სოხუმის სახელმწიფო უნივერსიტეტი

სოხუმის სახელმწიფო უნივერსიტეტის

შ რ ო მ ე ბ ი

VII

მათემატიკისა და კომპიუტერულ
მეცნიერებათა სერია

სოხუმის სახელმწიფო უნივერსიტეტის გამომცემლობა
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შრომების VII ტომი (*მათემატიკისა და კომპიუტერულ მეცნიერებათა სერია)* ეძღვნება პროფესორ **რევაზ აბსავას** ხსოვნას (გარდაცვალებიდან 5 წლისთავს) და მასში, ძირითადად, წარმოდგენილია ალბათობის თეორიისა და მათემატიკური სტატისტიკის, ასევე მომიჯნავე დარგების აქტუალური საკითხები.

კრებული განკუთვნილია შესაბამისი დარგების სპეციალისტებისათვის და სტუდენტებისათვის.

შრომების

მათემატიკისა და კომპიუტერულ მეცნიერებათა სერიის
მათემატიკისა და კომპიუტერულ მეცნიერებათა სერიის

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ს ა რ ჩ ე ვ ი

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ტ. VII, 2009
მათემატიკისა და კომპიუტერულ მეცნიერებათა სერია

პროფესორი რევაზ აბსავა

რევაზ მიხეილის ძე აბსავა დაიბადა 1947 წლის 9 თებერვალს ზუგდიდის რაიონის სოფელ გრიგოლიშში, მოსამსახურის ოჯახში.

1965 წელს დაამთავრა კორცხელის საშუალო სკოლა ოქროს მედალზე, 1969წ. კი მ. გორკის სახელობის სოხუმის სახელმწიფო პედაგოგიური ინსტიტუტის ფიზიკა-მათემატიკის ფაკულტეტი მათემატიკის სპეციალობით. 1969-1971 წლებში იგი მუშაობდა ზუგდიდის რაიონის სოფელ ჯიხაშკარის I საშუალო სკოლაში მათემატიკის მასწავლებლად, 1971 წლიდან კი სოხუმის პედაგოგიური ინსტიტუტის, 1979 წლიდან აფხაზეთის სახელმწიფო უნივერსიტეტის ფიზიკა-მათემატიკის ფაკულტეტის ალგებრა-გეომეტრიის კათედრაზე მასწავლებლის, უფროსი მასწავლებლის, დოცენტის თანამდებობებზე. 1989 წლიდან, აფხაზეთის სახელმწიფო უნივერსიტეტიდან ქართული ნაწილის გამოყოფისა და მის ბაზაზე ახალი საუნივერსიტეტო ცენტრის – ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის სოხუმის ფილიალის დაფუძნების შემდეგ, ის იყო ამ უმაღლესი სასწავლებლის მათემატიკისა და კომპიუტერულ მეცნიერებათა ფაკულტეტის დეკანი.

მშობლიურ ინსტიტუტში სამუშაოდ დაბრუნების დღიდანვე განისაზღვრა მისი პედაგოგიური და სამეცნიერო მუშაობის მიმართულება, ალბათობის თეორია და მათემატიკური სტატისტიკა.

რევაზ აბსავა, როგორც მეცნიერი, ივანე ჯავახიშვილის სახელობის თბილისის სახელმწიფო უნივერსიტეტის ალბათობის თეორიისა და მათემატიკური სტატისტიკის კათედრის აღზრდილია. ის მეცნიერების დიდ და ეკლიან გზაზე დააკვალიანა კათედრის გამგემ პროფესორმა **გვანჯი მანიამ**, გამოჩენილმა მეცნიერმა და მეცნიერების ბრწყინვალე ორგანიზატორმა. ამავე კათედრის პროფესორის, **ელიზბარ ნადარაიას** ხელმძღვანელობით შესრულდა საკანდიდატო დისერტაცია თემაზე: ***„მრავალგანზომილებიანი განაწილების სიმკვრივისა და რეგრესიის ფუნქციის არაპარამეტრული გულოვანი ტიპის შეფასებათა შესახებ“*** (სპეც.: 01.01.05 – *ალბათობის თეორია და მათემატიკური სტატისტიკა*), რომელიც ბრწყინვალედ დაიცვა მოსკოვის ელექტრონული მანქანათმშენებლობის ინსტიტუტის სპეციალიზირებული საბჭოს (K.063.68.05) სხდომაზე 1986 წლის 10 ოქტომბერს და მას მიენიჭა ფიზიკა-მათემატიკის მეცნიერებათა კანდიდატის სამეცნიერო ხარისხი. ხოლო, 1989წ. მარტში უმაღლესმა საატესტაციო კომისიამ მიანიჭა დოცენტის წოდება.

შემდგომ იყო პოლიტიკური კრიზისი აფხაზეთში, ომი, რომელმაც შეაფერხა არა მარტო თითოეული აფხაზეთელის წინსვლა...

1993 წლიდან თბილისში, მიუხედავად დიდი პრობლემებისა, პედაგოგიური და ადმინისტრაციული მუშაობის პარალელურად რ. აბსავამ განაგრძო სამეცნიერო მუშაობა (კონსულტანტი: თსუ ალბათობის თეორიისა და მათემატიკური სტატისტიკის კათედრის გამგე, საქართველოს მეცნიერებათა აკადემიის წევრ-კორესპონდენტი, პროფესორი **ელიზბარ ნადარაია**). კვლევის ობიექტი იყო: განაწილების კანონის ფუნქციონალური მახასიათებლების ზოგადი სახის არაპ-

არამეტრულ შეფასებათა კვადრატული გადახრის ზღვარითი განაწილების მიღების მეთოდის შემუშავება და მისი რეალიზება; განაწილების სიმკვრივის და რეგრესიის მრუდის შესახებ მარტივი და რთული ჰიპოთეზების შესამოწმებელი კრიტერიუმების აგება და მათი სიმძლავრის დადგენა; კლასიფიკაციის და იდენტიფიკაციის პრობლემებთან დაკავშირებული ზოგიერთი საკითხის გადაწყვეტა.

რევაზ აბსავა 30-ზე მეტი სამეცნიერო პუბლიკაციის ავტორია, მათ შორის არის ორი მეთოდური მითითება, სახელმძღვანელო ალბათობის თეორიაში და მონოგრაფია. რევაზ აბსავას მეცნიერულმა კვლევებმა დასრულებული სახე მიიღო სადოქტორო დისერტაციის სახით: ***„განაწილების კანონის ფუნქციონალური მახასიათებლების არაპარამეტრული შეფასებები და გამოყენება კლასიფიკაციის და იდენტიფიკაციის ამოცანებში“*** (სპეც.: 01.01.08 – *მათემატიკური კიბერნეტიკა*). ფიზიკა–მათემატიკის მეცნიერებათა დოქტორის სამეცნიერო ხარისხის მოსაპოვებლად დისერტაცია წარედგინა ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის Ph.M.01.08 სადისერტაციო საბჭოს. დაინიშნა დაცვის დღე...

დაცვამდე რამდენიმე დღით ადრე, 57 წლის ასაკში წავიდა ჩვენგან ბატონი რევაზი და დაგვიტოვა დიდი გულისტკივილი. იმდენად დიდი იყო ამ შემთხვევის რეზონანსი საქართველოს სამეცნიერო საზოგადოებაში, რომ მოხდა უპრეცედენტო შემთხვევა მათემატიკოსებს შორის საქართველოს ისტორიაში. დანიშნულ დროს შედგა ზემოთ აღნიშნული საბჭოს სხდომა, სადაც დისერტანტის როლში იყო კონსულტანტი, ბატონი ელიზბარ ნადარაია.

ნაშრომის მეცნიერული სიახლე მდგომარეობს შემდეგში:

- განაწილების ფუნქციონალური მახასიათებლების არაპარამეტრულ შეფასებათა ზოგადი კლასისათვის შესწავლილია ინტეგრალური კვადრატული გადახრის ზღვარითი განაწილება. არსებითად, ეს კლასი მოიცავს ბევრ ცნობილ შეფასებას, როგორიცაა, მაგალითად, განაწილების სიმკვრივის გულოვანი და პროექციული ტიპის შეფასებები, ჰისტოგრამები, რეგრესიის მრუდის გულოვანი შეფასება, რეგრესოგრამები, რისკის ფუნქციის გულოვანი შეფასება და ა.შ.
- მოძებნილია კვადრატული გადახრის ზღვარითი გადაწილება თითქმის ყველა ზემოთ ჩამოთვლილი ფუნქციონალური მახასიათებლების შეფასებისათვის ფრიად ზოგად პირობებში, ამასთანავე პირველად იქნა შესწავლილი კვადრატული გადახრის ზღვარითი განაწილება რეგრესოგრამის, „შეშფოთებული“ რეგრესიის (ბერნშტეინის ამოცანა) და რისკის ფუნქციისათვის.

- განაწილების სიმკვრივის და რეგრესიის მრუდისათვის აგებულია ინტეგრალური ტიპის კრიტერიუმები სტატისტიკურ ჰიპოთეზათა შესამოწმებლად.
- ჰელინგერის მანძილის ცნებაზე დაყრდნობით აგებულია თანხმობის კრიტერიუმი, რომელიც არ არის დამოკიდებული განაწილების სიმკვრივის სახეზე.
- შესწავლილია განაწილების სიმკვრივის და რეგრესიის მრუდის არაპარამეტრული გულოვანი ტიპის შეფასებათა გამოყენება კლასიფიკაციის და იდენტიფიკაციის ამოცანებში. კერძოდ, აგებულია ემპირიული გადამწყვეტი წესების კლასი, რომელთათვისაც ბაიესის რისკის შესაბამისი მიმდევრობები თითქმის ყველგან იკრიბება ოპტიმალურისაკენ.
- შესწავლილია რეგრესიის ფუნქციისათვის გასერ–მიულერის ტიპის არაპარამეტრული გულოვანი შეფასების გამოყენება იდენტიფიკაციის ამოცანებისათვის. მიღებულია ამ შეფასების კვადრატული გადახრის ზღვარითი განაწილება და მოყვანილია მათი გამოყენება რეგრესიის ფუნქციისათვის მარტივი და რთული ჰიპოთეზის შემოწმებასთან დაკავშირებულ ამოცანებთან. ამოხსნილია ი. ლინიკის განზოგადებული ამოცანა. შემოწმებულია ჰიპოთეზა მოცემული რიგის პარაბოლური ტიპის რეგრესიის არსებობის შესახებ.
- შემუშავებულია რეგრესიის ორი ფუნქციის ტოლობის ჰიპოთეზის შემოწმების კრიტერიუმის აგების მეთოდიკა, რომელიც მჭიდრო კავშირშია იდენტიფიკაციის ამოცანებთან და შესწავლილია აგებული კრიტერიუმის ასიმპტოტური თვისებები.
- რამდენიმე განაწილების შემთხვევაში აგებულია ერთგვაროვნების ჰიპოთეზის შემოწმების კრიტერიუმი, ამასთანავე განიხილება „მახლობელი" ალტერნატივების მიმდევრობა, რომელთათვისაც შემოთავაზებული კრიტერიუმები უფრო მძლავრია, ვიდრე კოლმოგოროვ–სმირნოვის ტიპის კრიტერიუმები.
- შემუშავებულია ოპტიმალური, ადაპტური გულოვანი შეფასების აგების მეთოდიკა მრავალგანზომილებიანი განაწილების სიმკვრივისათვის. მოძებნილია გული, რომლის საშუალებით შეგვიძლია მინიმუმამდე დავიყვანოთ განაწილების სიმკვრივის შეფასების ინტეგრალური კვადრატული გადახრის მათემატიკური ლოდინი. განზოგადებულია ვეის-ვოლფოვიცის შედეგი წერტილში განაწილების სიმკვრივის შეფასების შესახებ.

სადისერტაციო საბჭომ ერთხმად, ფარული კენჭისყრით, რევაზ აბსავას მიანიჭა ფიზიკა-მათემატიკის მეცნიერებათა დოქტორის სამეცნიერო ხარისხი სიკვდილის შემდეგ.

აღსანიშნავია, რომ გარდაცვალების შემდეგ, ბატონი ელიზბარ ნადარაიას ძალისხმევით 2005 წელს იყო გამოცემული ზემოთ აღნიშნული მონოგრაფია ***„დაკვირვებათა განაწილების კანონის ფუნქციონალური მახასიათებლების არაპარამეტრულ შეფასებათა თეორიის ზოგიერთი ამოცანა“.*** შემდგომ, 2008 წელს, მისივე თანაავტორობით თსუ გამომცემლობამ გამოსცა მონოგრაფიის განახლებული ვარიანტი.

2005 და 2009 წლებში თბილისის უნივერსიტეტმა ორჯერ გამოსცა რევაზ აბსავას თანაავტორობით ***„ალბათობის თეორიის“*** სახელმძღვანელოს განახლებული და შევსებული ვარიანტები.

