered for systems of impulsive equations with finite and fixed points of impulses actions. Sufficient conditions are given for the solvability and unique solvability of the problem.

რეზიუმე. სასრულ და ფიქსირებულ იმპულსირ წერტილებიან განტოლებათა სისტემებისთვის განხილულია ზოგადი სახის არაწრფივი სასაზღვრო ამოცანა. მოყვანილია ამ ამოცანის ამოხსნადობისა და ცალსახად ამოხსნადობის საკმარისი პირობები.

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In the present paper, we consider the system of nonlinear impulsive equations with a finite number of impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = I_l(x(\tau_l)) \ \ (l = 1,\dots,m_0), \tag{2}$$

with the general boundary value condition

$$h(x) = 0, \tag{3}$$

where $a < \tau_1 < \dots < \tau_{m_0} \le b$ (we will assume $\tau_0 = a$ and $\tau_{m_0+1} = b$, if necessary), $-\infty < a < b < +\infty$, m_0 is a natural number, $f = (f_i)_{i=1}^n$ belongs to Carathéodory class $Car([a,b] \times \mathbb{R}^n, \mathbb{R}^n)$, $I_l = (I_{li})_{i=1}^n : R^n \to \mathbb{R}^n$ $(l = 1,\dots,m_0)$ are continuous operators, and $h : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ is a continuous, nonlinear, in general, vector-functional.

In the paper, the sufficient (among them the effective sufficient) conditions are given for solvability and unique solvability of the general nonlinear impulsive boundary value problem $(1),(2);(3)$. We have established the Conti–Opial type theorems for the solvability and unique solvability of this problem. Analogous problems investigated in [8]– [11], [13] (see also the references therein) deal with the general nonlinear boundary value problems for ordinary differential and functional-differential systems.

Certain results obtained in the paper are more general than those already known even for ordinary differential case.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see e.g. [1]–[7], [12], [14] and the references therein). But the above-mentioned works, as we know, do not contain the results obtained in the present paper.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$; $[a, b]$ $(a, b \in R)$ is a closed segment.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$$|X| = (|x_{ij}|)_{i,j}^{n,m}, \quad [X]_+ = \frac{|X| + X}{2};$$

$$\mathbb{R}_+^{n\times m} = \Big\{(x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m)\Big\};$$

$$\mathbb{R}^{(n\times n)\times m} = \mathbb{R}^{n\times n} \times \cdots \times \mathbb{R}^{n\times n} \ \ (m\text{-times}).$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = R_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix, inverse to X, the determinant of X and the spectral radius of X; $I_{n\times n}$ is the identity $n \times n$-matrix.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X : [a, b] \to R^{n\times m}$, i.e., the sum of total variations of the latter components;

$$V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m},$$

where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = \overset{t}{\underset{a}{\vee}}(x_{ij})$ for $a < t \leq b$;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limit of the matrix-function $X : [a, b] \to R^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary);

$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [a, b]\big\}.$$

$\mathrm{BV}([a, b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a, b] \to R^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X) < +\infty$);

$C([a, b], D)$, where $D \subset R^{n\times m}$, is the set of all continuous matrix-functions $X : [a, b] \to D$;

$C([a, b], D; \tau_1, \dots, \tau_{m_0})$ is the set of all matrix-functions $X : [a, b] \to D$, having the one-sided limits $X(\tau_l-)$ $(l = 1, \dots, m_0)$ and $X(\tau_l+)$ $(l =$

$1,\dots,m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}l\}$ belong to $C([c,d],D)$;

$C_s([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0})$ is the Banach space of all $X \in C([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0})$ with the norm $\|X\|_s$.

$\widetilde{C}([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$;

$\widetilde{C}([a,b],D;\tau_1,\dots,\tau_{m_0})$ is the set of all matrix-functions $X:[a,b]\to D$, having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus\{\tau_k\}_{k=1}^m$ belong to $\widetilde{C}([c,d],D)$.

If B_1 and B_2 are the normed spaces, then the operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$.

The operator $\varphi: C([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0})\to R^n$ is called nondecreasing if for every $x,y\in C([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0})$ such that $x(t)\le y(t)$ for $t\in[a,b]$ the inequality $\varphi(x)(t)\le\varphi(y)(t)$ holds for $t\in[a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X:[a,b]\to D$.

If $D_1\subset R^n$ and $D_2\subset R^{n\times m}$, then $Car([a,b]\times D_1,D_2)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

(a) the function $f_{kj}(\cdot,x):[a,b]\to D_2$ is measurable for every $x\in D_1$;

(b) the function $f_{kj}(t,\cdot\,):D_1\to D_2$ is continuous for almost all $t\in[a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)|:\ x\in D_0\big\}\in L([a,b],R;g_{ik})\ \text{ for every compact }\ D_0\subset D_1.$$

$Car^0([a,b]\times D_1,D_2)$ is the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that the functions $f_{kj}(\cdot,x(\cdot))$ $(i=1,\dots,l;\ k=1,\dots,n)$ are measurable for every vector-function $x:[a,b]\to\mathbb{R}^n$ with a bounded total variation.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x\in\widetilde{C}([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})$ satisfying both the system (1) a.e. on $[a,b]\setminus\{\tau_1\dots,\tau_{m_0}\}$ and the relation (2) for every $k\in\{1,\dots,m_0\}$.

Definition 1. Let $\ell: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$ be a linear continuous operator, and let $\ell_0: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n_+$ be a positive homogeneous operator. We say that a pair $(P,\{J_l\}_{l=1}^{m_0})$, consisting of a matrix-function $P\in Car([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a finite sequence of continuous operators $J_l=(J_{li})_{i=1}^n:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,\dots,m_0)$, satisfy the Opial condition with respect to the pair (ℓ,ℓ_0) if:

(a) there exist a matrix-function $\Phi \in L([a,b],\mathbb{R}^n_+)$ and constant matrices $\Psi_l \in \mathbb{R}^{n\times n} (l=1,\dots,m_0)$ such that

$$|P(t,x)| \le \Phi(t) \text{ a.e. on } [a,b], \ x \in \mathbb{R}^n$$

and

$$|J_l(x)| \le \Psi_l \text{ for } x \in \mathbb{R}^n \ (l=1,\dots,m_0);$$

(b)

$$\det(I_{n\times n} + G_l) \ne 0 \ (l=1,\dots,m_0) \tag{4}$$

and the problem

$$\frac{dx}{dt} = A(t)x \text{ a.e. on } [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{5}$$

$$x(\tau_l+) - x(\tau_l-) = G_l x(\tau_l) \ (l=1,\dots,m_0), \tag{6}$$

$$|\ell(x)| \le \ell_0(x) \tag{7}$$

has only the trivial solution for every matrix-function $A \in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices G_l $(l=1,\dots,m_0)$ for which there exists a sequence $y_k \in \widetilde{C}([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty} \int_a^t P(\tau,y_k(\tau))\,d\tau = \int_a^t A(\tau)\,d\tau \text{ uniformly on } [a,b]$$

and

$$\lim_{k\to+\infty} J_l(y_k(\tau_l)) = G_l \ (l=1,\dots,m_0).$$

Remark 1. In particular, the condition (4) holds if

$$\|\Psi_l\| < 1 \ (l=1,\dots,m_0).$$

Below, we will assume that $f=(f_i)_{i=1}^n \in Car([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and, in addition, $f(\tau_l,x)$ is arbitrary for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$.

Theorem 1. *Let the conditions*

$$\big\|f(t,x)-P(t,x)x\big\| \le \alpha(t,\|x\|) \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ x\in\mathbb{R}^n, \tag{8}$$

$$\big\|I_l(x)-J_l(x)x\big\| \le \beta_l(\|x\|) \text{ for } x\in\mathbb{R}^n \ (l=1,\dots,m_0) \tag{9}$$

and

$$\big|h(x)-\ell(x)\big| \le \ell_0(x)+\ell_1(\|x\|_s) \text{ for } x\in \mathrm{BV}([a,b],\mathbb{R}^n) \tag{10}$$

hold, where

$$\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n \text{ and } \ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n_+$$

are, respectively, linear continuous and positive homogeneous continuous operators, the pair $(P,\{J_l\}_{l=1}^{m_0})$ satisfies the Opial condition with respect to the pair (ℓ,ℓ_0); $\alpha\in Car([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is a function, nondecreasing in the *second variable, and* $\beta_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that*

$$\limsup_{\rho\to+\infty} \frac{1}{\rho}\left(\|\ell_1(\rho)\| + \int\limits_a^b \alpha(t,\rho)\,dt + \sum_{l=1}^{m_0} \beta_l(\rho)\right) < 1. \tag{11}$$

Then the problem (1), (2); (3) *is solvable.*

Theorem 2. *Let the conditions* (8)–(10),

$$P_1(t) \le P(t,x) \le P_2(t) \ \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x \in \mathbb{R}^n,$$

and

$$J_{1l} \le I_k(x) \le J_{2l} \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0)$$

hold, where $P \in Car^0([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i \in L([a,b],\mathbb{R}^{n\times n})$ $(i = 1,2)$, $J_{il} \in \mathbb{R}^{n\times n}$ $(i = 1,2;\ l = 1,\dots,m_0)$, $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *are, respectively, linear continuous and positive homogeneous continuous operators;* $\alpha \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function, nondecreasing in the second variable, and* $\beta_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that the condition* (11) *holds. Let, moreover, the condition* (4) *hold and the problem* (5), (6); (7) *have only the trivial solution for every matrix-function* $A \in L([a,b],\mathbb{R}^{n\times n})$ *and constant matrices* $G_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *such that*

$$P_1(t) \le A(t) \le P_2(t) \ \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x \in \mathbb{R}^n$$

and

$$J_{1l} \le G_l \le J_{2l} \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0).$$

Then the problem (1), (2); (3) *is solvable.*

Remark 2. Theorem 1.2 is of interest only in the case where $P \notin Car([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because the theorem follows immediately from Theorem 1.1 in the case where $P \in Car([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$.