რევაზ აბსავამ პიროვნული თვისებებით, საქმიანობით, მეცნიერული მოღვაწეობით სამარადჟამოდ უკვდავყო თავისი სახელი. მან თავისი ცხოვრებით ბევრს უჩვენა მაგალითი ადამიანური თანადგომისა და უშვრეტელად სითბოს გაცემისა. მან ჩამოაყალიბა ბრწყინვალე, ძლიერი, აკადემიური კოლექტივი, მისი მშობლიური მათემატიკისა და კომპიუტერულ მეცნიერებათა ფაკულტეტი. ის იყო ფაკულტეტის ყველა წევრის მეგობარი, სოხუმის სახელმწიფო უნივერსიტეტის ერთ-ერთი აღიარებული ლიდერი. ყველა, ვინც **რეზო აბსავას** კარგად იცნობდა, ეცოდინება, რომ მან თავისი ხანმოკლე სიცოცხლის განმავლობაში შეძლო კაცურად ეცხოვრა, გამოევლინა სიყვარულისა და შრომის უდიდესი ნიჭი, გამხდარიყო მისაბაძი თავისი განუმეორებელი და უნიკალური ხასიათით...

სოხუმის სახელმწიფო უნივერსიტეტის შრომების მათემატიკისა და კომპიუტერულ მეცნიერებათა სერიის სარედაქციო კოლეგია

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**ELIZBAR NADARAYA, PETRE BABILUA,
GRIGOL SOKHADZE**

THE ESTIMATION OF A DISTRIBUTION FUNCTION BY AN INDIRECT SAMPLE

Abstract. The problem of estimation of a distribution function isconsidered when the observer has an access only to some indicator random values. Some basic asymptotic properties of the constructed estimates are studied. In this paper, the limit theorems are proved for continuous functionals related to the estimate of $\hat{F}_n(x)$ in the space $C[a,1-a]$.

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1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a s ample of independent observations of a random non-negative value *X* with a distribution function $F(x)$. In problems of the theory of censored observations, sample values are pairs $Y_i = (X_i \wedge t_i)$ and $Z_i = I(Y_i = X_i)$, $i = \overline{1,n}$, where t_i are given numbers ($t_i \neq t_j$ for $i \neq j$) or sample values independent of X_i, $i = \overline{1,n}$. Throughout the paper, $I(A)$ denotes the indicator of the set *A*.

Our present study deals with a somewhat different case: an observer has an access only to the values of random variables $\xi_i = I(X_i < t_i)$, with $t_i = c_F \dfrac{2i-1}{2n}$, $i = \overline{1,n}$, $c_F = \inf\{x \geq 0 : F(x) = 1\} < \infty$.

The problem consists in estimating the distribution function $F(x)$ by means of a sample $\xi_1, \xi_2, \ldots, \xi_n$. Such a problem arises for example from a region of corrosion investigations, see [1] where an experiment related to corrosion is described.

As an estimate for $F(x)$ we consider an expression of the form

$$\hat{F}_n(x) = \begin{cases} 0, & x \le 0, \\ F_{1n}(x) \cdot F_{2n}^{-1}(x), & 0 < x < c_F, \\ 1, & x \ge c_F, \end{cases} \tag{1}$$

$$F_{1n}(x) = \frac{1}{nh} \sum_{j=1}^{n} K\left(\frac{x - t_j}{h} \right) \xi_j,$$

$$F_{2n}(x) = \frac{1}{nh} \sum_{j=1}^{n} K\left(\frac{x - t_j}{h} \right),$$

where $K(x)$ is some weight function (kernel), $\{h = h(n)\}$ is a sequence of positive numbers converging to zero.

2. Conditions of asymptotic unbiasedness and consistency

In this subsection we give the conditions of asymptotic unbiasedness and consistency and the theorems on a limiting distribution $\hat{F}_n(x)$.

Lemma 1. *Assume that*

1^0. $K(x)$ *is some distribution density of bounded variation and* $K(x) = K(-x)$, $x \in R = (-\infty, \infty)$. *If* $nh \to \infty$, *then*

$$\frac{1}{nh} \sum_{j=1}^{n} K^{m_1 - 1}\left(\frac{x - t_j}{h} \right) F^{m_2 - 1}(t_j) = \frac{1}{c_F h} \int_0^{c_F} K^{m_1 - 1}\left(\frac{x - u}{h} \right) F^{m_2 - 1}(u)\,du + O\left(\frac{1}{nh} \right), \tag{2}$$

uniformly with respect to $x \in [0, c_F]$; m_1, m_2 *are natural numbers.*

Proof. Let $P(x)$ be a uniform distribution function on $[0, c_F]$, and $P_n(x)$ be an empirical distribution function of "the sample" $t_1, t_2, \ldots, t_n$, i.e., $P_n(x) = n^{-1} \sum_{j=1}^{n} I(t_j < x)$. It is obvious that

$$\sup_{0\le x\le c_F}\left|P_n(x)-P(x)\right| = \sup_{0\le x\le c_F}\left|\frac{1}{n}\left[n\frac{x}{c_F}+\frac{1}{2}\right]-\frac{x}{c_F}\right| \le \frac{1}{2n}. \tag{3}$$

We have

$$\frac{1}{nh}\sum_{i=1}^{n}K^{m_1-1}\left(\frac{x-t_i}{h}\right)F^{m_2-1}(t_i)-\frac{1}{c_F h}\int_0^{c_F}K^{m_1-1}\left(\frac{x-u}{h}\right)F^{m_2-1}(u)du =$$

$$=\frac{1}{h}\int_0^{c_F}K^{m_1-1}\left(\frac{x-u}{h}\right)F^{m_2-1}(u)d\left(P_n(u)-P(u)\right). \tag{4}$$

Applying the integration by parts formula to the integral in the right-hand part of (4) and taking (3) into account, we obtain (2).

Below it is assumed without loss of generality that the interval $[0, c_F] = [0,1]$.

Theorem 1. *Let* $F(x)$ *be continuous and the conditions of the lemma be fulfilled. Then the estimate* (1) *is asymptotically unbiased and consistent at all points* $x \in [0,1]$. *Moreover,* $\hat{F}_n(x)$ *has an asymptotically normal distribution, i.e.,*

$$\sqrt{nh}\left(\hat{F}_n(x)-E\hat{F}_n(x)\right)\sigma^{-1}(x)\xrightarrow{d}N(0,1),$$
$$\sigma^2(x)=F(x)(1-F(x))\int K^2(u)du,$$

where d denotes convergence in distribution, and $N(0,1)$ *a random value having a normal distribution with mean* 0 *and variance* 1.

Proof. By Lemma 1 we have

$$\begin{aligned} EF_{1n}(x) &= \int_{\frac{x-1}{h}}^{\frac{x}{h}}K(t)F(x+ht)dt+O\left(\frac{1}{nh}\right), \\ F_{2n}(x) &= \frac{1}{h}\int_0^1 K\left(\frac{x-u}{h}\right)du+O\left(\frac{1}{nh}\right), \end{aligned} \tag{5}$$

and for $n\to\infty$

$$\frac{1}{h}\int_0^1 K\left(\frac{x-u}{h}\right)du \to F_2(x) = \begin{cases} 1, & x \in (0,1), \\ \frac{1}{2}, & x = 0,\ x = 1, \end{cases}$$

$$\int_{\frac{x-1}{h}}^{\frac{x}{h}} K(t)F(x+th)dt \to F(x)F_2(x).$$

Hence it follows that $E\hat{F}_n(x) \to F(x),\ x \in [0,1]$ as $n \to \infty$.

Analogously, it is not difficult to show that

$$Var\,\hat{F}_n(x) = \left[\frac{1}{nh^2}\int_0^1 K^2\left(\frac{x-u}{h}\right)F(u)(1-F(u))du + O\left(\frac{1}{(nh)^2}\right)\right]F_{2n}^{-2}(x).$$

Hence we readily derive

$$nh\ Var\,\hat{F}_n(x) \sim \sigma^2(x) = F(x)(1-F(x))\int K^2(u)du \tag{6}$$

for $x \in [0,1]$.

Thus $\hat{F}_n(x)$ is a consistent estimate for $F(x),\ x \in [0,1]$, and therefore

$$P\left\{\hat{F}_n(x_1) \le \hat{F}_n(x_2)\right\} \to 1 \text{ as } n \to \infty,\ x_1 < x_2,\ x_1, x_2 \in [0,1].$$

Let us now establish that $\hat{F}_n(x)$ has an asymptotically normal distribution. Since, by virtue of (5), $F_{2n}(x) \to F_2(x)$, it remains for us to verify the condition of the Liapunov central limit theorem for $F_{1n}(x)$.

Let us denote

$$\eta_i = \eta_i(x) = (nh)^{-1}K\left(\frac{x-t_i}{h}\right)\xi_i$$

and show that

$$L_n = \sum_{j=1}^n E\left|\eta_j - E\eta_j\right|^{2+\delta}\left(Var\,F_{1n}(x)\right)^{-1-\frac{\delta}{2}} \to 0,\ \delta > 0. \tag{7}$$

We have

$$\sum_{j=1}^n E\left|\eta_j - E\eta_j\right|^{2+\delta} \le 2M^{1+\delta}(nh)^{-(2+\delta)}\sum_{j=1}^n K\left(\frac{x-t_j}{h}\right)F(t_j),\quad M = \max_{x\in R} K(x).$$

Hence, taking (2) into account, we find

$$\sum_{j=1}^{n} E\left|\eta_j - E\eta_j\right|^{2+\delta} \le c_1 (nh)^{-(1+\delta)}. \tag{8}$$

Using the relation (6) and the inequality (8), we establish that $L_n = O\left((nh)^{-\frac{\delta}{2}}\right)$, i.e., (7) holds.

3. Uniform consistency

In this subsection we define the conditions, under which the estimate $\hat{F}_n(x)$ uniformly converges in probability (a.s.) to true $F(x)$.

Let us introduce the Fourier transform of the function $K(x)$

$$\varphi(t) = \int e^{itx} K(x)\,dx$$

and assume that

2^0. $\varphi(t)$ is absolutely integrable. Following E. Parzen [2], $F_{1n}(x)$ can be represented as

$$F_{1n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\frac{x}{h}} \varphi(u) \frac{1}{nh}\sum_{j=1}^{n} \xi_j e^{iu\frac{t_j}{h}}\, du.$$

Thus

$$F_{1n}(x) - EF_{1n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\frac{x}{h}} \varphi(u) \frac{1}{nh}\sum_{j=1}^{n} \left(\xi_j - F(t_j)\right) e^{iu\frac{t_j}{h}}\, du.$$

Denote

$$d_n = \sup_{x\in\Omega_n}\left|\hat{F}_n(x) - E\hat{F}_n(x)\right|, \quad \Omega_n = \left[h^{\alpha}, 1-h^{\alpha}\right], \quad 0 < \alpha < 1.$$

Theorem 2. *Let $K(x)$ satisfy conditions 1^0 and 2^0.*

(a) *Let $F(x)$ be continuous and $n^{\frac{1}{2}} h_n \to \infty$, then*

$$D_n = \sup_{x\in\Omega_n}\left|\hat{F}_n(x) - F(x)\right| \xrightarrow{P} 0;$$

(b) *If $\sum_{n=1}^{\infty} n^{-\frac{p}{2}} h^{-p} < \infty$, $p > 2$, then $D_n \to 0$ a.s.*

Proof. We have

$$\sup_{x\in\Omega_n}\left(1-\frac{1}{h}\int_0^1 K\left(\frac{x-u}{h}\right)du\right)\le\int_{-\infty}^{-h^{\alpha-1}}K(u)du+\int_{h^{\alpha-1}}^{\infty}K(u)du\to 0. \tag{9}$$

This and (5) imply that

$$\sup_{x\in\Omega_n}\left|F_{2n}(x)-1\right|\to 0, \tag{10}$$

i.e., due to the uniform convergence for any $\varepsilon_0>0$, $0<\varepsilon_0<1$, and sufficiently large $n\ge n_0$, we have $F_{2n}(x)\ge 1-\varepsilon_0$ uniformly with respect to $x\in\Omega_n$. Therefore

$$d_n\le(1-\varepsilon_0)^{-1}\sup_{x\in\Omega_n}\left|F_{1n}(x)-EF_{1n}(x)\right|\le$$

$$\le(1-\varepsilon_0)^{-1}\frac{1}{2\pi}\int|\varphi(u)|\frac{1}{nh}\left|\sum_{j=1}^n\overline{\eta}_j e^{iu\frac{t_j}{h}}\right|du,\quad \overline{\eta}_j=\xi_i-F(t_j).$$

Hence, by Hölder's inequality, we obtain

$$d_n^p\le(1-\varepsilon_0)^{-p}\frac{1}{(2\pi)^p}\int|\varphi(u)|\left|\sum_{j=1}^n\overline{\eta}_j e^{iu\frac{t_j}{h}}\right|^p du\left(\int|\varphi(u)|du\right)^{\frac{p}{q}},\quad \frac{1}{p}+\frac{1}{q}=1,\quad p>2$$

.

Thus

$$Ed_n^p\le c(\varepsilon,p,\varphi)\frac{1}{(nh)^p}\int|\varphi(u)|E\left|\sum_{j,k}\cos\left(\left(\frac{t_j-t_k}{h}\right)u\right)\overline{\eta}_j\overline{\eta}_k\right|^{\frac{p}{2}}du, \tag{11}$$

where

$$c(\varepsilon,p,\varphi)=(1-\varepsilon_0)^{-p}\frac{1}{(2\pi)^p}\left(\int|\varphi(u)|du\right)^{\frac{p}{q}}.$$

Denote

$$A(u)=\sum_{j,k}\cos\left(\left(\frac{t_j-t_k}{h}\right)u\right)\overline{\eta}_j\overline{\eta}_k.$$

Then by (11) we write

$$Ed_n^p\le 2^{\frac{p}{2}-1}c(\varepsilon_0,p,\varphi)\frac{1}{(nh)^p}\left[\int|\varphi(u)|\left|EA(u)\right|^{\frac{p}{2}}du+\int|\varphi(u)|E\left|A(u)-EA(u)\right|^{\frac{p}{2}}du\right]. \tag{12}$$

Using Whittle's inequality [3] for moments of quadratic form, we obtain

$$E\left|A(u)-EA(u)\right|^{\frac{p}{2}} \le 2^{\frac{3}{2}p} c\left(\frac{p}{2}\right)\left[c(p)\right]^{\frac{1}{2}}\left(\sum_{i,j}\cos^2\left(\left(\frac{t_j-t_k}{h}\right)u\right)\gamma_j^2(p)\gamma_k^2(p)\right)^{\frac{p}{4}},$$

where

$$\gamma_k(p)=\left(E\left|\overline{\eta}_k\right|^p\right)^{\frac{1}{p}} \le 1, \quad c(s)=\frac{2^{\frac{p}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right).$$

Hence it follows that

$$E\left|A(u)-EA(u)\right|^{\frac{p}{2}} = O\left(n^{\frac{p}{2}}\right) \tag{13}$$

uniformly with respect to $u \in (-\infty,\infty)$. It is also clear that

$$\left|EA(u)\right|^{\frac{p}{2}} = O\left(n^{\frac{p}{2}}\right) \tag{14}$$

uniformly with respect to $u \in (-\infty,\infty)$.

Having combined the relations (12), (13) and (14), we obtain

$$Ed_n^p = O\left(\frac{1}{\left(\sqrt{n}\,h\right)^p}\right), \quad p > 2.$$

Therefore

$$P\left\{\sup_{x\in\Omega_n}\left|\hat{F}_n(x)-E\hat{F}_n(x)\right| \ge \varepsilon\right\} \le \frac{c_3}{\varepsilon^p\left(\sqrt{n}\,h\right)^p}. \tag{15}$$

Furthermore, we have

$$\sup_{x\in\Omega_n}\left|E\hat{F}_n(x)-F(x)\right| \le \frac{1}{1-\varepsilon_0}\left(\sup_{x\in\Omega_n}\left|EF_{1n}(x)-F(x)\right| + \sup_{x\in\Omega_n}\left|1-F_{2n}(x)\right|\right). \tag{16}$$

By virtue of (10), the second summand in the right-hand part of (16) tends to 0, whereas the first summand is estimated as follows:

$$\sup_{x\in\Omega_n}\left|EF_{1n}(x)-F(x)\right| \le S_{1n}+S_{2n}+O\left(\frac{1}{nh}\right), \tag{17}$$

$$S_{1n} = \sup_{0\le x\le 1}\left|\frac{1}{h}\int_0^1 \left(F(y)-F(x)\right)K\left(\frac{x-y}{h}\right)dy\right|,$$

$$S_{2n} = \sup_{x\in\Omega_n}\left(1-\frac{1}{h}\int_0^1 K\left(\frac{x-y}{h}\right)dy\right),$$

and, by virtue of (9),

$$S_{2n} \to 0 \text{ as } n \to \infty . \tag{18}$$

Let us now consider S_{1n}. Note that

$$S_{1n} \le \sup_{0\le x\le 1} \left| \int_0^1 \left| F(y) - F(x) \right| \frac{1}{h} K\left(\frac{x-y}{h} \right) dy \right| =$$

$$= \sup_{0\le x\le 1} \int_{x-1}^{x} \left| F(x-u) - F(x) \right| \frac{1}{h} K\left(\frac{u}{h} \right) du \le$$

$$\le \sup_{0\le x\le 1} \int_{-\infty}^{\infty} \left| F(x-u) - F(x) \right| \frac{1}{h} K\left(\frac{u}{h} \right) du . \tag{19}$$

Assume that $\delta > 0$ and divide the integration domain in (19) into two domains $|u| \le \delta$ and $|u| > \delta$. Then

$$S_{1n} \le \sup_{0\le x\le 1} \int_{|u|\le\delta} \left| F(x-u) - F(x) \right| \frac{1}{h} K\left(\frac{u}{h} \right) du +$$

$$+ \sup_{0\le x\le 1} \int_{|u|>\delta} \left| F(x-u) - F(x) \right| \frac{1}{h} K\left(\frac{u}{h} \right) du \le$$

$$\le \sup_{x\in R} \sup_{|u|\le\delta} \left| F(x-u) - F(x) \right| + 2 \int_{|u|\ge\frac{\delta}{h}} K(u) du . \tag{20}$$

By a choice of $\delta > 0$ the first summand in the right-hand part of (20) can be made arbitrarily small. Choosing $\delta > 0$ and letting $n \to \infty$, we find that the second summand tends to zero. Therefore

$$\lim_{n\to\infty} S_{1n} = 0 . \tag{21}$$

Finally, from the relations (15)-(18) and (21) the proof of the theorem follows.

Remark 1.

(1) If $K(x) = 0$, $|x| \ge 1$ and $\alpha = 1$, i.e., $\Omega_n = [h, 1-h]$, then $S_{2n} = 0$.

(2) In the conditions of Theorem 2

$$\sup_{x\in[a,b]}\left|\hat{F}_n(x)-F(x)\right|\to 0$$

in probability (a.s.) for any fixed interval $[a,b]\subset[0,1]$ since there may exist n_0 such that $[a,b]\subset\Omega_n$, $n\geq n_0$.

Assume that $h=n^{-\gamma}$, $\gamma>0$. The conditions of Theorem 2 are fulfilled:

$$n^{\frac{1}{2}}h_n\to\infty \text{ if } 0<\gamma<\frac{1}{2},$$

and

$$\sum_{n=1}^{\infty} n^{-\frac{p}{2}}h_n^{-p}<\infty \text{ if } 0<\gamma<\frac{p-2}{2p},\ p>2.$$

4. Estimation of moments

In considering the problem, there naturally arises a question of estimation of the integral functionals of $F(x)$, for example, moments μ_m, $m\geq 1$:

$$\mu_m=m\int_0^1 t^{m-1}(1-F(t))dt.$$

As estimates for μ_m we consider the statistics

$$\hat{\mu}_{nm}=1-\frac{m}{n}\sum_{j=1}^{n}\xi_j\frac{1}{h}\int_h^{1-h}t^{m-1}K\left(\frac{t-t_j}{h}\right)F_{2n}^{-1}(t)dt.$$

Theorem 3. *Let* $K(x)$ *satisfy condition* 1^0 *and, in addition to this,* $K(x)=0$ *outside the interval* $[-1,1]$*. If* $nh\to\infty$ *as* $n\to\infty$*, then* $\hat{\mu}_{nk}$ *is an asymptotically unbiased, consistent estimate for* μ_m *and moreover*

$$\frac{\sqrt{n}(\hat{\mu}_{nm}-E\hat{\mu}_{nm})}{\sigma}\xrightarrow{d}N(0,1),\quad \sigma^2=m^2\int_0^1 t^{2m-2}F(t)(1-F(t))dt.$$

Proof. Since $K(x)$ has $[-1,1]$ as a support, we establish from (5) that

$$F_{2n}(n) = 1 + O\left(\frac{1}{nh}\right)$$

uniformly with respect to $x \in [h, 1-h]$.

Hence, by Lemma 1 we have

$$E\hat{\mu}_{nm} = 1 - \frac{m}{n}\sum_{j=1}^{n} F(t_j)\frac{1}{h}\int_h^{1-h} t^{m-1}K\left(\frac{t-t_j}{h}\right)F_{2n}^{-1}(t)dt =$$

$$= 1 - m\int_h^{1-h}\left[\frac{1}{h}\int_0^1 K\left(\frac{t-u}{h}\right)F(u)du\right]t^{m-1}dt + O\left(\frac{1}{nh}\right) =$$

$$= 1 - m\int_h^{1-h}\left(\int_{-1}^{1} K(v)F(t+vh)dv\right)t^{m-1}dt + O\left(\frac{1}{nh}\right) =$$

$$= 1 - m\int_0^1 t^{m-1}\left[\int_{-1}^{1} K(v)F(t+vh)dv\right]dt + O(h) + O\left(\frac{1}{nh}\right). \qquad 22)$$

By the Lebesgue theorem on majorized convergence, from (22) we establish that

$$E\hat{\mu}_{nm} \to 1 - m\int_0^1 F(t)t^{m-1}dt = m\int_0^1 t^{m-1}(1-F(t))dt = \mu_m, \quad m \ge 1. \qquad (23)$$

Therefore $\hat{\mu}_{nm}$ is an asymptotically unbiased estimate for μ_m.

Further, analogously to (22), it can be shown that

$$Var\,\hat{\mu}_{nm} = \frac{m^2}{n}\int_0^1 F(t)(1-F(t))t^{2m-2}\left[\mathrm{K}\left(\frac{1-t}{h}-1\right) - \mathrm{K}\left(1-\frac{t}{h}\right)\right]^2 dt + O\left(\frac{h}{n}\right) + O\left(\frac{1}{(nh)^2}\right),$$

where

$$\mathrm{K}(v) = \int_{-\infty}^{v} K(u)du.$$

By the same Lebesgue theorem we see that

$$n\,Var\,\hat{\mu}_{nm} \sim \sigma^2 = m^2\int_0^1 t^{2m-2}F(t)(1-F(t))dt. \qquad (24)$$

Therefore (23) and (24) imply that $\hat{\mu}_{nm} \xrightarrow{P} \mu_m$.

To complete the proof of the theorem it remains to show that the statistics $\sqrt{n}(\hat{\mu}_{nm} - E\hat{\mu}_{nm})$ have an asymptotically normal distribution with mean 0 and dispersion 1. For this it suffices to show that the Liapunov fraction $L_n \to 0$. Indeed,

$$L_n = n^{-(2+\delta)} m^{2+\delta} \sum_{j=1}^{n} E\left|\xi_j - F(t_j)\right|^{2+\delta} \left|\frac{1}{h}\int_h^{1-h} t^{m-1} K\left(\frac{t-t_j}{h}\right) F_{2n}^{-1}\, dt\right|^{2+\delta} \left(Var\hat{\mu}_{nm}\right)^{-\left(1+\frac{\delta}{2}\right)} \le$$

$$\le c_6 n^{-(2+\delta)} \sum_{j=1}^{n} E\left|\xi_j - F(t_j)\right|^{2+\delta} \left(Var\hat{\mu}_{nm}\right)^{-\left(1+\frac{\delta}{2}\right)} \le$$

$$\le c_7 n^{-1-\delta} \left(Var\hat{\mu}_{nm}\right)^{-1-\frac{\delta}{2}} = O\left(n^{-\frac{\delta}{2}}\right).$$

The theorem is proved.

5. Limit theorems of functionals related to the estimate $\hat{F}_n(x)$.

In this subsection the kernel $K(x) \ge 0$ is chosen so that it would be a function of finite variation and satisfy the conditions

$$K(-u) = K(u), \quad \int K(u)du = 1, \quad K(u) = 0 \;\; for \;\; |u| \ge 1.$$

Theorem 4. *Let* $g(x) \ge 0$, $x \in [a, 1-a]$, $0 < a < \frac{1}{2}$, *be a measurable and bounded function.*

(a) *If* $F(a) > 0$ *and* $nh^2 \to \infty$ *as* $n \to \infty$, *then*

$$\overline{T}_n = \sqrt{n} \int_a^{1-a} g_1(x)\left[\hat{F}_n(x) - E\hat{F}_n(x)\right] dx \xrightarrow{d} N\left(0, \sigma^2\right), \tag{25}$$

where

$$g_1(x) = g(x)\psi(F(x)), \quad \psi(t) = \frac{1}{\sqrt{t(1-t)}};$$

(b) *If* $F(a) > 0$, $nh^2 \to \infty$, $nh^4 \to 0$ *as* $n \to \infty$ *and* $F(x)$ *has bounded derivatives up to second order, then as* $n \to \infty$

$$T_n = \sqrt{n}\int_a^{1-a} g_1(x)\left[\hat{F}_n(x) - F(x)\right] dx \xrightarrow{d} N\left(0,\sigma^2\right),$$

$$\sigma^2 = \int_a^{1-a} g^2(u)du.$$

Remark 2. We have introduced $a > 0$ in (25) in order to avoid the boundary effect of the estimate $\hat{F}_n(x)$ since near the interval boundary the estimate $\hat{F}_n(x)$ being a kernel type estimate behaves worse in the sense of order of bias tendency to zero than on any inner interval $[a,1-a] \subset [0,1]$, $0 < a < \frac{1}{2}$.