Theorem 3. *Let the conditions* (10),

$$|f(t,x) - P_0(t)x| \le$$
$$\le Q(t)|x| + q(t,\|x\|) \ \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x \in \mathbb{R}^n,$$

and

$$\big|I_l(x) - J_{0l}\cdot x\big| \le H_l|x| + h_l(\|x\|) \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0)$$

hold, where $P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $Q \in L([a,b],\mathbb{R}^{n\times n}_+)$, J_{0l} *and* $H_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices,* $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *are, respectively, the linear continuous and positive homogeneous continuous operators;* $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function, nondecreasing in the second variable, and*

$h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function such that

$$\det(I_{n\times n} + J_{0l}) \neq 0 \ \ (l = 1,\dots,m_0) \tag{12}$$

and

$$\|H_l\| \cdot \|(I_{n\times n} + J_{0l})^{-1}\| < 1 \ \ (j = 1,2;\ l = 1,\dots,m_0) \tag{13}$$

hold, and the system of impulsive inequalities

$$\Big|\frac{dx}{dt} - P_0(t)x\Big| \le Q(t)x \ \ \text{a.e. on } [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{14}$$

$$\big|x(\tau_l+) - x(\tau_l-) - J_{0l}x(\tau_l)\big| \le H_l \cdot x(\tau_l) \ \ (l = 1,\dots,m_0) \tag{15}$$

have only the trivial solution under the condition (7)*. Then the problem* (1), (2)*;* (3) *is solvable.*

Corollary 1. *Let the conditions* (12)

$$\big|f(t,x) - P_0(t)x\big| \le q(t,\|x\|) \ \ \text{a.e. on } [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \ \ x \in \mathbb{R}^n, \tag{16}$$

and

$$\big|I_l(x) - J_{0l} \cdot x\big| \le h_l(\|x\|) \ \ \text{for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0) \tag{17}$$

hold, where $P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are the constant matrices, $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ and $\ell_0 : C_s([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators; $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ is a vector-function, nondecreasing in the second variable, and $h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function such that the condition (11) *holds. Let, moreover,*

$$\big|h(x) - \ell(x)\big| \le \ell_1(\|x\|_s) \ \ \text{for } x \in \mathrm{BV}([a,b],\mathbb{R}^n) \tag{18}$$

and the impulsive system

$$\begin{aligned} &\frac{dx}{dt} = P_0(t)x \ \ \text{a.e. on } [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \\ &x(\tau_l+) - x(\tau_l-) = J_{0l}x(\tau_l) \ \ (l = 1,\dots,m_0) \end{aligned}$$

have only the trivial solution under the condition

$$\ell(x) = 0.$$

Then the problem (1), (2)*;* (3) *is solvable.*

For every matrix-function $X \in L([a,b],\mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k=1,\dots,m_0)$ we introduce the operators

$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_0 = I_n \ \text{ for } \ a \le t \le b,$$
$$\big[(X,Y_1,\dots,Y_{m_0})(a)\big]_i = O_{n\times n} \ \ (i=1,2,\dots),$$
$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_{i+1} = \int_a^t X(\tau)\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau)\big]_i\,d\tau +$$
$$+\sum_{a\le\tau_l<t} Y_l\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau_l)\big]_i \ \text{ for } \ a<t\le b \ \ (i=1,2,\dots). \qquad (19)$$

Corollary 2. *Let the conditions* (12), (16)–(18) *hold, where*

$$\ell(x) \equiv \int_a^b dL(t)\cdot x(t),$$

$P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}(l=1,\dots,m_0)$ *are constant matrices,* $L \in L([a,b],\mathbb{R}^{n\times n})$, $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *is a positive homogeneous continuous operator;* $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function, nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that the condition* (11) *holds. Let, moreover, there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\int_a^b dL(t)\cdot\big[(P_0,G_l,\dots,G_{m_0})(t)\big]_i$$

is nonsingular and

$$r(M_{k,m}) < 1, \qquad (20)$$

where the operators $[(P_0,G_1,\dots,G_{m_0})(t)]_i$ $(i=0,1,\dots)$ *are defined by* (19), *and*

$$M_{k,m} = \Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(b)\Big]_m +$$
$$+\sum_{i=0}^{m-1}\Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(b)\Big]_i\times$$
$$\times\int_a^b dV(M_k^{-1}L)(t)\cdot\Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(t)\Big]_k.$$

Then the problem (1), (2); (3) *is solvable.*

Corollary 3. *Let the conditions* (12), (16)–(18) *and*

$$\ell(x) \equiv \sum_{j=1}^{n_0} L_j x(t_j) \qquad (21)$$

hold, where $P_0 \in L([a,b],\mathbb{R}^{n\times n})$*,* $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are constant matrices,* $t_j \in [a,b]$ *and* $L_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$*,* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *is a positive homogeneous continuous operator;* $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function, nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that the condition* (11) *holds. Let, moreover, there exist natural numbers* l *and* m *such that the matrix*

$$M_k = \sum_{j=1}^{n_0}\sum_{i=0}^{k-1} L_j\big[(P_0,G_l,\dots,G_{m_0})(t_j)\big]_i$$

is nonsingular and the inequality (20) *holds, where*

$$M_{k,m} = \Big[\big(|P_0|,|G_l|,\dots,|G_{m_0}|\big)(b)\Big]_m + \\ + \Big(\sum_{i=0}^{m-1}\Big[\big(|P_0|,|G_l|,\dots,|G_{m_0}|\big)(b)\Big]_i\Big)\times \\ \times \sum_{j=1}^{n_0}|M_k^{-1}L_j|\cdot\Big[\big(|P_0|,|G_l|,\dots,|G_{m_0}|\big)(t_j)\Big]_k.$$

Then the problem (1), (2)*;* (3) *is solvable.*

Corollary 4. *Let the conditions* (12)*,* (16)*–*(18) *and* (21) *hold, where* $P_0 \in L([a,b],\mathbb{R}^{n\times n})$*,* $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices,* $t_j \in [a,b]$ *and* $L_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$*,* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *is a positive homogeneous continuous operator;* $q \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function, nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are nondecreasing, respectively, functions and vector-function such that the condition* (11) *holds. Let, moreover,*

$$\det\Big(\sum_{j=1}^{n_0}L_j\Big) \neq 0 \ \ \text{and} \ \ r(L_0\cdot V(A)(b)) < 1,$$

where

$$L_0 = I_{n\times n} + \Big|\Big(\sum_{j=1}^{n_0}L_j\Big)^{-1}\Big|\cdot\sum_{j=1}^{n_0}|L_j| \ \ \text{and} \ \ A_0 = \int_a^b |P_0(t)|\,dt + \sum_{l=1}^{m_0}|G_l|.$$

Then the problem (1), (2)*;* (3) *is solvable.*

Theorem 4. *Let the conditions* (12)*,* (13)*,*

$$\big|f(t,x) - f(t,y) - P_0(t)(x-y)\big| \le Q(t)|x-y| \\ \text{a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x,y\in\mathbb{R}^n, \\ \big|I_l(x)-I_l(y)-J_{0l}\cdot(x-y)\big| \le H_k\cdot|x-y| \ \ \text{for } x,y\in\mathbb{R}^n \ \ (k=l,\dots,m_0)$$

and

$$\left|h(x) - h(y) - \ell(x - y)\right| \le \ell_0(x - y) \quad \textit{for } x, y \in \mathrm{BV}([a,b], \mathbb{R}^n)$$

hold, where $P_0 \in L([a,b], \mathbb{R}^{n\times n})$*,* $Q \in L([a,b], \mathbb{R}_+^{n\times n})$*,* J_{0k} *and* $H_l \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are the constant matrices,* $\ell : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}_+^n$ *are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the system of impulsive inequalities* (14)*,* (15) *have only the trivial solution under the condition* (7)*. Then the problem* (1)*,* (2)*;* (3) *is solvable.*

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Author's address:

M. Ashordia

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

2. Sukhumi State University, 12 Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: ashord@rmi.ge

G. Ekhvaia, N. Kekelia

Sukhumi State University, 12 Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: goderdzi.ekhvaia@mail.ru; nest.kek@mail.ru

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Ivan Kiguradze

POSITIVE SOLUTIONS OF NONLOCAL PROBLEMS FOR NONLINEAR SINGULAR DIFFERENTIAL SYSTEMS

Abstract. For nonlinear differential systems with singularities with respect to phase variables, sufficient conditions for the existence of positive solutions of nonlocal problems are established.