Proof of Theorem 4. We have

$$\overline{T}_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\xi_j - F\left(t_j\right)\right)\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t_j}{h}\right) g_{2n}(u)du\ ,$$

where

$$g_{2n}(u) = g_1(u)F_{2n}^{-1}(u).$$

Hence

$$\sigma_n^2 = Var\,\overline{T}_n = \frac{1}{n}\sum_{j=1}^{n}\psi^{-2}\left(F\left(t_j\right)\right)\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t_j}{h}\right) g_{2n}(u)du\right)^2. \qquad (26)$$

Since $K(u)$ has $[-1,1]$ as a support and $0 < a \le u \le 1-a$, it can be easily verified that

$$F_{2n}(u) = 1 + O\left(\frac{1}{nh}\right) \quad \text{and} \quad g_{2n}(u) = g_1(u) + O\left(\frac{1}{nh}\right)$$

uniformly on $u \in [a,1-a]$. Therefore from (26) we have

$$\sigma_n^2 = \frac{1}{n}\sum_{j=1}^{n}\psi^{-2}\left(F\left(t_j\right)\right)\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t_j}{h}\right) g_1(u)du\right)^2 + O\left(\frac{1}{nh}\right).$$

By virtue of Lemma 1, we can easily show that

$$\frac{1}{n}\sum_{j=1}^{n}\psi^{-2}\left(F\left(t_j\right)\right)\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t_j}{h}\right)g_1(u)du\right)^2 =$$

$$=\int_0^1 \psi^{-2}\left(F(t)\right)dt\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t}{h}\right)g_1(u)du\right)^2 + O\left(\frac{1}{nh^2}\right).$$

Therefore

$$\sigma_n^2 = \int_a^{1-a}\psi^{-2}\left(F(t)\right)dt\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t}{h}\right)g_1(u)du\right)^2 + \varepsilon_n^{(1)} + \varepsilon_n^{(2)} + O\left(\frac{1}{nh^2}\right), \tag{27}$$

$$\varepsilon_n^{(1)} = \int_0^a \psi^{-2}\left(F(t)\right)dt\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t}{h}\right)g_1(u)du\right)^2,$$

$$\varepsilon_n^{(2)} = \int_{1-a}^1 \psi^{-2}\left(F(t)\right)dt\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t}{h}\right)g_1(u)du\right)^2.$$

Since by $F(u)(1-F(u)) \le \frac{1}{4}$, $g(u) \le c_8$ and

$$\psi\left(F(u)\right) \le \frac{1}{F(a)\left(1-F(1-a)\right)},\quad a \le u \le 1-a,$$

it follows that $g_1(u) \le c_9$, we have

$$\varepsilon_n^{(1)} \le c_{10}\int_0^a dt\left(\int_{\frac{a-t}{h}}^{\frac{1-a-t}{h}} K(u)du\right)^2, \tag{28}$$

where $a-t \ge 0$ and $1-a-t \ge 0$. The first inequality is obvious, whereas the second one follows from the inequalities $0 \le t \le a$ and $0 < a < \frac{1}{2}$.

Therefore

$$\lim_{n\to\infty}\int_{\frac{a-t}{h}}^{\frac{1-a-t}{h}} K(u)du = \begin{cases} 0, & 0 \le t < a, \\ \frac{1}{2}, & t = a. \end{cases}$$

By the Lebesgue theorem on bounded convergence, from the latter expression and (28) we obtain

$$\varepsilon_n^{(1)} \to 0 \text{ as } n \to \infty. \tag{29}$$

Analogously,

$$\varepsilon_n^{(1)} \to 0 \text{ as } n \to \infty. \tag{30}$$

Now let us establish that

$$\int_a^{1-a} \psi^{-2}(F(t))dt\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t}{h}\right)g_1(u)du\right)^2 \to \sigma^2 = \int_a^{1-a} g^2(u)du \text{ as } n \to \infty$$

.

We have

$$\left|\int_a^{1-a} \psi^{-2}(F(t))dt\left(\frac{1}{h}\int_a^{1-a} g_1(u)K\left(\frac{u-t}{h}\right)du\right)^2 - \int_a^{1-a} \psi^{-2}(F(t))g_1^2(t)dt\right| \le$$

$$\le c_{11}\int_a^{1-a} \psi^{-2}(F(t))dt\left[\left|\frac{1}{h}\int_a^{1-a} g_1(u)K\left(\frac{u-t}{h}\right)du - g_1(t)\right|\right] \le$$

$$\le c_{12}\int_a^{1-a} dt\left[\left|\frac{1}{h}\int_a^{1-a} g_1(u)K\left(\frac{u-t}{h}\right)du - g_1(t)\int_a^{1-a} \frac{1}{h}K\left(\frac{u-t}{h}\right)du\right|\right] +$$

$$+ c_{13}\int_a^{1-a}\left|\int_a^{1-a} \frac{1}{h}K\left(\frac{u-t}{h}\right)du - 1\right|dt = A_{1n} + A_{2n}. \tag{31}$$

Since

$$\int_a^{1-a} \frac{1}{h}K\left(\frac{u-t}{h}\right)du \to 1$$

for all $t \in (a, 1-a)$, we have

$$A_{2n} \to 0 \text{ as } n \to \infty. \tag{32}$$

Further, we continue the function $g_1(u)$ so that that outside $[a, 1-a]$ it has zero values and denote the continued function by $\bar{g}_1(u)$. Then

$$A_{1n} \le c_{14}\left|\int_0^1\left(\int_{-\infty}^{\infty}\left|\bar{g}_1(x+y) - \bar{g}_1(y)\right|dy\right)\frac{1}{h}K\left(\frac{x}{h}\right)dx\right| \le$$

$$\le c_{15}\left|\int_{-1}^1\left(\int_{-\infty}^{\infty}\left|\bar{g}_1(y+uh) - \bar{g}_1(y)\right|dy\right)K(u)du\right| =$$

$$= c_{15}\int_{-1}^1 \omega(uh)K(u)du \to 0 \text{ as } n \to \infty, \tag{33}$$

where

$$\omega(y)=\int_{-\infty}^{\infty}\left|\bar{g}_1(y+x)-\bar{g}_1(x)\right|dx\,.$$

The (33) holds by virtue of the Lebesgue theorem on majorized convergence and the fact that $\omega(uh)\le 2\|\bar{g}_1\|_{L_1(-\infty,\infty)}$ and $\omega(uh)\to 0$ as $n\to\infty$. Thereby, taking (27)-(33) into account, we have proved that

$$\sigma_n^2\to\sigma^2=\int_a^{1-a}g^2(u)du\,. \tag{34}$$

Now let us verify the fulfillment of the conditions of the central limit theorems for the sums

$$\bar{T}_n=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}a_{jn}\left(\xi_j-F(t_j)\right),$$

$$a_{jn}=\int_a^{1-a}\frac{1}{h}K\left(\frac{x-t_j}{h}\right)g_{2n}(x)dx.$$

We have

$$L_n=\frac{n^{-\left(1+\frac{\delta}{2}\right)}\sum\limits_{j=1}^{n}a_{jn}^{2+\delta}E\left|\xi_j-F(t_j)\right|^{2+\delta}}{\left(\sqrt{Var\,\bar{T}_n}\right)^{2+\delta}}=O\left(n^{-\frac{\delta}{2}}\right),$$

since $a_{jn}\le c_{16}$, $E\left|\xi_j-F(t_j)\right|^{2+\delta}\le 1$ for all $1\le j\le n$ and $Var\,\bar{T}_n\to\sigma^2$.

Finally, the statement (b) of the theorem follows from (a) if we take into account that

$$\sqrt{n}\int_a^{1-a}g_1(x)\left[E\hat{F}_n(x)-F(x)\right]dx=\sqrt{n}\int_a^{1-a}g_1(x)\left[\int_{-1}^{1}K(u)\left(F(x-uh)-F(x)\right)du\right]dx=$$

$$=O\left(\sqrt{n}\,h^2\right)+O\left(\frac{h}{n}\right). \tag{35}$$

The theorem is proved.

Lemma 2.

(1) *In the conditions of the item* (a) *of Theorem* 4,

$$E\left|\bar{T}_n\right|^s\le c_{17}\left(\int_a^{1-a}g(u)du\right)^{\frac{s}{2}},\quad s>2. \tag{36}$$

(2) *In the conditions of the item* (b) *of Theorem* 4,

$$E|T_n|^s \le c_{18}\left(\int_a^{1-a} g(u)du\right)^{\frac{s}{2}}, \quad s > 2. \tag{37}$$

Proof. $\overline{T}_n$ is the linear form of $\eta_j = \xi_j - F(t_j)$, $E\eta_j = 0$, $1 \le j \le n$. Hence to prove (36) we use Whittle's inequality [2].

It is obvious that $E|\eta_j|^s \le 1$, $j = \overline{1,n}$. Therefore by Whittle's inequality

$$E|\overline{T}_n|^s \le c(s)2^s\left[\frac{1}{h^2}\sum_{j=1}^{n}\left(\int_a^{1-a} K\left(\frac{u-t_j}{h}\right)g_{2n}(u)du\right)^2\right]^{\frac{s}{2}},$$

where $g_{2n}(u) = g_1(u)F_{2n}^{-1}(u)$.

This, by virtue of Lemma 1, yields

$$E|\overline{T}_n|^s \le c(s)2^s\left[\int_0^1\left(\frac{1}{n}\int_a^{1-a} K\left(\frac{u-t}{h}\right)g_{2n}(u)du\right)^2 dt + O\left(\frac{1}{nh^2}\right)\left(\int_a^{1-a} g_{2n}(u)du\right)^2\right]^{\frac{s}{2}}. \tag{38}$$

Further, since

$$g_{2n}(u) \le g(u)\left[\frac{1}{F(a)(1-F(1-a))}\right]\left[1+O\left(\frac{1}{nh}\right)\right] \le c_{19}g(u), \quad a \le u \le 1-a,$$

from (38) it follows that

$$E|\overline{T}_n|^s \le c_{20}\left[\sup_{0\le t\le 1}\left(\frac{1}{h}\int_a^{1-a} K\left(\frac{u-t}{h}\right)g_{2n}(u)du\right)\int_0^1 dt\int_a^{1-a}\frac{1}{h}K\left(\frac{u-t}{h}\right)g_{2n}(u)du\right]^{\frac{s}{2}} +$$

$$+O\left(\frac{1}{nh^2}\right)^{\frac{s}{2}}\left(\int_a^{1-a} g_{2n}(u)du\right)^{\frac{s}{2}} \le$$

$$\le c_{21}\left(\int_a^{1-a} g(u)du\right)^{\frac{s}{2}}[1+o(1)] \le c_{22}\left(\int_a^{1-a} g(u)du\right)^{\frac{s}{2}}, \quad s > 2.$$

Next we obtain

$$E\left|T_n\right|^s \le 2^{s-1}\left(E\left|\overline{T}_n\right|^s + \left|\sqrt{n}\int_a^{1-a} g_1(u)\left[E\hat{F}_n(u) - F(u)\right] du\right|\right)^s \le$$

$$\le c_{23}\left(\int_a^{1-a} g(u)du\right)^{\frac{s}{2}} + \left|O\left(\sqrt{n}\, h^2\right)\int_a^{1-a} g(u)du\right|^s \le$$

$$\le c_{24}\left(\int_a^{1-a} g(u)du\right)^{\frac{s}{2}}.$$

The lemma is proved.

Let us introduce the following random processes:

$$\overline{T}_n(t) = \sqrt{n}\int_a^t \left(\hat{F}_n(u) - E\hat{F}_n(u)\right)\psi(F(u))du,$$

$$T_n(t) = \sqrt{n}\int_a^t \left(\hat{F}_n(u) - F(u)\right)\psi(F(u))du.$$

Theorem 5.

1^0. *Let the conditions of the item* (a) *of Theorem 4 be fulfilled. Then for all continuous functionals* $f(\cdot)$ *on* $C[a,1-a]$, *the distribution* $f\left(\overline{T}_n(t)\right)$ *converges to the distribution* $f(W(t-a))$ *where* $W(t-a)$, $a \le t \le 1-a$, *is a Wiener process with a correlation function* $r(s,t) = \min(t-a, s-a)$, $W(t-a) = 0$, $t = a$.

2^0. *Let the conditions of the item* (b) *of Theorem* 4 *be fulfilled. Then for all continuous functionals* $f(\cdot)$ *on* $C[a,1-a]$, *the distribution* $f(T_n(t))$ *converges to the distribution* $f(W(t-a))$.

Proof. First we will show that the finite-dimensional distributions of processes $\overline{T}_n(t)$ converge to the finite-dimensional distribution of a process $W(t-a)$, $t \ge a$. Let us consider one moment of time t_1. We have to show that

$$\overline{T}_n(t_1) \xrightarrow{d} W(t_1 - a). \tag{39}$$

To prove (39), it suffices to take $g(x) = I_{[a,t_1)}(x)$ in (25). Then, by virtue of Theorem 4,

$$\overline{T}_n(t_1) \xrightarrow{d} N\left(0, \int_a^{1-a} I_{[a,t_1)}(x)dx\right) = N(0, t_1 - a).$$

Let us now consider two moments of time t_1, t_2, $t_1 < t_2$. We have to show that

$$\left(\overline{T}_n(t_1), \overline{T}_n(t_2)\right) \xrightarrow{d} \left(W(t_1 - a), W(t_2 - a)\right). \tag{40}$$

To prove (40), it suffices to take in (25)

$$g(x) = (\lambda_1 + \lambda_2) I_{[a,t_1)}(x) + \lambda_2\, I_{[t_1,t_2)}(x),$$

where λ_1 and λ_2 are arbitrary finite numbers. Then, by virtue of Theorem 4,

$$\lambda_1 \overline{T}_n(t_1) + \lambda_2 \overline{T}_n(t_2) \xrightarrow{d} N\left(0, (\lambda_1 + \lambda_2)^2 (t_1 - a) + \lambda_2^2 (t_2 - t_1)\right).$$

On the other hand,

$$\lambda_1 W(t_1 - a) + \lambda_2 W(t_2 - a) = (\lambda_1 + \lambda_2)[W(t_1 - a) - W(0)] + \lambda_2\, [W(t_2 - a) - W(t_1 - a)]$$

is distributed as $N\left(0, (\lambda_1 + \lambda_2)^2 (t_1 - a) + \lambda_2^2 (t_2 - t_1)\right)$. Therefore (40) holds. The case of three and more number of moments is considered analogously. Therefore the finite-dimensional distributions of processes $\overline{T}_n(t)$ converge to the finite-dimensional distributions of a Wiener process $W(t-a)$, $a \le t \le 1-a$ with a correlation function

$$r(t_1, t_2) = \min(t_1 - a, t_2 - a), \quad W(t - a) = 0, \quad t = a.$$

Now we will show that the sequence $\{\overline{T}_n(t)\}$ is dense, i.e., the sequence of the corresponding distributions is dense. For this it suffices to show that for any $t_1, t_2 \in [a, 1-a]$ and all n

$$E\left|\overline{T}_n(t_1) - \overline{T}_n(t_2)\right|^s \le c_{25}|t_1 - t_2|^{\frac{s}{2}}, \quad s > 2 .$$

Indeed, this inequality is obtained from (36) for $g(x) = I_{[t_1,t_2]}(x)$.

Further, taking (35), (37) and the statement (b) of Theorem 4 into account, we easily ascertain that the finite-dimensional distributions of processes $T_n(t)$ converge to the finite-dimensional distributions of a Wiener process $W(t-a)$, and also that

$$E\left|T_n(t_1) - T_n(t_2)\right|^s \le c_{26}|t_1 - t_2|^{\frac{s}{2}}, \quad s > 2 .$$

Hence, from Theorem 2 of the monograph [3, p. 583] the proof of the theorem follows.

6. Application

By virtue of Theorem 5 and the Corollary of Theorem 1 from [3, p. 371] we can write that

$$P\left\{ T_n^+ = \max_{a \le t \le 1-a} T_n(t) > \lambda \right\} \to G(\lambda) = \frac{2}{\sqrt{2\pi(1-2a)}} \int_{\lambda}^{\infty} \exp\left\{ -\frac{x^2}{2(1-2a)} \right\} dx$$

(a is a prescribed number, $0 < a < 1/2$) as $n \to \infty$.

This result makes it possible to construct tests of a level α, $0 < \alpha < 1$, for testing the hypothesis H_0 by which

$$H_0 : \ \lim_{n\to\infty} E\hat{F}_n(x) = F_0(x), \quad a \le x \le 1-a ,$$

in the presence of the alternative hypothesis

$$H_1 : \ \int_a^{1-a} \psi(F_0(x)) \left(\lim_{n\to\infty} E\hat{F}_n(x) - F_0(x) \right) dx > 0 .$$

Let λ_α be the critical value, $G(\lambda_\alpha) = \alpha$. If as a result of the experiment it turns out that $T_n^+ \geq \lambda_\alpha$, then the hypothesis H_0 must be rejected.

Remark 3. Let t_i be the partitioning points of an interval $[0, c_F]$, $c_F = \inf \{x \geq 0 : F(x) = 1\} < \infty$, chosen from the relation $H(t_j) = \frac{2j-1}{2n}$, $j = \overline{1, n}$, where

$$H(x) = \int_0^x h(u)\,du,$$

$h(u)$ is some known density of a distribution on $[0, c_F]$ and $h(x) \geq \mu > 0$ for all $x \in [0, c_F]$. In that case, by a reasoning analogous to that used above we can obtain a generalization of the results of the present study.

Remark 4. Some ideas of the proof of Theorem 4 are borrowed from the interesting paper by A. V. Ivanov [5].

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ელიზბარ ნადარაია, პეტრე ბაბილუა, გრიგოლ სოხაძე

განაწილების ფუნქციის შეფასება არაპირდაპირი შერჩევით

ნაშრომში აგებულია განაწილების ფუნქციის შეფასება, როდესაც დამკვირვებლისთვის მისაწვდომია ზოგიერთი ინდიკატორული შემთხვევითი სიდიდის მნიშვნელობები. შესწავლილია აგებული შეფასების ზოგიერთი ძირითადი თვისება. ნაშრომში ასევე დამტკიცებულია $\hat{F}_n(x)$ შეფასებასთან დაკავშირებული $C[a,1-a]$ სივრცეში უწყვეტი ფუნქციონალებისათვის ზღვარითი თეორემები.

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**ELIZBAR NADARAYA, MZIA PATSATSIA,
GRIGOL SOKHADZE**

ON THE MAXIMUM PSEUDO-LIKELIHOOD ESTIMATION OF A DISTRIBUTION PARAMETERS BY GROUPED OBSERVATIONS WITH CENSORING

Abstract. The paper considers the problem of estimation of probability distribution parameters by using grouped observations. In some groups, observation involves not individual values of a random variable, but only their total quantity, whereas some groups are censored for the observer so that even the quantity of elements contained in them is unknown. For such a sampling, the maximum pseudo-likelihood method is used, the question of asymptotic consistency and asymptotic effectiveness of such estimators is investigated. The consideration is concretized for the estimator of a mean normal distribution.

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1. Introduction

In mathematical statistics, in particular in the point estimation theory, a special place is held by the maximum likelihood method (in the sequel abbreviated to “mlm”). Various modifications of this method and its generalizations are successfully applied in many situations. In the course of many years the mlm has been used for grouped samplings, as well as in censoring problems and estimator truncation problems (see [1] and [5] and the references therein). The results obtained are satisfactory from both the mathematical and the computational standpoint.

In the present paper, we consider the problem of estimation of probability distribution parameters by using a specific variant of observations. It is assumed that observations are grouped and there might be cases of partial or full non-observation of individual realization variables. We propose to use a somewhat more refined (in a certain sense) version of the mlm and show that it leads to asymptotically consistent and effective estimators.

2. Statement of the problem and the method its solution.

Let X be a random variable with a distribution function $F(x) = F(x, \theta)$, where $\theta \in \Theta$ is an unknown vector parameter in a finite-dimensional space $\Theta \subset R^q$. Assume that Θ is a compactum. We need to construct the consistent estimator θ using observation data on the random variable X. The experiment is set up so that we do not know the actual number of realizations, but know only a part of them.

Let the fixed points $-\infty \le t_1 \le t_2 \le \cdots \le t_n \le \infty$ be given on the straight line R (we do not exclude the case where the first or the last point takes an infinite value). These points form intervals which may be of three categories:

0) an interval интервал (t_i, t_{i+1}) belongs to the zero-th category if in this interval neither individual values of the sample nor the total quantity of sample values of the random value X are known:

1) an interval (t_i, t_{i+1}) belongs to the first category if in this category individual values of the sampling are unknown, but the number of sample values of the random value X is known. As usual, this number is denoted by n_i.

2) an interval (t_i, t_{i+1}) belongs to the second category if in this interval individual values of the sampling $x_{i1}, x_{i2}, \ldots, x_{in_i}$ are known.

In the sequel, summation or integration over the intervals of the zero-th, first or second categories will be denoted by the symbols (0), (1) или (2), respectively.

We call a sampling of this type a partially grouped sampling with censoring. Censored samplings of both types, as well as truncated samplings are obviously a particular case of the problem stated. The absence of information in the intervals of the zero-th category creates a difficulty which we will try to overcome by assuming that we know the type of the distribution $F(x,\theta)$ and the quantity of sample values not occurring in an interval of the first category: $n = \sum\limits_{(1),(2)} n_i$.

Let $A_i = (t_i, t_{i+1})$ be an interval of the zero-th category. Denote by m_i the quantity of sampling terms occurring in A_i. Then $r = n + \sum\limits_{(0)} m_i$ is the total quantity of observations. Note that $\dfrac{m_i}{n + \sum\limits_{(0)} m_i}$ is a relative frequency of the occurrence of X in A_i. If $\hat{F}_r(x) = \hat{F}_r(x,\theta)$ is an empirical distribution function, then

$$\frac{m_i}{n + \sum\limits_{(0)} m_i} = \hat{F}_r(t_{i+1}) - \hat{F}_r(t_i) \tag{1}$$

and, by the strengthened Bernoulli law of large numbers, it reduces to $p_i(\theta) = F(t_{i+1},\theta) - F(t_i,\theta)$ with probability 1.

By summing equalities (1) over all intervals of the zero-th category we find

$$\sum_{(0)} m_i = n \frac{\sum\limits_{i\in(0)} \left[\hat{F}_r(t_{i+1}) - \hat{F}_r(t_i)\right]}{1 - \left[\sum\limits_{i\in(0)} \left[\hat{F}_r(t_{i+1}) - \ddot{F}_r(t_i)\right]\right]} .$$

Hence we obtain

$$m_i = n \frac{\hat{F}_r(t_{i+1}) - \hat{F}_r(t_i)}{1 - \left[\sum\limits_{i\in(0)} \left[\hat{F}_r(t_{i+1}) - \hat{F}_r(t_i)\right]\right]} . \tag{2}$$

Let us apply the maximum pseudo-likelihood method. Assume that the random value X has a distribution density $f(x) = f(x,\theta)$ with respect to the Lebesgue measure. Then a likelihood function has the form

$$L_n(x;\theta) = \prod_{i\in(0)} \left[F(t_{i+1}) - F(t_i)\right]^{m_i} \prod_{i\in(1)} \left[F(t_{i+1}) - F(t_i)\right]^{n_i} \prod_{j=1}^{n_i} f(x_{il}) . \tag{3}$$

where m_i, $i \in (0)$, are defined by (2). The finding of points of a maximum of the function $L(x;\theta)$ is complicated by the difficulty of studying the smoothness properties of empirical functions. For this reason we consider a slightly modified likelihood function

$$\overline{L}_n(x;\theta)=\prod_{i\in(0)}\left[F(t_{i+1})-F(t_i)\right]^{n\frac{F(t_{i+1})-F(t_i)}{1-[F(t_{i+1})-F(t_i)]}}\prod_{i\in(1)}\left[F(t_{i+1})-F(t_i)\right]^{n_i}\prod_{j=1}^{n_i} f(x_{il}). \quad (4)$$

Lemma. *Let the following conditions be fulfilled:*

(a) *the distribution function* $F(x,\theta)$ *is continuous with respect to both variables and has the continuous derivative* $f(x,\theta)=\dfrac{\partial F(x,\theta)}{\partial x}$*;*

(b) *the function* $\overline{L}_n(x,\theta)$ *has the absolute maximum* $\theta=\overline{\theta}_n$.

Then $\overline{\theta}_n$ *is an asymptotically consistent and asymptotically effective estimator of the true value of the parameter* $\theta=\theta_0$.

The proof follows from the respective theorems of [6], [7].