რეზიუმე. არაწრფივი დიფერენციალური სისტემებისათვის სინგულარობებით ფაზური ცვლადების მიმართ დადგენილია არალოკალური ამოცანების დადებითი ამონახსნების არსებობის საკმარისი პირობები.

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Let $-\infty < a < b < +\infty$, $\mathbb{R}_+^n$ be the set of n-dimensional real vectors $(x_i)_{i=1}^n$ with nonnegative components $x_1, \dots, x_n$,

$$\mathbb{R}_{0+}^n = \big\{(x_i)_{i=1}^n : \ x_1 > 0, \dots, x_n > 0\big\},$$

and let $C([a,b];\mathbb{R}_+^n)$ be the set of continuous vector functions $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}_+^n$. Consider the nonlocal problem

$$\frac{du_i}{dt} = f_i(t, u_1, \dots, u_n) \ \ (i = 1, \dots, n), \tag{1}$$

$$u_i(t_i) = \varphi_i(u_1, \dots, u_n) \ \ (i = 1, \dots, n), \tag{2}$$

where $f_i :]a,b[\times \mathbb{R}_{0+}^n \to \mathbb{R}$ are functions satisfying the local Carathéodory conditions, $a \le t_i \le b$ $(i = 1, \dots, n)$, and $\varphi_k : C([a,b];\mathbb{R}_+^n) \to \mathbb{R}_+$ $(k = 1, \dots, n)$ are continuous and bounded on every bounded subset of $C([a,b];\mathbb{R}_+^n)$ functionals.

In the case where the functions f_i $(i = 1, \dots, n)$ have no singularities with respect to phase variables, boundary value problems of the type (1), (2) have been studied in [1]–[4].

The present paper deals with the case not investigated yet, when f_i $(i = 1, \dots, n)$ have singularities with respect to the phase variables, that is the case, where

$$\lim_{x_k \to 0} \big|f_i(t, x_1, \dots, x_n)\big| = +\infty \ \ (i, k = 1, \dots, n).$$

Throughout the paper, along with the above-introduced we will use the following notations.

$(x_{ik})_{i,k=1}^n$ is the matrix with components x_{ik} $(i, k = 1, \dots, n)$.

$r(X)$ is the spectral radius of the $n \times n$ matrix X.

If $u : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\|u\|_C = \max\big\{\|u(t)\| : \ a \leq t \leq b\big\}.$$

If $\delta_k : [a, b] \to [0, +\infty[$ $(k = 1, \dots, n)$ are continuous functions satisfying the conditions

$$\delta_k(t) > 0 \ \text{ for almost all } \ t \in [a, b] \ \ (k = 1, \dots, n),$$

and $\rho > 0$, then

$$f^*(\delta_1, \dots, \delta_n, \rho)(t) = \sup\Big\{ \sum_{i=1}^{n} \big|f_i(t, x_1, \dots, x_n)\big| : \\ \delta_1(t) < x_1 < \delta_1(t) + \rho, \dots, \delta_n(t) < x_n < \delta_n(t) + \rho \Big\}.$$

Along with (1), (2), we consider the auxiliary problem

$$\frac{du_i}{dt} = \lambda f_i(t, u_1, \dots, u_n) + (1 - \lambda)\delta_i(t) \ \ (i = 1, \dots, n), \tag{3}$$

$$u_i(t_i) = \lambda\varphi_i(u_1, \dots, u_n) \ \ (i = 1, \dots, n), \tag{4}$$

$$u_i(t) \geq \delta_i(t) \ \text{ for } \ a \leq t \leq b, \tag{5}$$

depending on the parameter $\lambda \in]0, 1]$ and on absolutely continuous functions $\delta_i : [a, b] \to [0, +\infty[$ $(i = 1, \dots, n)$.

An absolutely continuous vector function $(u_i)_{i=1}^n : [a, b] \to \mathbb{R}_+^n$ is said to be a positive solution of the system (1) (of the system (3)) if it almost everywhere on $[a, b]$ satisfies this system and

$$u_i(t) > 0 \ \text{ for almost all } \ t \in [a, b] \ \ (i = 1, \dots, n).$$

A positive solution $(u_i)_{i=1}^n$ of the system (1) (of the system (3)), satisfying the conditions (2) (the conditions (4) and (5)), is called a positive solution of the problem (1), (2) (a solution of the problem (3), (4), (5)).

The following theorem is valid.

Theorem 1 (The Principle of a Priori Boundedness)**.** *Let for any $i \in \{1, \dots, n\}$ on the set*

$$\Big\{(t, x_1, \dots, x_n) : \ t \in [a, b] \setminus I_0, \ x_k > \delta_k(t) \ \textit{for} \ k \neq i, \ x_i = \delta_i(t)\Big\}$$

the inequality

$$\big[f_i(t, x_1, \dots, x_n) - \delta_i'(t)\big] \operatorname{sgn}(t - t_i) \geq 0$$

hold, where I_0 is a set of zero measure, and $\delta_k : [a, b] \to [0, +\infty[$ $(k = 1, \dots, n)$ are absolutely continuous functions such that

$$\delta_i(t) > 0 \ \textit{ for } \ t \in [a, b] \setminus I_0 \ \ (i = 1, \dots, n),$$

$$\varphi_i(u_1, \dots, u_n) \geq \delta_i(t_i) \ \textit{ for } \ (u_k)_{k=1}^n \in C([a, b]; \mathbb{R}_+^n) \ \ (i = 1, \dots, n).$$

Let, moreover,

$$\int_a^b f^*(\delta_1, \dots, \delta_n; \rho)(t)\, dt < +\infty \ \ \textit{for} \ \ \rho > 0$$

and there exist a positive constant ρ_0 *such that for any* $\lambda \in]0, 1]$ *every solution of the problem* $(3), (4), (5)$ *admits the estimate*

$$\sum_{i=1}^n \|u_i\|_C \le \rho_0.$$

Then the problem $(1), (2)$ *has at least one positive solution.*

The operator $(\varphi_{0i})_{i=1}^n : C([a, b]; \mathbb{R}_+^n) \to \mathbb{R}_+^n$ is said to be positively homogeneous if for any $i \in \{1, \dots, n\}$, $\lambda > 0$ and $(u_k)_{k=1}^n \in C([a, b]; \mathbb{R}_+^n)$ the equality

$$\varphi_{0i}(\lambda u_1, \dots, \lambda u_n) = \lambda \varphi_{0i}(u_1, \dots, u_n)$$

is satisfied.

Following [1], we introduce

Definition 1. We say that the pair $((p_{ik})_{i,k=1}^n; (\varphi_{0i})_{i=1}^n)$, consisting of the matrix function $(p_{ik})_{i,k=1}^n$ with the Lebesgue integrable components $p_{ik} : [a, b] \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ and the positively homogeneous nondecreasing operator $(\varphi_{0i})_{i=1}^n : C([a, b]; \mathbb{R}_+^n) \to \mathbb{R}_+^n$ belongs to the set $\mathcal{U}(t_1, \dots, t_n)$ if the problem

$$u_i'(t) \operatorname{sgn}(t - t_i) \le \sum_{k=1}^n p_{ik}(t) u_k(t) \ \ (i = 1, \dots, n),$$
$$u_i(t_i) \le \varphi_{0i}(u_1, \dots, u_n) \ \ (i = 1, \dots, n)$$

has no a nonzero, nonnegative solution.

On the basis of Theorem 1, the following theorem can be proved.