3. Estimation of a mean for a normal distribution with incomplete observation.

Let X be a normally distributed random value with density $p(t)=\dfrac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}$, where μ is the unknown mean and σ is known.. Let the interval $[a,b]$ be inaccessible (a and b may be infinite, too) for the observer, and also the quantity of individual observations in this interval be unknown. However we have observations outside this interval: $X_1, X_2,..., X_n$. It is required to estimate μ by these observations. For this we use maximum pseudo-likelihood estimators.

To construct the likelihood function, note that if we denote by k the number of terms of the total sampling which have occurred in $[a,b]$, then $\dfrac{k}{k+n}$ will be the frequency of occurrences in the interval $[a,b]$. Therefore, by the Bernoulli-Kholmogorov theorem, $\dfrac{k}{k+n}\to p$ a.s., where $p = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$, $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^3}{2}} du$. Thus, as k, we take

$$\hat{k} = \frac{n\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]}{1 - \left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]}.$$

Since the probability that exactly k elements from the sampling will occur in the «black hole » is $\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^k$, we can write the pseudo-likelihood function in the form

$$L = \prod_{i=1}^{n} \frac{1}{\sigma} \varphi\left(\frac{X_i - \mu}{\sigma}\right) \cdot \left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]^{\frac{n\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]}{1 - \left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]}}, \qquad (5)$$

where as usual we denote $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$. Note that the power in (5) may turn out to be a non-integral number. However it is always positive. Note also that the multiplication sign is related to the expression in front of the large bracket.

From (5) we obtain

$$\ln L = -n \ln\left(\sqrt{2\pi}\sigma\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(X_i - \mu\right)^2 +$$

$$+ \frac{n\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]}{1 - \left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} \cdot \ln\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]$$

.

Hence we calculate

$$\frac{d}{d\mu} \ln L = \frac{1}{\sigma^2} \sum_{i=1}^{n} X_i - \frac{n}{\sigma^2} \mu -$$

$$-\frac{\frac{1}{\sigma}n\left[\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)\right]\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}}{\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}^2}\cdot\ln\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]-$$

$$-\frac{\frac{1}{\sigma}n\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\cdot\left[\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)\right]}{\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}^2}\cdot\ln\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]-$$

$$-\frac{\frac{1}{\sigma}n\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\cdot\left[\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)\right]}{\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}\cdot\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]}=$$

$$=\frac{1}{\sigma^2}\sum_{i=1}^{n}X_i-\frac{n}{\sigma^2}\mu-\frac{n}{\sigma}\frac{\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)}{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]}-$$

$$-\frac{n}{\sigma}\frac{\left[\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)\right]\ln\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]}{\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}^2}=$$

$$=\frac{1}{\sigma^2}\sum_{i=1}^{n}X_i-\frac{n}{\sigma^2}\mu-\frac{n}{\sigma}\cdot\frac{\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)}{\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}^2}\cdot\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\right.\right.$$

$$\left.\left.-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]+\ln\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}. \qquad (6)$$

Let us study the expression (6) for $\mu \to -\infty$. Denote, for the sake of brevity, $\dfrac{b-\mu}{\sigma} = t$, $\dfrac{a-\mu}{\sigma} = s$ and note that for $\mu \to -\infty$ we obtain $t \to \infty$, $s \to \infty$ and $\dfrac{t}{s} \to 1$. Therefore

$$\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \downarrow 0, \quad \varphi\left(\frac{b-\mu}{\sigma}\right) - \varphi\left(\frac{a-\mu}{\sigma}\right) \uparrow 0$$

and

$$\ln\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right] \downarrow -\infty .$$

By the Cauchy mean value theorem,

$$\frac{\varphi(t) - \varphi(s)}{\Phi(t) - \Phi(s)} = -\tau \quad \text{where} \quad s < \tau < t .$$

Note that for $\mu \to -\infty$ there exists the limit $\dfrac{\tau}{t} \to 1$.

The denominator in (6) tends to 1 and the first two summands in the braces also tend to 1. So, we have to calculate the limit of the expression

$$[\varphi(t) - \varphi(s)] \ln[\Phi(t) - \Phi(s)] \quad \text{as} \quad \mu \to -\infty .$$

By l'Hopital's rule,

$$0 \le \lim_{\mu \to -\infty} [\varphi(t) - \varphi(s)] \cdot \ln[\Phi(t) - \Phi(s)] = \lim_{\mu \to -\infty} \frac{\ln[\Phi(t) - \Phi(s)]}{\dfrac{1}{\varphi(t) - \varphi(s)}} = \lim_{\mu \to -\infty} \frac{-\dfrac{1}{\sigma} \dfrac{\varphi(t) - \varphi(s)}{\Phi(t) - \Phi(s)}}{-\dfrac{1}{\sigma} \dfrac{t\varphi(t) - s\varphi(s)}{[\varphi(t) - \varphi(s)]^2}} =$$

$$= \lim_{\mu \to -\infty} \frac{[\varphi(t) - \varphi(s)]^3}{[\Phi(t) - \Phi(s)] \cdot [t\varphi(t) - s\varphi(s)]} \le \lim_{\mu \to -\infty} \frac{[\varphi(t) - \varphi(s)]^2}{t \cdot [\Phi(t) - \Phi(s)]} = -\lim_{\mu \to -\infty} \frac{\tau}{t} \cdot [\varphi(t) - \varphi(s)] = 0.$$

Therefore $\lim\limits_{\mu \to -\infty} \dfrac{d}{d\mu}(t) \ln L = \infty$.

Analogously, for $\mu \to \infty$ we have $t \to -\infty$, $s \to -\infty$ and

$$\lim_{\mu \to \infty} [\varphi(t) - \varphi(s)] \cdot \ln[\Phi(t) - \Phi(s)] = 0 .$$

Therefore $\lim\limits_{\mu\to\infty}\dfrac{d}{d\mu}\ln L=-\infty$.

The continuous function $\dfrac{d}{d\mu}\ln L$ changes the sign and therefore there exists a point $\hat{\mu}$ such that $\left.\dfrac{d}{d\mu}\ln L\right|_{\mu=\hat{\mu}}=0$. Let us verify that the second derivative at this point is negative.

We write the second derivative in the form

$$\frac{d^2}{d\mu^2}\ln L=$$

$$=-\frac{n}{\sigma^2}-\frac{n}{\sigma^2}\cdot\frac{[t\varphi(t)-s\varphi(s)]\cdot\{1-[\Phi(t)-\Phi(s)]+\ln[\varphi(t)-\varphi(s)]\}\cdot\{1-[\Phi(t)-\Phi(s)]\}}{\{1-[\Phi(t)-\Phi(s)]\}^3}+$$

$$+\frac{n}{\sigma^2}\cdot\frac{[\varphi(t)-\varphi(s)]\cdot\left\{\varphi(t)-\varphi(s)-\dfrac{\varphi(t)-\varphi(s)}{\Phi(t)-\Phi(s)}\right\}}{\{1-[\Phi(t)-\Phi(s)]\}^2}+$$

$$+\frac{n}{\sigma^2}\cdot\frac{2\cdot[\varphi(t)-\varphi(s)]^2\cdot\{1-[\Phi(t)-\Phi(s)]+\ln[\Phi(t)-\Phi(s)]\}}{\{1-[\Phi(t)-\Phi(s)]\}^3}.$$

We have to show that these four fractions, when summed, have a negative value. The first fraction is negative. For the third fraction

$$\frac{n}{\sigma^2}\cdot\frac{[\varphi(t)-\varphi(s)]^2\cdot\left\{1-\dfrac{1}{\Phi(t)-\Phi(s)}\right\}}{\{1-[\Phi(t)-\Phi(s)]\}^2}$$

we have a negative value since it is obvious that $\Phi(t)-\Phi(s)\le 1$. It remains to show that

$$\{1-[\Phi(t)-\Phi(s)]+\ln[\Phi(t)-\Phi(s)]\}\cdot\left\{[t\varphi(t)-s\varphi(s)]\cdot[1-[\Phi(t)-\Phi(s)]]+2[\varphi(t)-\varphi(s)]^2\right\}\ge 0.$$

Since for $0<x\le 1$ we have $1-t+\ln t\ge 0$, we need to prove that

$$[t\varphi(t)-s\varphi(s)]\cdot\{1-[\Phi(t)-\Phi(s)]\}+2[\varphi(t)-\varphi(s)]^2\ge 0. \qquad (7)$$

In this expression only $t\varphi(t)-s\varphi(s)$ may be negative. This may happen in two cases: when $s<-1$ and $t<0$ or $s>0$ and $t>1$. Because of symmetry we consider the second case, in which it is assumed that $\begin{cases}\hat{\mu}<a\\ \hat{\mu}<b-\sigma\end{cases}$. We use Kulldorf's inequality (see [1, p. 134]), by which

$$[t\varphi(t)-s\varphi(s)]\cdot[\Phi(t)-\Phi(s)]+[\varphi(t)-\varphi(s)]^2>(t-s)\cdot\varphi(t)\varphi(s).$$

Since $1-[\Phi(t)-\Phi(s)]>\Phi(t)-\Phi(s)$, we have

$$[t\varphi(t)-s\varphi(s)]\cdot\{1-[\Phi(t)-\Phi(s)]\}+2[\varphi(t)-\varphi(s)]^2>[t\varphi(t)-s\varphi(s)]\cdot[\Phi(t)-\Phi(s)]+$$
$$+2[\varphi(t)-\varphi(s)]^2>(t-s)\varphi(t)\varphi(s)+[\varphi(t)-\varphi(s)]>0.$$

The second case with $s<-1$ and $t<0$ is considered in analogous manner.

Thus $\frac{d^2}{d\mu^2}\ln L<0$ and therefore $\hat{\mu}$ is a unique maximum point for the likelihood function (5). Taking Lemmas 1 and 2 into account, we can state that the following theorem is valid.

Theorem. *Assume that we have the sampling of a normal random value* $X_1, X_2, ..., X_n$ *with the unknown mathematical expectation* μ *and the known dispersion* σ^2*. Observation is carried out outside the interval* $[a,b]$ *where neither sampling terms nor their quantity are registered. Then the estimator of a maximum pseudo-likelihood exists for* μ *and is the unique root of the equation*

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}X_i-\frac{n}{\sigma^2}\mu-\frac{n}{\sigma}\cdot\frac{\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)}{\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}^2}\cdot\left\{1-\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\right.\right.$$
$$\left.\left.-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]+\ln\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]\right\}=0. \quad (8)$$

Moreover, this estimator is asymptotically consistent and effective..

Remark 1. While proving the theorem, we see that a or b may be infinite. The case $a=-\infty$ and $-\infty<b<\infty$ correspond to the case of left censoring, whereas for $-\infty<a<\infty$ and $b=\infty$ we have the right censoring.

Example 1. If $a=-\infty$ and $b=0$, then we obtain the following equation for defining $\hat{\mu}$:

$$\mu+\sigma\frac{\varphi\left(\frac{\mu}{\sigma}\right)}{\Phi\left(\frac{\mu}{\sigma}\right)}-\frac{\sigma^2}{n}\ln\left[1-\Phi\left(\frac{\mu}{\sigma}\right)\right]=\frac{1}{n}\sum_{i=1}^{n}X_i\,.$$

Example 2. As has been shown in proving the theorem, for $b\to a$ the equation (8) implies the classical case $\hat{\mu}=\frac{1}{n}\sum_{i=1}^{n}X_i$.

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ელიზბარ ნადარაია, მზია ფაცაცია,
გრიგოლ სოხაძე

განაწილების პარამეტრების ფსევდო-მაქსიმალური დასაჯერობის მეთოდით შეფასების შესახებ ცენზურირებული დაჯგუფებული მონაცემებისათვის

ნაშრომში განხილულია ალბათური განაწილების პარამეტრების შეფასება დაჯგუფებული მონაცემებისათვის. ამასთან, ზოგიერთ ჯგუფებში დაკვირვებადია შემთხვევითი სიდიდის არა ინდივიდუალური მნიშვნელობები, არამედ მათი საერთო რაოდენობა, ხოლო ზოგიერთ ჯგუფში დამკვირვებლისათვის მიუწვდომელია მათი რაოდენობაც კი. ასეთი ტიპის შერჩევისათვის გამოყენებულია ფსევდო-მაქსიმალური დასაჯერობის მეთოდი, შესწავლილია შეფასებების ასიმპტოტურად ძალდებულობა და ასიმპტოტურად ეფექტურობა. მაგალითის სახით განხილულია ნორმალური განაწილება.

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GOGI PANTSULAIA

ON A RIEMANN INTEGRABILITY OF FUNCTIONS DEFINED ON INFINITE-DIEMENSIONAL RECTANGLES

Abstract. We announce a result asserted that for a Riemann integrable function f defined on the infinite-dimensional rectangle $\prod_{i=1}^{\infty}[a_i,b_i]$ OB , an infinite-dimensional version of the Weyl theorem is valid. This fact allows us to give an effective algorithm for a calculation of the Riemann integral $(R)\ \mathrm{T}_{\mathrm{X}_{i=1}^{\Gamma}[a_i,b_i]} f(x)dl(x)$, where l denotes an infinite-dimensional 'Lebesgue measure' constructed by R.Baker in 1991.