Theorem 2. *Let*

$$\varphi_i(u_1, \dots, u_n) \le \varphi_{0i}\big(u_1, \dots, u_n\big) + \gamma \ \ \textit{for} \ \ (u_k)_{k=1}^n \in C([a, b]; \mathbb{R}_+^n) \quad (i = 1, \dots, n)$$

and

$$0 \le \big(f_i(t, x_1, \dots, x_n) - p_i(t) x_i^{\lambda_i}\big) \operatorname{sgn}(t - t_i) \le \\ \le \sum_{k=1}^n p_{ik}(t) x_k \ \ \textit{for} \ \ t \in [a, b] \setminus I_0, \ \ (x_k)_{k=1}^n \in \mathbb{R}_{0+}^n \ \ (i = 1, \dots, n), \quad (6)$$

where I_0 *is a set of zero measure,* γ *is a nonnegative constant,* $\lambda_i < 1$ $(i = 1, \dots, n)$*,* $p_i : [a, b] \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ *are the Lebesgue integrable functions and*

$$\big((p_{ik})_{i,k=1}^n; (\varphi_{0i})_{i=1}^n\big) \in \mathcal{U}(t_1, \dots, t_n).$$

Then the problem $(1), (2)$ *has at least one positive solution.*

The above Theorem 2 and Lemma 5.4 of [1] result in

Corollary 1. *Let*

$$\varphi_i(u_1,\dots,u_n) \le \sum_{k=1}^n \ell_{ik}\|u_k\|_C + \gamma \ \ \textit{for} \ \ (u_k)_{k=1}^n \in C([a,b];\mathbb{R}_+) \quad (i=1,\dots,n),$$

and the inequalities (6) *be fulfilled, where* I_0 *is a set of zero measure,* ℓ_{ik} $(i,k=1,\dots,n)$ *and* γ *are nonnegative constants,* $\lambda_i < 1$ $(i=1,\dots,n)$, $p_i : [a,b] \to \mathbb{R}_{0+}$ *and* $p_{ik} : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$ *are the Lebesgue integrable functions. If, moreover,*

$$r(\Lambda) < 1, \ \ \textit{where} \ \ \Lambda = \Big(\ell_{ik} + \int_a^b p_{ik}(t)\,dt\Big)_{i,k=1}^n,$$

then the problem (1), (2) *has at least one positive solution.*

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Author's address:

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

E-mail: kig@rmi.ge

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K. Mansimov, T. Melikov, and T. Tadumadze

VARIATION FORMULAS OF SOLUTION FOR A CONTROLLED DELAY FUNCTIONAL-DIFFERENTIAL EQUATION TAKING INTO ACCOUNT DELAYS PERTURBATIONS AND THE MIXED INITIAL CONDITION

Abstract. Variation formulas of solution are obtained for a nonlinear controlled delay functional-differential equation with respect to perturbations of initial moment, constant delays, initial vector, initial functions and control function. The effects of delay perturbations and the mixed initial condition are discovered in the variation formulas.

რეზიუმე. სამართი დაგვიანებულ არგუმენტიანი ფუნქციონალურ-დიფერენციალური განტოლებისათვის მიღებულია ამონახსნის ვარიაციის ფორმულები საწყისი მომენტის, მუდმივი დაგვიანებების, საწყისი ვექტორის, საწყისი ფუნქციებისა და მართვის ფუნქციის შეშფოთებების მიმართ. ვარიაციის ფორმულებში გამოვლენილია დაგვიანებების შეშფოთებისა და შერეული საწყისი პირობის ეფექტები.

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Key words and phrases. Controlled delay functional-differential equation, variation formula of solution, effect of delay perturbation, effect of the mixed initial condition.

1. Introduction

In the present paper, variation formulas of solution (variation formulas) are obtained for a nonlinear controlled delay functional-differential equation under perturbations of initial moment, constant delays, initial vector, initial functions and control function. The effects of delays perturbations and the mixed initial condition are discovered in the variation formulas. The mixed initial condition means that at the initial moment, some coordinates of the trajectory do not coincide with the corresponding coordinates of the initial function, whereas the others coincide. The variation formula allows one to construct an approximate solution of the perturbed equation in an analytical form on the one hand, and in the theory of optimal control it plays the basic role in proving the necessary conditions of optimality [1]–[11], on the other. Variation formulas for various classes of functional-differential equations without perturbation of delay are given in [2], [6], [7] and [9]–[13]. Variation formulas for delay functional-differential equations with the continuous and discontinuous initial condition taking into account constant delay perturbation are proved in [14] and [15], respectively. Variation formulas for controlled delay functional-differential equations with the continuous initial condition taking into account constant delay perturbation are proved in [16].

2. Formulation of the Main Results

Let R^n_x be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T denotes transposition; suppose $P \subset R^k_p$, $Z \subset R^m_z$ and $W \subset R^r_u$ are open sets and $O = (P, Z)^T = \{x = (p, z)^T \in R^n_x : \ p \in P, z \in Z\}$, with $k+m = n$. Let the n-dimensional function $f(t, x, p, z, u)$ satisfy the following conditions: for almost all $t \in I = [a, b]$, the function $f(t, \cdot) : O \times P \times Z \times W \to R^n_x$ is continuously differentiable; for any $(x, p, z, u) \in O \times P \times Z \times W$, the functions $f(t, x, p, z, u), f_x(\cdot), f_p(\cdot), f_z(\cdot) f_u(\cdot)$ are measurable on I; for arbitrary compacts $K \subset O$, $U \subset W$ there exists a function $m_{K,U}(\cdot) \in L(I, [0, \infty))$, such that for any $x \in K$, $(p, z)^T \in K$, $u \in U$ and for almost all $t \in I$ the inequality

$$|f(t, x, p, z, u)| + |f_x(\cdot)| + |f_p(\cdot)| + |f_z(\cdot)| + |f_u(\cdot)| \le m_{K,U}(t)$$

is fulfilled.

Let $0 < \tau_1 < \tau_2$, $0 < \sigma_1 < \sigma_2$ be the given numbers and $E_\varphi = E_\varphi(I_1, R^k_p)$ be the space of continuous functions $\varphi : I_1 \to R^k_p$, where $I_1 = [\widehat{\tau}, b], \widehat{\tau} = a - \max\{\tau_2, \sigma_2\}$. Further,

$$\Phi = \big\{\varphi \in E_\varphi : \ \varphi(t) \in P\big\} \ \text{ and } \ G = \big\{g \in E_g = E_g(I_1, R^m_z) : \ g(t) \in Z\big\}$$

are the sets of initial functions. Let E_u be the space of bounded measurable functions $u : I \to R^r_u$ and $\Omega = \{u \in E_u : \ u(t) \in W, \ t \in I, \ \mathrm{cl}\, u(I) \subset W\}$ be a set of control functions, where $u(I) = \{u(t) : \ t \in I\}$ and $\mathrm{cl}\, u(I)$ is the closure of the set $u(I)$.

To any element

$$\mu = (t_0, \tau, \sigma, p_0, \varphi, g, u) \in \Lambda = (a, b) \times (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times P \times \Phi \times G \times \Omega,$$

we assign the controlled delay functional-differential equation

$$\dot{x}(t) = (\dot{p}(t), \dot{z}(t))^T = f\big(t, x(t), p(t - \tau), z(t - \sigma), u(t)\big) \tag{2.1}$$

with a mixed initial condition

$$x(t) = (\varphi(t), g(t))^T, \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = (p_0, g(t_0))^T. \tag{2.2}$$

The condition (2.2) is said to be a mixed initial condition; it consists of two parts: the first part is $p(t) = \varphi(t)$, $t \in [\widehat{\tau}, t_0)$, $p(t_0) = p_0$, the discontinuous part, since generally $p(t_0) \neq \varphi(t_0)$; the second part is $z(t) = g(t)$, $t \in [\widehat{\tau}, t_0]$, the continuous part, since always $z(t_0) = g(t_0)$.

Definition 2.1. Let $\mu = (t_0, \tau, \sigma, p_0, \varphi, g, u) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b)$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies the condition (2.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (2.1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \sigma_0, p_{00}, \varphi_0, g_0, u_0) \in \Lambda$ be a fixed element. In the space $E_\mu = R^1_{t_0} \times R^1_\tau \times R^1_\sigma \times R^k_p \times E_\varphi \times E_g \times E_u$ we introduce the set of variations

$$V = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta\sigma, \delta p_0, \delta\varphi, \delta g, \delta u) \in E_\mu - \mu_0 : \ |\delta t_0| \le \alpha,$$

$$|\delta\tau| \le \alpha, \ |\delta\sigma| \le \alpha, \ |\delta p_0| \le \alpha, \ \delta\varphi = \sum_{i=1}^{\nu} \lambda_i \delta\varphi_i,$$

$$\delta g = \sum_{i=1}^{\nu} \lambda_i \delta g_i, \ \delta u = \sum_{i=1}^{\nu} \lambda_i \delta u_i, \ |\lambda_i| \le \alpha, \ i = \overline{1, \nu} \Big\},$$

where $\delta\varphi_i \in E_\varphi - \varphi_0$, $\delta g_i \in E_g - g_0$, $\delta u_i \in E_u - u_0$, $i = \overline{1, \nu}$, are the fixed functions; $\alpha > 0$ is a fixed number.

Let $x_0(t) = (p_0(t), z_0(t))^T$ be the solution corresponding to the element μ_0 and defined on the interval $[\widehat{\tau}, t_{10}]$, with $t_{10} < b$. There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$ we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, to this element there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ (see Theorem 5.3 in [17, p. 111]).

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \ \ (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times [0, \varepsilon_1] \times V.$$

Theorem 2.1. *Let the following conditions hold:*

2.1. $t_{00} + \tau_0 < t_{10}$;

2.2. *the functions $\varphi_0(t)$, $g_0(t)$, $t \in I_1$, are absolutely continuous and $\dot\varphi_0(t)$, $\dot g_0(t)$ are bounded; there exist compact sets $K_0 \subset O$ and $U_0 \subset W$ containing neighborhoods of sets $(\varphi_0(I_1), g_0(I_1))^T \cup x_0([t_{00}, t_{10}])$ and $\operatorname{cl} u_0(I)$, respectively, such that the function $f(t, x, p, z, u)$, $(t, x) \in I \times K_0$, $(p, z)^T \in K_0$, $u \in U_0$, is bounded;*

2.3. *there exist the limits*

$$\lim_{t \to t_{00}-} \dot g_0(t) = \dot g_0^-,$$

$$\lim_{w \to w_0} f(w, u_0(t)) = f_0^-, \ \ w \in (t_{00} - \tau_0, t_{00}] \times O \times P \times Z,$$

$$\lim_{(w_1, w_2) \to (w_{01}, w_{02})} \big[f(w_1, u_0(t)) - f(w_2, u_0(t)) \big] = f_{01}^-,$$

$$w_1, w_2 \in (t_{00}, t_{00} + \tau_0] \times O \times P \times Z,$$

where

$$\begin{aligned}
w &= (t, x, p, z),\\
w_0 &= \big(t_{00}, x_{00}, \varphi_0(t_{00}-\tau_0), g_0(t_{00}-\sigma_0)\big),\\
x_{00} &= (p_{00}, g_0(t_{00}))^T,\\
w_{01} &= \big(t_{00}+\tau_0, x_0(t_{00}+\tau_0), p_{00}, z_0(t_{00}+\tau_0-\sigma_0)\big),\\
w_{02} &= \big(t_{00}+\tau_0, x_0(t_{00}+\tau_0), \varphi_0(t_{00}), z_0(t_{00}+\tau_0-\sigma_0)\big).
\end{aligned}$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that*

$$\Delta x(t;\varepsilon\delta\mu) = \varepsilon\delta x(t;\delta\mu) + o(t;\varepsilon\delta\mu) \tag{2.3}$$

for arbitrary

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2, t_{10}+\delta_2]\times[0,\varepsilon_2]\times\big\{\delta\mu\in V : \delta t_0 \le 0,\ \delta\tau\le 0,\ \delta\sigma\le 0\big\},$$

where

$$\begin{aligned}
\delta x(t;\delta\mu) &= \Big\{Y(t_{00};t)\big[(\Theta_{k\times 1}, \dot g_0^-)^T - f_0^-\big] - Y(t_{00}+\tau_0;t)f_{01}^-\Big\}\delta t_0 -\\
&\qquad - Y(t_{00}+\tau_0;t)f_{01}^-\delta\tau + \beta(t;\varepsilon\delta\mu), \qquad (2.4)\\
\beta(t;\varepsilon\delta\mu) &= Y(t_{00};t)(\delta p_0, \delta g(t_{00}))^T -\\
&\qquad - \Big\{\int_{t_{00}}^{t} Y(\xi;t)f_p[\xi]\dot p_0(\xi-\tau_0)\,d\xi\Big\}\delta\tau -\\
&\qquad - \Big\{\int_{t_{00}}^{t} Y(\xi;t)f_z[\xi]\dot z_0(\xi-\sigma_0)\,d\xi\Big\}\delta\sigma +\\
&\qquad + \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_p[\xi+\tau_0]\delta\varphi(\xi)\,d\xi +\\
&\qquad + \int_{t_{00}-\sigma_0}^{t_{00}} Y(\xi+\sigma_0;t)f_z[\xi+\sigma_0]\delta g(\xi)\,d\xi +\\
&\qquad + \int_{t_{00}}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi; \qquad (2.5)
\end{aligned}$$

$$\lim_{\varepsilon\to 0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon} = 0$$

uniformly for

$$(t,\delta\mu)\in[t_{10}-\delta_2, t_{10}+\delta_2]\times\big\{\delta\mu\in V:\ \delta t_0\le 0,\ \delta\tau\le 0,\ \delta\sigma\le 0\big\};$$

$\Theta_{k\times 1}$ is the $k\times 1$ zero matrix, $Y(s;t)$ is the $n\times n$ matrix function satisfying on the interval $[t_{00},t]$ the equation

$$Y_\xi(\xi;t) = -Y(\xi;t)f_x[\xi] - \Big(Y(\xi+\tau_0;t)f_p[\xi+\tau_0], Y(\xi+\sigma_0;t)f_z[\xi+\sigma_0]\Big),$$

and the condition

$$Y(\xi;t) = \begin{cases} H_{n\times n} & \textit{for } \xi = t, \\ \Theta_{n\times n} & \textit{for } \xi > t. \end{cases}$$

Here, $H_{n\times n}$ is the $n\times n$ identity matrix,

$$f_x[\xi] = f_x\Big(\xi, x_0(\xi), p_0(\xi-\tau_0), z_0(\xi-\sigma_0), u_0(\xi)\Big), \quad \dot p_0(\xi-\tau_0) = \dot p_0(s)_{|s=\xi-\tau_0},$$

under $\dot p_0(s)$ is assumed derivative of the function $p_0(s)$ on the set $[\widehat\tau, t_{00})\cup(t_{00}, t_{10}+\delta_2]$.

Some comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t)$ on the interval $[t_{10}-\delta_2, t_{10}+\delta_2]$ and the expression (2.4) is called the variation formula.

c 1) Theorem 2.1 corresponds to the case where the variations at the points t_{00},τ_0,σ_0 are performed simultaneously on the left.

c 2) The addend

$$-\Big\{Y(t_{00}+\tau_0;t)f_{01}^- + \int_{t_{00}}^t Y(\xi;t)f_p[\xi]\dot p_0(\xi-\tau_0)\,d\xi\Big\}\delta\tau - \\ -\Big\{\int_{t_{00}}^t Y(\xi;t)f_z[\xi]\dot z_0(\xi-\sigma_0)\,d\xi\Big\}\delta\sigma$$

is the effect of perturbations of the delays τ_0 and σ_0 (see (2.4) and (2.5)).

c 3) The expression

$$Y(t_{00};t)(\delta p_0, \delta g(t_{00}))^T + \\ + \Big\{Y(t_{00};t)\big[(\Theta_{k\times 1}, \dot g_0^-)^T - f_0^-\big] - Y(t_{00}+\tau_0;t)f_{01}^-\Big\}\delta t_0$$

is the effect of the mixed initial condition (2.2) under perturbations of initial moment t_{00}, initial vector p_{00} and function $g_0(t)$.

c 4) The expression

$$\int\limits_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_p[\xi+\tau_0]\delta\varphi(\xi)\,d\xi+$$

$$+\int\limits_{t_{00}-\sigma_0}^{t_{00}} Y(\xi+\sigma_0;t)f_z[\xi+\sigma_0]\delta g(\xi)\,d\xi+\int\limits_{t_{00}}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$

in the formula (2.5) is the effect of perturbations of the initial functions $\varphi_0(t)$, $g_0(t)$ and the control function $u_0(t)$.

c 5) The variation formula allows one to obtain an approximate solution of the perturbed functional-differential equation

$$\dot{x}(t)=f\Big(t,x(t),p(t-\tau_0-\varepsilon\delta\tau),z(t-\sigma_0-\varepsilon\delta\sigma),u_0(t)+\varepsilon\delta u(t)\Big)$$

with the perturbed initial condition

$$x(t)=\big(\varphi_0(t)+\varepsilon\delta\varphi(t),g_0(t)+\varepsilon\delta g(t)\big)^T,\ \ t\in[\widehat{\tau},t_{00}+\varepsilon\delta t_0),$$
$$x(t_{00}+\varepsilon\delta t_0)=\big(p_{00}+\varepsilon\delta p_0,g_0(t_{00})+\varepsilon\delta g(t_{00})\big)^T.$$

In fact, for a sufficiently small $\varepsilon\in(0,\varepsilon_2]$ from (2.3) it follows that

$$x(t;\mu_0+\varepsilon\delta\mu)\approx x_0(t)+\varepsilon\delta x(t;\delta\mu).$$

Theorem 2.2. *Let the conditions* 2.1 *and* 2.2 *of Theorem* 2.1 *hold. Moreover, there exist the limits*

$$\lim_{t\to t_{00}+}\dot{g}_0(t)=\dot{g}_0^+,$$
$$\lim_{w\to w_0}f(w,u_0(t))=f_0^+,\ \ w\in[t_{00},t_{10})\times O\times P\times Z,$$
$$\lim_{(w_1,w_2)\to(w_{01},w_{02})}\big[f(w_1,u_0(t))-f_0(w_2,u_0(t))\big]=f_{01}^+,$$
$$w_1,w_2\in[t_{00}+\tau_0,t_{10})\times O\times P\times Z.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that for arbitrary*

$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times\{\delta\mu\in V:\ \delta t_0\geq 0,\ \delta\tau\geq 0,\ \delta\sigma\geq 0\}$

the formula (2.3) *holds, where*

$$\delta x(t;\delta\mu)=\Big\{Y(t_{00};t)\big[(\Theta_{k\times1},\dot{g}_0^+)^T-f_0^+\big]-Y(t_{00}+\tau_0;t)f_{01}^+\Big\}\delta t_0-$$
$$-Y(t_{00}+\tau_0;t)f_{01}^+\delta\tau+\beta(t;\varepsilon\delta\mu).$$

Theorem 2.2 corresponds to the case where the variations at the points t_{00}, τ_0, σ_0 are performed simultaneously on the right.