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Key words and phrases: Riemann integrability, Lebesgue measure, infinite-dimensional rectangles

1. Introduction

Various questions combinatorial or discrete type frequently arise in different domains of modern mathematics (especially, in Mathematical analysis, Measure theory, Differential equations, Game theory, Set Theory, Graph Theory etc.) and are important from the theoretical view-point and from the view-point of their numerous applications. In particular, these questions play a key role in applications of algorithms and computer science. For example, the notion of a equidistributed, or uniformly distributed sequences $(a_n)_{nON}$ in an interval $[a,b]$ describes a certain discrete mathematical structure which has various interesting applications from the view-point of its applications in the theory of algorithms and computer science. In particular, by Weyl well known theorem, for every Rieman integrable function f on $[a,b]$, the following equality

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(a_k) = \frac{(R)\int_a^b f(x)dx}{b-a}$$

holds if and only if the sequence $(a_n)_{n\in N}$ is equidistributed, or uniformly distributed in an interval $[a,b]$, where $(R)\int_a^b f(x)dx$ denotes Riemann integral of the f over $[a,b]$. Moreover, in common cases, it is possible to estimate the velocity of such a convergence.

To various applications of a equidistributed, or uniformly distributed sequences is devoted the well known monograph of L. Kuipers and H. Niederreiter [1]. Firstly, such sequences have been used by Hardy and Littlewood [2] in Diophantine approximation theory.

It is natural to consider

Problem 1.1. *Whether one can elaborate a theory of* uniformly distributed sequences *for infinite-dimensional rectangles ?.*

Since the theory of uniformly distributed sequences for finite-dimensional rectangles essentially implies the technique of the Lebesgue measure, here arises the following

Problem 1.2. *What measures in infinite-dimensional topological vector space R^{∞} can be assumed as partial analogs of the n-dimensional classical Lebesgue measure (defined on the Euclidean vector space R^n) ?*

In this direction, we must say that a partial analog of the Lebesgue measure on R^{∞} is not defined uniquely. Problem 1.2 was solved by African mathematician R.Baker [3]. He introduced a notion of "Lebesgue measure" on R^{∞} as follows: a measure λ being a completion of a shift-invariant Borel measure on R^{∞} is called a "Lebesgue measure" on R^{∞} if for any infinite-dimensional rectangle

$$\prod_{i=1}^{\infty}[a_i,b_i]\ (-\infty<a_i<b_i<+\infty)\ \text{with}\ 0\le\prod_{i=1}^{\infty}(b_i-a_i)<+\infty$$

the equality

$$\lambda(\prod_{i=1}^{\infty}(a_i,b_i))=\prod_{i=1}^{\infty}(b_i-a_i)$$

holds, where

$$\prod_{i=1}^{\infty}(b_i-a_i):=\lim_{n\to+\infty}\prod_{i=1}^{n}(b_i-a_i).$$

In 2004, simultaneously have been published articles [4] and [5] which also contain solution of Problem 1.2. In [4] has been posed a question asking whether measures [3] and [4] coincide. The negative answer to this question has been obtained in [6]. In this manuscript has been introduced the notion of generators of shy sets in Polish topological vector spaces and has been done their various interesting applications. In particular, here has been demonstrated that this class contains specific measures which naturally generate early implicitly introduced classes of null sets. For example, here has been constructed : 1) Mankiewicz generator which generate exactly the class of all cube null sets; 2) Preiss - Tiser generators which generate exactly the class of all Preiss -Tiser null sets , etc. Moreover, such measures (unlike s -finite Borel measures) possess many interesting, sometimes unexpected, geometric properties. New concepts of the "Lebesgue measure" on R^{Γ} (the so called, a -standard and a -ordinary Lebesgue measures) have been proposed and their some realizations in the **ZFC** theory have been considered in [7] . Also, it has been shown that Baker's both measures [3] and [4], Mankiewicz and Preiss - Tiser generators [6] and the measure [5] are not an a -standard Lebesgue measure on R^{Γ} for $a = (1,1,L)$.

Under such a rich class of shifp-invariant Borel measures in R^{Γ} it is natural to consider the following problems:

Problem 1.3. W*hether one can elaborate the theory of Riemann integration in* R^{Γ} ?

Problem 1.4. *In terms of partial analogs of the Lebesgue measure in* R^{Γ} *, whether one can elaborate theory of uniformly destributed sequences in infinite-dimensional rectangles*?

Note that the solution of Problems **1.**3-**1.**4 assumes an infinite generalization of well known classical results (for example, Lebesgue theorem about Riemann integrability, Weyl theorem, etc) formulated in terms of Lebesgue measure (on R^n) to an infinite-dimensional topological vector spaces of all real valued sequences R^{Γ} (equipped with Tikhonov topology) in terms of partial analogs of the Lebesgue measure (for example, Mankiewicz generator, Baker measures, standard Lebesgue measure, etc).

The purpose of the present manuscript is to consider a certain concept for a partial solution of the Problems **1.**3-**1.**4 in terms of the measure λ [3] and to announce main results established in [9].

The paper is organized as follows.

In Section 2 we give some auxiliary definitions from the Riemann integrability theory. In Section 3 we announce our main results established in [9].

2. Auxiliary definitions and notions

In order to solve Problem **1.**3 for "Lebesgue measure" λ [3], we need a notion of Riemann integrability on infinite-dimensional rectangle in terms of the measure λ.

We denote by B a class of all measurable rectangle

$\prod_{i=1}^{\infty}[a_i,b_i]$ ($-\infty<a_i<b_i<+\infty$) with $0<\prod_{i=1}^{\infty}(b_i-a_i)<+\infty$.

It is clear that an equality $\lambda(\prod_{i=1}^{\infty}(a_i,b_i))=\prod_{i=1}^{\infty}(b_i-a_i)$ holds.

A set U is called an elementary rectangle in the $\prod_{i=1}^{\infty}[a_i,b_i]$ if it admits the following representation

$$U=\prod_{k=1}^{m}[c_k,d_k]\times\prod_{k=m+1}^{\infty}[a_k,b_k]$$

where $a_k \le c_k < d_k \le b_k$ for $1 \le k \le m$ ($m \in N$).

It is obvious that

$$\lambda(U)=\prod_{k=1}^{m}(d_k-c_k)\times\prod_{k=m+1}^{\infty}(b_k-a_k)$$

for the elementary rectangle U.

Also, a number $d(U)$, defined by

$$d(U)=\sum_{k=1}^{m}\frac{d_k-c_k}{2^{k+1}(1+d_k-c_k)}+\sum_{k=m+1}^{\infty}\frac{b_k-a_k}{2^{k+1}(1+b_k-a_k)},$$

is diameter of the U with respect to Tikhonov metric in R^{∞}.

A family of pairwise disjoint elementary rectangles $\tau = (U_k)_{1 \le k \le n}$ of the $\prod_{i=1}^{\infty}[a_i,b_i]$ is called Riemann partition of the $\prod_{i=1}^{\infty}[a_i,b_i]$ if $\prod_{i=1}^{\infty}[a_i,b_i]=\bigcup_{k=1}^{n} U_k$.

A number $d(\tau)$, defined by

$$d(\tau)=\max\{d(U_k):1 \le k \le n\}$$

is called mesh or norm of the Riemann partition τ.

Let f be a real-valued bounded function defined on $\prod_{i=1}^{\infty}[a_i,b_i]$.

Definition 2.1 We say that the f is Riemann-integrable on $\prod_{i=1}^{\infty}[a_i,b_i]$ if there exists a real number s such that for every positive real number ε there exists a real number $\delta > 0$ such that, for every Riemann partition $\tau = (U_k)_{1 \le k \le n}$ of the $\prod_{i=1}^{\infty}[a_i,b_i]$ with $d(\tau) < \delta$ and for every sample $(t_k)_{1 \le k \le n}$ with $t_k \in U_k (1 \le k \le n)$, we have

$$\left|\sum_{k=1}^{n} f(t_k)\lambda(U_k) - s\right| < \varepsilon$$

The number s is called Riemann integral from the f over $\prod_{i=1}^{\infty}[a_i,b_i]$ and is denoted by $(R)\int_{\prod_{i=1}^{\infty}[a_i,b_i]} f(x)d\lambda(x)$.

Definition 2.2. We say that a real-valued function $f:\mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is Riemann integrable with respect to the measure λ if there exists a countable family $(B_k)_{k \in N}$ of pairwise disjoint elements of the $\mathcal{B}$

that

(i) $(\forall k)(k \in N \to f|_{B_k}$ is Riemann integrable on $B_k)$, where $f|_{B_k}$ denotes a restriction of the f on B_k;

(ii) $(\forall x)(x \in \mathbb{R}^{\infty} \setminus \cup_{k\in N} B_k \to f(x)=0)$;

(iii) The series $\sum_{k\in N} (R)\int_{B_k} f|_{B_k}(x)d\lambda(x)$ is absolutely convergent.

If for the function f the conditions (i)-(iii) fulfilled, then a number

$$\sum_{k\in N} (R)\int_{B_k} f|_{B_k}(x)d\lambda(x)$$

is called Riemann integral of the f over $\mathbb{R}^{\infty}$ with respect to λ and is denoted by

$$(R)\int_{\mathbb{R}^{\infty}} f(x)d\lambda(x).$$

Note, that a first step to forward for resolution of Problem 1.3 will be an investigation of the following

Problem 2.1 Describe in terms of the measure λ a class of all *real-valued Riemann integrable functions on* $\prod_{i=1}^{\infty}[a_i,b_i]$.

Definition 2.3. An increasing sequence $(Y_n)_{n\in N}$ of finite subsets of the infinite-dimensional rectangle $\prod_{i=1}^{\infty}[a_i,b_i] \in B$ is said to be uniformly distributed in the $\prod_{i=1}^{\infty}[a_i,b_i]$, if, for every elementary rectangle U in the $\prod_{i=1}^{\infty}[a_i,b_i]$, we have

$$\lim_{n\to\infty} \frac{\#(Y_n \cap U)}{\#(Y_n)} = \frac{\lambda(U)}{\lambda(\prod_{i=1}^{\infty}[a_i,b_i])}.$$

2. Main Results

Present section we announce the main results obtained in [9].

Theorem 3.1 *Let* $\prod_{i=1}^{\infty}[a_i,b_i] \in B$. *Let* $(x_n^{(k)})_{n\in N}$ *be uniformly distributed in the interval* $[a_k,b_k]$ *for* $k \in N$. *We set* $Y_n = \prod_{k=1}^{n}(\cup_{j=1}^{n} x_j^{(k)}) \times \prod_{k>n}\{a_k\}$. *Then* $(Y_n)_{n\in N}$ *is uniformly distributed in the* $\prod_{i=1}^{\infty}[a_i,b_i]$,

Theorem 3.2 *Let* f *be a continuous (w.r.t. Tikhonov metric) function on* $\prod_{i=1}^{\infty}[a_i,b_i]$. *Then the* f *is Riemann-integrable on* $\prod_{i=1}^{\infty}[a_i,b_i]$.

We have the following infinite-dimensional version of the Lebesgue theorem (see, [10], Lebesgue Theorem , p.359).

Theorem 3.3 *Let* f *be a bounded real-valued function on* $\prod_{i=1}^{\infty}[a_i,b_i]$. *Then* f *is Riemann integrable on* $\prod_{i=1}^{\infty}[a_i,b_i]$ *if and only if* f *is* λ *-almost continuous (w.r.t. Tikhonov metric) on* $\prod_{i=1}^{\infty}[a_i,b_i]$.

Theorem 3.4 *For* $\prod_{i=1}^{\infty}[a_i,b_i] \in B$, *let* $(Y_n)_{n\in N}$ *be an increasing family its finite subsets. Then* $(Y_n)_{n\in N}$ *is uniformly distributed in the* $\prod_{i=1}^{\infty}[a_i,b_i]$ *if and only if for every continuous (w.r.t. Tikhonov metric) function* f *on* $\prod_{i=1}^{\infty}[a_i,b_i]$ *the following equality*

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n} f(y)}{\#(Y_n)} = \frac{(R)\int_{\prod_{i=1}^{\infty}[a_i,b_i]} f(x)d\lambda(x)}{\lambda(\prod_{i=1}^{\infty}[a_i,b_i])}$$

holds.