Theorem 2.3. *Let the conditions of Theorems* 2.1 *and* 2.2 *hold. Moreover,*

$$(\Theta_{k\times 1},\dot g_0^-)^T - f_0^- = (\Theta_{k\times 1},\dot g_0^+)^T - f_0^+ =: \widehat{f}_0, f_{01}^- = f_{01}^+ =: \widehat{f}_{01}.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V$ *the formula* (2.3) *holds, where*

$$\delta x(t;\delta\mu)=\Big\{Y(t_{00};t)\widehat{f}_0 - Y(t_{00}+\tau_0;t)\widehat{f}_{01}\Big\}\delta t_0 - \\ - Y(t_{00}+\tau_0;t)\widehat{f}_{01}\delta\tau + \beta(t;\varepsilon\delta\mu).$$

Theorem 2.3 corresponds to the case where at the points t_{00}, τ_0, σ_0 the two-sided variations are simultaneously performed. Theorems 2.1–2.3 are proved by the method given in [10]. If $t_{00}+\tau_0>t_{10}$, then Theorems 2.1–2.3 are also valid. In this case the number δ_2 is so small that $t_{00}+\tau_0>t_{10}+\delta_2$, therefore in the variation formulas we have $Y(t_{00}+\tau_0;t)=\Theta_{n\times n}$, $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$. If $t_{00}+\tau_0=t_{10}$, then Theorem 2.1 is valid on the interval $[t_{10},t_{10}+\delta_2]$ and Theorem 2.2 is valid on the interval $[t_{10}-\delta_2,t_{10}]$.

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Author's address:

K. Mansimov

1. Baku State University, 23 Z. Khalilov St., Baku-Az 1148, Azerbaijan.

2. Institute of Cybernetics of NAS Azerbaijan, 9 F. Agaev St., Baku-Az 1141, Azerbaijan.

E-mail: `mansimov@front.ru; mansimov kamil@mail.ru`

T. Melikov

Azerbaijan Technological University, 103, Sh. I. Khatai Ave., Ganja-Az 2011, Azerbaijan

E-mail: `t.melikov@rambler.ru`

T. Tadumadze

I. Javakhishvili Tbilisi State University, Department of Mathematics & I.Vekua Institute of Applied Mathematics, 2 University St., Tbilisi 0186, Georgia.

E-mail: `tamaztad@yahoo.com; tamaz.tadumadze@tsu.ge`

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Nino Partsvania

ON TWO–POINT BOUNDARY VALUE PROBLEMS FOR TWO–DIMENSIONAL NONLINEAR DIFFERENTIAL SYSTEMS WITH STRONG SINGULARITIES

Abstract. For two-dimensional nonlinear differential systems with strong singularities with respect to a time variable, unimprovable sufficient conditions for the solvability and unique solvability of two-point boundary value problems are established.

რეზიუმე. ორგანზომილებიანი არაწრფივი დიფერენციალური სისტემებისათვის ძლიერი სინგულარობებით დროითი ცვლადის მიმართ დადგენილია ორწერტილოვან სასაზღვრო ამოცანათა ამოხსნადობისა და ცალსახად ამოხსნადობის არაგაუმჯობესებადი საკმარისი პირობები.

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Let $-\infty < a < b < +\infty$, and let $f_i :]a,b[\times R \to R$ $(i=1,2)$ be continuous functions. In the open interval $]a,b[$, we consider the two-dimensional nonlinear differential system

$$\frac{du_1}{dt} = f_1(t,u_2), \quad \frac{du_2}{dt} = f_2(t,u_1) \tag{1}$$

with boundary conditions of one of the following two types:

$$\lim_{t\to a} u_1(t) = 0, \quad \lim_{t\to b} u_1(t) = 0, \tag{2_1}$$

and

$$\lim_{t\to a} u_1(t) = 0, \quad \lim_{t\to b} u_2(t) = 0. \tag{2_2}$$

A vector function (u_1,u_2) with continuously differentiable components $u_i :]a,b[\to R$ $(i=1,2)$ is said to be a solution of the system (1) if it satisfies that system at each point of $]a,b[$.

A solution of the system (1), satisfying the boundary conditions (2_1) (the boundary conditions (2_2)), is said to be a solution of the problem $(1),(2_1)$ (a solution of the problem $(1),(2_2)$).

A solution of the problem $(1),(2_1)$ (of the problem $(1),(2_2)$), satisfying the condition

$$\int\limits_a^b u_2^2(t)\,dt < +\infty, \tag{3}$$

is said to be a solution of the problem $(1), (2_1), (3)$ (a solution of the problem $(1), (2_2), (3)$).

Let

$$[f_2(t,x)]_- = \frac{1}{2}\left(|f_2(t,x)| - f_2(t,x)\right).$$

Theorems below on the solvability and unique solvability of the problem $(1), (2_1), (3)$ cover the case, where

$$\int_a^{t_0} (t-a)[f_2(t,x)]_- dt =$$

$$= \int_{t_0}^b (b-t)[f_2(t,x)]_- dt = +\infty \quad \text{for} \;\; t_0 \in]a,b[, \quad x \neq 0. \tag{4_1}$$

Analogous theorems for the problem $(1), (2_2), (3)$ cover the case, where

$$\int_a^b (t-a)[f_2(t,x)]_- dt = +\infty \quad \text{for} \;\; x \neq 0. \tag{4_2}$$

In the case, where the condition (4_1) (the condition (4_2)) is satisfied, we say that the system (1) has strong singularities at the points a and b (at the point a). In both cases, roughly speaking, the orders of singularity of the function f_2 with respect to the time variable are no less than 2, i.e., no less than the dimension of the considered differential system. Just because of that reason these singularities are said to be strong in the Agarwal–Kiguradze sence [1]. The above-mentioned cases essentially differ from so-called weak singular cases, where for arbitrary $t_0 \in]a,b[$ and $x \neq 0$, the following conditions

$$\int_a^{t_0} [f_2(t,x)]_- dt = \int_{t_0}^b [f_2(t,x)]_- dt = +\infty \quad \text{or} \quad \int_a^b [f_2(t,x)]_- dt = +\infty$$

hold but

$$\int_a^b (t-a)(b-t)[f_2(t,x)]_- dt < +\infty.$$

In the case of strong singularity, in contrast to the case of weak singularity, the problem $(1), (2_1), (3)$, generally speaking, is not equivalent to the problem $(1), (2_1)$. Analogously, the problem $(1), (2_2), (3)$ is not equivalent to the problem $(1), (2_2)$. To convince ourselves that this is so, let us consider the case, where the system (1) has the form

$$\frac{du_1}{dt} = u_2, \quad \frac{du_2}{dt} = -\frac{\mu}{(t-a)^2}\, u_1. \tag{$1'$}$$

If μ satisfies the inequality

$$0 < \mu < \frac{1}{4},$$

then the problem $(1'), (2_1)$,(3) has only the trivial solution whereas the problem $(1'), (2_1)$ has infinite set of solutions

$$u_1(t) = c\big[(t-a)^{\lambda_1} - (b-a)^{\lambda_1-\lambda_2}(t-a)^{\lambda_2}\big], \quad u_2(t) =$$
$$= c\big[\lambda_1(t-a)^{\lambda_1-1} - \lambda_2(b-a)^{\lambda_1-\lambda_2}(t-a)^{\lambda_2-1}\big], \quad c \in R,$$

where

$$\lambda_1 = \frac{1+\sqrt{1-4\mu}}{2}, \quad \lambda_2 = \frac{1-\sqrt{1-4\mu}}{2}.$$

Analogously, the problem $(1'), (2_2), (3)$ has only the trivial solution, while the problem $(1'), (2_2)$ has infinite set of solutions

$$u_1(t) = c\Big[(t-a)^{\lambda_1} - \frac{\lambda_1}{\lambda_2}\,(b-a)^{\lambda_1-\lambda_2}(t-a)^{\lambda_2}\Big], \quad u_2(t) =$$
$$= c\lambda_1\big[(t-a)^{\lambda_1-1} - (b-a)^{\lambda_1-\lambda_2}(t-a)^{\lambda_2-1}\big], \quad c \in R.$$

For the weakly singular system (1) and its various particular cases, unimprovable in a certain sense sufficient conditions for the solvability and well-posedness of problems of the type $(1), (2_1)$ and $(1), (2_2)$ are contained in [2]–[7], [11]–[14], [17]–[19]. Two-point boundary value problems for higher order differential equations with strong singularities are investigated in detail by I. Kiguradze and R. P. Agarwal (see, [1], [8]–[10]). Conditions, guaranteeing the existence of extremal solutions of two-point boundary value problems for second order nonlinear differential equations with strong singularities, are contained in [16]. The Agarwal–Kiguradze type theorems for two-dimensional linear differential systems are given in [15]. Below we give analogous results for the problems $(1), (2_1), (3)$ and $(1), (2_2), (3)$.

First we consider the problem $(1), (2_1), (3)$. The following theorems are valid.

Theorem 1. *Let in the domain* $]a,b[\times R$ *the inequalities*

$$\delta|x| \le \big[f_1(t,x) - f_1(t,0)\big]\operatorname{sgn} x \le \ell_0|x|, \tag{5}$$

$$\big[f_2(t,x) - f_2(t,0)\big]\operatorname{sgn} x \ge -\ell\bigg(\frac{1}{(t-a)^2} + \frac{1}{(b-t)^2}\bigg)|x| \tag{6}$$

be fulfilled, where δ, ℓ_0, *and* ℓ *are positive constants such that*

$$4\ell\ell_0 < 1. \tag{7}$$

If, moreover,

$$\int_a^b f_1^2(t,0)dt < 0, \quad \int_a^b (t-a)^{1/2}(b-t)^{1/2}|f_2(t,0)|dt < +\infty, \tag{8}$$

then the problem $(1), (2_1), (3)$ *has at least one solution.*

Theorem 2. *Let in the domain* $]a,b[\times R$ *the conditions*

$$\delta|x-y| \le \big[f_1(t,x)-f_1(t,y)\big]\operatorname{sgn}(x-y) \le \ell_0|x-y|, \tag{9}$$

$$\big[f_2(t,x)-f_2(t,y)\big]\operatorname{sgn}(x-y) \ge -\ell\bigg(\frac{1}{(t-a)^2}+\frac{1}{(b-t)^2}\bigg)|x-y| \tag{10}$$

be fulfilled, where δ, ℓ_0, *and* ℓ *are positive constants, satisfying the inequality* (7)*. If, moreover, the condition* (8) *holds, then the problem* $(1), (2_1), (3)$ *has one and only one solution.*

Note that the condition (7) in Theorems 1 and 2 is unimprovable in the sense that it cannot be replaced by the non-strict inequality

$$4\ell\ell_0 \le 1. \tag{7$'$}$$

Indeed, consider the case, where

$$f_1(t,x)=x, \quad f_2(t,x)=-\frac{1}{4(t-a)^2}\,x+9.$$

Then the conditions (5), (6), (8)–(10) are satisfied, where $\delta=\ell_0=1$ and $\ell=\frac{1}{4}$. Consequently, all the conditions of Theorems 1 and 2 are fulfilled except the condition (7), instead of which the inequality (7$'$) is satisfied. Nevertheless, in the considered case the problem $(1), (2_1), (3)$ has no solution. The fact is that in that case an arbitrary solution of the system (1) admits the representation

$$u_1(t)=c_1(t-a)^{1/2}+c_2(t-a)^{1/2}\ln(t-a)+4(t-a)^2,$$

$$u_2(t)=\frac{1}{2}\,c_1(t-a)^{-1/2}+c_2(t-a)^{-1/2}\bigg(\frac{1}{2}\,\ln(t-a)+1\bigg)+8(t-a),$$

where c_1 and c_2 are arbitrary real numbers, and consequently,

$$\int\limits_a^b u_2^2(t)dt=+\infty \quad \text{for } |c_1|+|c_2|\ne 0.$$

Consider now the problem $(1), (2_2), (3)$. Suppose

$$f_2^*(t,x)=\max\big\{|f_2(t,y)|:\ |y|\le x\big\} \quad \text{for } a<t<b, \ x>0.$$

Theorem 3. *Let in the domain* $]a,b[\times R$ *the inequalities* (5) *and*

$$\big[f_2(t,x)-f_2(t,0)\big]\operatorname{sgn} x \ge -\frac{\ell}{(t-a)^2}\,|x|$$

be fulfilled, where δ*,* ℓ_0*, and* ℓ *are positive constants, satisfying the condition* (7)*. If, moreover,*

$$\int\limits_a^b f_1^2(s,0)ds < +\infty, \quad \int\limits_a^b (s-a)^{1/2}|f_2(s,0)|ds <$$

$$< +\infty, \quad \int\limits_t^b f_2^*(s,x)ds < +\infty \quad \text{for } a<t<b, \ \ x>0, \tag{11}$$

then the problem (1), (2_2), (3) *has at least one solution.*

Theorem 4. *Let in the domain* $]a,b[\times R$ *the conditions* (5) *and*

$$\big[f_2(t,x)-f_2(t,y)\big]\operatorname{sgn}(x-y) \geq -\frac{\ell}{(t-a)^2}\,|x-y|$$

be fulfilled, where δ*,* ℓ_0*, and* ℓ *are positive constants, satisfying the inequality* (7)*. If, moreover, the conditions* (11) *hold, then the problem* (1), (2_2), (3) *has one and only one solution.*

Note that the condition (7) in Theorems 3 and 4 is unimprovable and it cannot be replaced by the condition $(7')$.

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Author's addresses:

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. International Black Sea University, 2 David Agmashenebeli Alley 13km, Tbilisi 0131, Georgia.

E-mail: ninopa@rmi.ge

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B. Půža and Z. Sokhadze

ON THE WEIGHTED INITIAL PROBLEM FOR SINGULAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. For singular functional differential systems, sufficient conditions for solvability and well-posedness of the weighted initial problem are established.

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In a finite interval $]a,b[$ we consider the functional differential system

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1}$$

with the weighted initial condition

$$\limsup_{t\to a} \big\|\phi^{-1}(t)x(t)\big\| < +\infty. \tag{2}$$

Here, $f : C([a,b];\mathbb{R}^n) \to L_{loc}(]a,b];\mathbb{R}^n)$ is a singular operator satisfying the local Carathéorory conditions, $\phi(t) = \operatorname{diag}\big(\varphi_1(t),\dots,\varphi_n(t)\big)$, and $\varphi_i : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$ are continuous non-decreasing functions such that $\varphi_i(a) = 0$, $\varphi_i(t) > 0$ for $a < t \le b$ $(i=1,\dots,n)$.

The initial problem for the singular system (1) has been thoroughly investigated in the cases, in which f is either the Nemytski's operator [1]–[6], or the evolutionary operator [7]–[9]. The weighted initial problem for higher order singular functional differential equations is studied in [11]–[14]. As for the weighted singular problem (1), (2), it is not studied well enough. In the present paper unimprovable in a certain sense conditions are given which, respectively, guarantee solvability and well-posedness of this problem.

Throughout the paper, the use will be made of the following notation.

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$.

$\mathbb{R}^n$ is the space of n-dimensional real column-vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i|.$$

If $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, then

$$[x]_+ = \Big(\frac{x_i + |x_i|}{2}\Big)_{i=1}^n.$$

$r(X)$ is the spectral radius of the $n \times n$ matrix X, and X^{-1} is the inverse to X matrix.

$\mathrm{diag}(x_1, \dots, x_n)$ is the diagonal $n \times n$-matrix with diagonal elements $x_1, \dots, x_n$.

If $X = \mathrm{diag}(x_1, \dots, x_n)$, then $\mathrm{Sgn}(X) = \big(\mathrm{sgn}(x_1), \dots, \mathrm{sgn}(x_n)\big)$.

$\mathbb{R}_+^n$ and $\mathbb{R}_+^{n\times n}$ are the sets of n-dimensional vectors and $n \times n$-matrices with nonnegative components.

$C([a,b];\mathbb{R}^n)$ is the space of continuous vector functions $x : [a,b] \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\Big\{\|x(t)\| : \; a \le t \le b\Big\}.$$

$C_\phi([a,b];\mathbb{R}^n)$ is the space of continuous vector functions $x : [a,b] \to \mathbb{R}^n$, satisfying the condition (2), with the norm

$$\|x\|_{C_\phi} = \sup\Big\{\big\|\phi^{-1}(t)x(t)\big\| : \; a < t \le b\Big\}.$$

If $x = (x_i)_{i=1}^n \in C_\phi([a,b];\mathbb{R}^n)$, then

$$|x|_{C_\phi} = \big(\|x_i\|_{C_{\varphi_i}}\big)_{i=1}^n.$$

$L([a,b];\mathbb{R}^n)$ is the space of vector functions with Lebesgue integrable on $[a,b]$ components.

$L_{loc}(]a,b];\mathbb{R}^n)$ is the space of vector functions whose components are Lebesgue integrable on $[a+\varepsilon, b]$ for an arbitrarily small $\varepsilon > 0$.

$K_{loc}(]a,b] \times \mathbb{R}^k;\mathbb{R}^m)$ and $K_{loc}(C([a,b];\mathbb{R}^k);L_{loc}(]a,b];\mathbb{R}^m))$ are the sets of vector functions $g :]a,b] \times \mathbb{R}^k \to \mathbb{R}^m$ and operators $f : C([a,b];\mathbb{R}^k) \to L_{loc}(]a,b];\mathbb{R}^m)$, satisfying the local Carathéodory conditions (see [15]).

An important particular case of the functional differential system (1) is the differential system with a deviating argument

$$\frac{dx(t)}{dt} = g\big(t, x(t), x(\tau(t))\big). \tag{3}$$

Along with the problem (1), (2), we consider the problem (3), (2). Everywhere below, when the question concerns these problems, it will be assumed that

$$f \in K_{loc}\big(C([a,b];\mathbb{R}^n); L_{loc}(]a,b];\mathbb{R}^n)\big), \quad g \in K_{loc}(]a,b] \times \mathbb{R}^{2n};\mathbb{R}^n),$$

and $\tau : [a,b] \to [a,b]$ is a measurable function.

We are mainly interested in the case, where the systems (1) and (3) are singular, i.e., in the case in which

$$\int_a^b f_\rho^*(t)\,dt = +\infty \;\text{ and }\; \int_a^b g_\rho^*(t)\,dt = +\infty \;\text{ for }\; \rho > 0,$$

where

$$f_\rho^*(t) = \sup\Big\{\big\|f(x)(t)\big\| : \ \|x\|_C \le \rho\Big\},$$

$$g_\rho^*(t) = \max\Big\{\big\|g(t,x,y)\big\| : \ \|x\| + \|y\| \le \rho\Big\}.$$

For an arbitrary positive number δ, we put

$$\chi(t,\delta,\lambda) = \begin{cases} 0 & \text{for } \ a \le t < a+\delta \\ \lambda & \text{for } \ t > a+\delta \end{cases},$$

and consider the auxiliary initial problem

$$\frac{dx(t)}{dt} = \chi(t,\delta,\lambda) f(x)(t), \tag{4}$$

$$x(a) = 0, \tag{5}$$

depending on the parameters $\lambda \in]0,1]$ and $\delta > 0$.

On the basis of Corollary 2 in [16], the following theorem can be proved.

Theorem 1. *Let there exist a positive number ρ_0 such that for arbitrary $\lambda \in]0,1]$ and $\delta > 0$ every solution x of the problem $(4),(5)$ admits the estimate*

$$\|x\|_{C_\phi} \le \rho_0.$$

Then the problem $(1),(2)$ has at least one solution.

This theorem allows one to get efficient sufficient conditions for the solvability of the problems $(1),(2)$ and $(3),(2)$. In particular, the following propositions are valid.

Theorem 2. *Let there exist a matrix $\mathcal{P} \in \mathbb{R}_+^{n\times n}$ and a vector function $q : \mathbb{R}_+ \to \mathbb{R}_+^n$ such that*

$$r(\mathcal{P}) < 1, \quad \lim_{\rho\to+\infty} \frac{\|q(\rho)\|}{\rho} = 0, \tag{6}$$

and for an arbitrary vector function $x \in C_\phi([a,b];\mathbb{R}^n)$ on the interval $[a,b]$ the inequality

$$\int_a^t \Big[\operatorname{sgn}(x(s)) f(x)(s)\Big]_+ ds \le \phi(t)\Big(\mathcal{P}|x|_{C_\phi} + q\big(\|x\|_{C_\phi}\big)\Big)$$

is fulfilled. Then the problem $(1),(2)$ has at least one solution.

Corollary 1. *Let the functions φ_i $(i = 1,\dots,n)$ be absolutely continuous and let there exist a set of zero measure $I_0 \subset [a,b]$, matrices $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k = 1,2)$ and a vector function $q : \mathbb{R}_+ \to \mathbb{R}_+^n$ with non-decreasing components such that on the set $([a,b]\setminus I_0)\times\mathbb{R}^{2n}$ the inequality*

$$\operatorname{Sgn}(x) g(t,x,y) \le \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x| + \mathcal{P}_2\phi^{-1}(\tau(t))|y|\Big) + \\ + \phi'(t) q\Big(\big\|\phi^{-1}(t)|x| + \phi^{-1}(\tau(t))|y|\,\big\|\Big)$$

is fulfilled. If, moreover, the conditions (6) *are fulfilled, where* $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$*, then the problem* (3), (2) *has at least one solution.*

Remark 1. In Theorem 2 and Corollary 1, the condition $r(\mathcal{P}) < 1$ is unimprovable and it cannot be replaced by the condition $r(\mathcal{P}) \leq 1$. The validity of that fact follows directly from the theorem below.

Theorem 3. *Let the functions* φ_i $(i = 1, \dots, n)$ *be absolutely continuous and let there exist a set of zero measure* $I_0 \subset [a, b]$*, matrices* $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2)$ *and a vector* $q_0 = (q_{0i})_{i=1}^n$ *with positive components* q_{0i} $(i = 1, \dots, n)$ *such that on the set* $([a, b] \setminus I_0) \times \mathbb{R}^{2n}$ *the inequality*

$$g(t, x, y) \geq \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x| + \mathcal{P}_2\phi^{-1}(\tau(t))|y| + q_0\Big)$$

is fulfilled. If, moreover, $r(\mathcal{P}_1 + \mathcal{P}_2) \geq 1$*, then the problem* (3), (2) *has no solution.*

Along with the problem (1), (2), we consider the perturbed problem

$$\frac{dy(t)}{dt} = f(y)(t) + h(t), \tag{7}$$

$$\limsup_{t\to a} \big\|\phi^{-1}(t)y(t)\big\| < +\infty, \tag{8}$$

and introduce the following

Definition. The problem (1), (2) is called well-posed if there exists a positive number ρ such that for an arbitrary function $h \in L([a, b]; \mathbb{R}^n)$, satisfying the condition

$$\nu_\phi(h) = \sup\left\{\left\|\phi^{-1}(t)\int_a^t |h(s)|\, ds\right\| :\ a < t \leq b\right\} < +\infty,$$

the problem (7), (8) is uniquely solvable and its solution admits the estimate

$$\|y - x\|_{C_\phi} \leq \rho\nu_\phi(h),$$

where x is a solution of the problem (1), (2).

Theorem 4. *Let there exist a matrix* $\mathcal{P} \in \mathbb{R}_+^{n\times n}$ *such that* $r(\mathcal{P}) < 1$*, and for arbitrary vector functions* x *and* $y \in C_\phi([a, b]; \mathbb{R}^n)$ *in the interval* $[a, b]$ *the inequality*

$$\int_a^t \Big[\operatorname{sgn}(y(s))\big(f(x + y)(s) - f(x)(s)\big)\Big]_+ ds \leq \phi(t)\mathcal{P}|y|_{C_\phi}$$

is fulfilled. If, moreover,

$$\sup\left\{\left\|\phi^{-1}(t)\int_a^t \big|f(0)(s)\big|\, ds\right\| :\ a < t \leq b\right\} < +\infty,$$

then the problem (1), (2) *is well-posed.*

Corollary 2. *Let the functions* φ_i $(i = 1, \dots, n)$ *be absolutely continuous and let there exist a set of zero measure* $I_0 \subset [a, b]$ *and matrices* $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2)$ *such that* $r(\mathcal{P}_1 + \mathcal{P}_2) < 1$*, and for any* $t \in [a, b] \setminus I_0$*,* x*,* $\overline{x}$*,* y *and* $\overline{y} \in \mathbb{R}^n$ *the inequality*

$$\operatorname{sgn}(\overline{x})\Big(g\big(t, x+\overline{x}, y+\overline{y}\big) - g(t, x, y)\Big) \le \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|\overline{x}| + \mathcal{P}_2\phi^{-1}(\tau(t))|\overline{y}|\Big)$$

is fulfilled. If, moreover,

$$\sup\left\{\left\|\phi^{-1}(t)\int_a^t \big|g(s, 0, 0)\big|\,ds\right\| :\ a < t \le b\right\} < +\infty,$$

then the problem (3), (2) *is well-posed.*

From Theorem 3 and Corollary 2 it follows

Corollary 3. *Let the functions* φ_i $(i = 1, \dots, n)$ *be absolutely continuous and*

$$g(t, x, y) = \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x| + \mathcal{P}_2\phi^{-1}(\tau(t))|y| + q_0\Big),$$

where $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2)$*, and* $q_0 \in \mathbb{R}_+^n$ *is the vector with positive components. Then the problem* (3), (2) *is well-posed if and only if*

$$r(\mathcal{P}_1 + \mathcal{P}_2) < 1.$$

Remark 2. According to Corollary 3, the inequality $r(\mathcal{P}) < 1$ $(r(\mathcal{P}_1 + \mathcal{P}_2) < 1)$ in Theorem 4 (in Corollary 2) is unimprovable and it cannot be replaced by the inequality $r(\mathcal{P}) \le 1$ $(r(\mathcal{P}_1 + \mathcal{P}_2) \le 1)$.

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Author's address:

B. Půža

Mathematical Institute, Academy of Sciences of the Czech Republic, branch in Brno, Žižkova 22, 616 62 Brno, Czech Republic.

E-mail: puza@math.muni.cz

Z. Sokhadze

Akaki Tsereteli State University, 59, Queen Tamar St., Kutaisi 4600, Georgia.

E-mail: z.soxadze@gmail.com

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შ ი ნ ა ა რ ს ი

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