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გოგი ფანცულაია

უსასრულო-განზომილებიან პარალელეპიპედზე განსაზღვრული ფუნქციების რიმანის აზრით ინტეგრებადობის შესახებ

ნაშრომში ანონსირებულია ის ფაქტი, რომ უსასრულო-განზომილებიან $\prod_{i=1}^{\infty}[a_i,b_i]$ OB პარალელეპიპედზე განსაზღვრული რიმანის აზრით ინტეგრებადი f ფუნქციისათვის მართებულია ვეილის თეორემის უსასრულო-განზომილებიანი ანალოგი. აღნიშნული ფაქტი იძლევა რიმანის

$$(R)\ \mathrm{T}_{\overset{\Gamma}{\underset{i=1}{\mathrm{X}}}[a_i,b_i]}\ f(x)dl\,(x)$$

ინტეგრალის გამოთვლის ეფექტურ ალგორითმს, სადაც l აღნიშნავს რ. ბეიკერის მიერ 1991 წელს აგებულ უსასრულო-განზომილებიან „ლებეგის ზომას“.

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USHANGI GOGINAVA

CONVERGENCE IN MEASURE OF TWO-DIMENSIONAL CONJUGATE WALSH-FEJER MEANS

Abstract. The main aim of this paper is to prove that for any Orlicz space, which is not a subspace of $L\log L(I^2)$, the set of the functions that quadratic conjugate Fejer means $(\alpha = \beta = 2/3)$ of the double Walsh-Fourier series converges in measure is of first Baire category.

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Key words and phrases: Conjugate Walsh-Fourier series, Orlicz space, Convergence in measure

1. Introduction

The partial sums $S_n(f)$ of the Walsh-Fourier series of a function $f \in L(I), \quad I = [0,1)$ converges in measure on I [9, 14]. The condition $f \in L\ln^+ L(I^2)$ provides convergence in measure on I^2 of the rectangular partial sums $S_{n,m}(f)$ of double Fourier-Walsh series [19]. The first example of a function from classes wider than $L\ln^+ L(I^2)$ with $S_{n,n}(f)$ divergent in measure on I^2 was obtained by Getsadze in [5]. Moreover, Tkebuchava [15] proved that in each Orlicz space wider than $L\ln^+ L(I^2)$ the set of functions with quadratic Walsh-Fourier sums converge in measure on I^2 is of first Baire category.

Weisz [16, 17] proved that conjugate Fejйr means of double Walsh-Fourier $\tilde{\sigma}_{n,m}^{(\alpha,\beta)} f \quad ,\alpha,\beta \in [0,1)$ converges a. e. for each $f \in L\ln^+ L(I^2)$. In particular, the condition $f \in L\ln^+ L(I^2)$ provides the convergence in measure on I^2 of the $\tilde{\sigma}_{n,m}^{(\alpha,\beta)} f, \alpha, \beta \in [0,1)$. In this paper we prove that the Conjugate

Fejйr means of the double Walsh-Fourier series does not improve the convergence in measure. In other words, we prove that for any Orlicz space, which is not a subspace of $L\log L(I^2)$, the set of the functions that quadratic conjugate Fejer means $(\alpha = \beta = 2/3)$ of the double Walsh-Fourier series converges in measure is of first Baire category.

For results with respect to divergence of measure of double Walsh-Fourier series

see [2- 4, 6- 8, 10, 12].

2. Definitions and Notation

We denote by $L^0 = L^0(I^2)$ the Lebesque space of functions that are measurable and finite almost everywhere on $I^2 = [0,1)\times[0,1)$. $mes(A)$ is the Lebesque measure of the set $A \subset I^2$. The constants appearing in the article denoted by c.

Let $L_\Phi = L_\Phi(I^2)$ be the Orlicz space [11] generated by Young function Φ, i.e. Φ is convex continuous even function such that $\Phi(0) = 0$ and

$$\lim_{u\to+\infty}\frac{\Phi(u)}{u} = +\infty,\ \lim_{u\to 0}\frac{\Phi(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_\Phi(I^2)} = \inf\{k>0 : \int_{I^2}\Phi(|f(x,y)|/k)\,dxdy \le 1\}.$$

In particular in case if $\Phi(u) = u\ln(1+u),\ u>0$, then corresponding space will denote by $L\ln^+ L(I^2)$.

It is well-known [11] that $L_\Phi \subset L_\Psi \Leftrightarrow \underline{\lim}_{u\to\infty}\dfrac{\Phi(u)}{\Psi(u)} > 0.$

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1,\ if\ \ x\in[0,1/2) \\ -1,\ if\ \ x\in[1/2,1) \end{cases},\quad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x),\ n\ge 1 \qquad and \qquad x\in[0,1).$$

Let $w_0, w_1, \ldots$ represent the Walsh functions [13], i.e. $w_0(x) = 1$ and if $k = 2^{k_1} + \cdots + 2^{k_s}$ is a positive integer with $k_1 > k_2 > \cdots > k_s \geq 0$ then

$$w_k(x) = r_{k_1}(x) \cdots r_{k_s}(x).$$

Every point $\alpha \in [0,1)$ can be written in the following way:

$$\alpha = \sum_{k=0}^{\infty} \frac{\alpha_k}{2^{k+1}}, \alpha_k = 0,1.$$

In case there are two different forms we choose the one for which $\lim_{k \to \infty} \alpha_k = 0.$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, if \quad x \in [0, 1/2^n), \\ 0, \; if \quad x \in [1/2^n, 1). \end{cases}$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m = 0,1,2,\ldots\}$ on the unit square I^2.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}(f, x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m,n) w_m(x) w_n(y),$$

where the number

$$\hat{f}(m,n) = \int_{I^2} f(x,y) w_m(x) w_n(y) dxdy$$

is (m,n)th Walsh-Fourier coefficient.

Let $\rho_0 := r_0, \rho_k := r_n$ if $2^n \leq k < 2^{n+1}$. Then the (N,M)th partial sums of the conjugate transforms is given by

$$\tilde{S}_{n,m}^{(\alpha,\beta)}(f, x, y) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \rho_k(\alpha) \rho_l(\beta) \hat{f}(k,l) w_k(x) w_l(y).$$

The conjugate Fejér means of a function f are defined be

$$\tilde{\sigma}_{n,m}^{(\alpha,\beta)}(f,x,y) := \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{S}_{n,m}^{(\alpha,\beta)}(f,x,y).$$

It is simple to show that

$$\tilde{\sigma}_{N,M}^{(\alpha,\beta)}(f,x,y) = \int_{I^2} f(t,u)\tilde{K}_M^{(\alpha)}(x\oplus t)\tilde{K}_N^{(\beta)}(y\oplus u)\,dtdu,$$

where

$$\tilde{K}_M^{(\alpha)}(x) := \frac{1}{n}\sum_{j=1}^{M}\tilde{D}_j^{(\alpha)}(x),$$

$$\tilde{D}_j^{(\alpha)}(x) := \sum_{i=0}^{j-1} r_i(\alpha)\left(D_{2^{i+1}}(x) - D_{2^i}(x)\right)$$

and $\oplus$ denotes dyadic addition (see [9, 14]).

3. Main Results

The main results of this paper are presented in the following proposition.

Theorem 1 *Let* $L_\Phi(I^2)$ *be an Orlicz space, such that*

$$L_\Phi(I^2) \not\subseteq L\ln^+ L(I^2).$$

Then the set of the functions from the Orlicz space $L_\Phi(I^2)$ with conjugate Fejйr means $\tilde{\sigma}_{n,n}^{(2/3,2/3)} f$ convergent in measure on I^2 is of first Baire category in $L_\Phi(I^2)$.

Corollary 1 *Let* $\varphi : [0,\infty[\to [0,\infty[$ *be a nondecreasing function satisfying for* $x \to +\infty$ *the condition*

$$\varphi(x) = o(x \log x).$$

Then there exists the function $f \in L(I^2)$ such that a)

$$\int_{I^2} \varphi(|f(x,y)|)\,dxdy < \infty;$$

b) conjugate Fejйr means $\tilde{\sigma}_{n,n}^{(2/3,2/3)} f$ of the function f diverges in measure

on I^2.

4. Auxiliary Results

Lemma 1 *Let* $x \in \left(\frac{1}{2^l}, \frac{1}{2^{l-1}}\right), l = 1,2,\ldots,m.$ *Them*

$$\left|\tilde{K}_{2m}^{(2/3)}(x)\right| \geq \frac{c}{x}.$$

Proof. Let

$$\tilde{D}_n^{(\alpha)}(x) := \sum_{j=0}^{n-1} r_j(\alpha)\left(D_{2^{j+1}}(x) - D_{2^j}(x)\right)$$

$$= \sum_{j=0}^{n-1} (-1)^{\alpha_j}\left(D_{2^{j+1}}(x) - D_{2^j}(x)\right)$$

$$= \sum_{j=0}^{n-1} (-1)^{\alpha_j} r_j(x) D_{2^j}(x).$$

On the other hand

$$\sum_{k=0}^{n-1} \alpha_k r_k(x) D_{2^k}(x) = \sum_{k=0}^{n-1} \frac{1-(-1)^{\alpha_k}}{2} r_k(x) D_{2^k}(x)$$

$$= \frac{1}{2}\sum_{k=0}^{n-1} r_k(x) D_{2^k}(x) - \frac{1}{2}\sum_{k=0}^{n-1} (-1)^{\alpha_k} r_k(x) D_{2^k}(x)$$

$$= \frac{1}{2}\left(D_{2^n}(x) - 1\right) - \frac{1}{2}\tilde{D}_n^{(\alpha)}(x).$$

Hence

$$\tilde{D}_n^{(\alpha)}(x) = D_{2^n}(x) - 1 - 2\sum_{k=0}^{n-1} \alpha_k r_k(x) D_{2^k}(x),$$

$$\tilde{K}_m^{(\alpha)} := \frac{1}{m}\sum_{n=1}^{m}\tilde{D}_n^{(\alpha)}(x) = \frac{1}{m}\sum_{n=1}^{m}\left(D_{2^n}(x)-1\right)-\frac{2}{m}\sum_{n=1}^{m}\sum_{k=0}^{n-1}\alpha_k r_k(x) D_{2^k}(x)$$

$$= \frac{1}{m}\sum_{n=1}^{m}D_{2^n} - 1 - \frac{2}{m}\sum_{k=0}^{m-1}(m-k)\alpha_k r_k(x) D_{2^k}(x).$$

Let $x \in \left(\frac{1}{2^l}, \frac{1}{2^{l-1}}\right), l = 1,2,\ldots,m.$ Then from (1) we have

$$\tilde{K}_{2m}^{(\alpha)}(x) = \frac{1}{2m}\sum_{n=1}^{l-1}D_{2^n}(x) - 1 - \frac{1}{m}\sum_{k=0}^{l-1}(2m-k)\alpha_k r_k(x) D_{2^k}(x).$$

It is evident that for $\alpha = \frac{2}{3}$ we have

$$\alpha_k = \begin{cases} 1, k = 2j \\ 0, k = 2j+1 \end{cases}$$

Let l is odd number. Then

$$\tilde{K}_{2m}^{(2/3)}(x) = \frac{1}{2m}\sum_{n=1}^{l-1}D_{2^n}(x) - 1 - \frac{1}{m}\sum_{k=0}^{(l-1)/2}(2m-2k) r_{2k}(x) D_{2^{2k}}(x),$$

$$\left|\tilde{K}_{2m}^{(2/3)}\right| \geq \frac{1}{m}(2m-l)2^{l-1} - \frac{1}{2m}\sum_{n=1}^{l-1}D_{2^n} - 1 - \frac{1}{m}\sum_{k=0}^{(l-1)/2-1}(2m-2k)2^{2k}$$

$$\geq 2^{l-1} - \frac{1}{2m}2^l - 1 - \frac{2^l}{3}$$

$$\geq c2^l$$

$$\geq \frac{c}{x}.$$

Let l is even number. Then

$$
\begin{aligned}
\left|\tilde{K}_{2m}^{(2/3)}\right| &\geq \frac{1}{m}\left(2m-l+2\right)2^{l-2}-\frac{1}{2m}\sum_{n=1}^{l-1}D_{2^n}-1-\frac{1}{m}\sum_{k=0}^{(l-2)/2-1}\left(2m-2k\right)2^{2k}\\
&\geq 2^{l-2}-\frac{1}{2m}2^{l}-1-\frac{2^{l-1}}{3}\\
&\geq c2^{l}\\
&\geq \frac{c}{x}.
\end{aligned}
$$

Combining (2) and (3) we complete the proof of Lemma 1.

We apply the reasoning of [1] formulated as the following proposition in particular case.

Lemma 2 *[1]Let* $H: L^1(I^2)\to L^0(I^2)$ *be a linear continuous operator, which commutes with family of translations* E*, i. e.* $\forall E\in \mathrm{E}\quad \forall f\in L^1(I^2)\quad HEf = EHf$*. Let* $\|f\|_{L^1(I^2)}=1$ *and* $\lambda>1$*. Then for any* $1\leq r\in \mathrm{N}$ *under condition* $mes\{(x,y)\in I^2 : |Hf|>\lambda\}\geq \frac{1}{r}$ *there exist* $E_1,\ldots,E_r,E_1^{'},\ldots,E_r^{'}\in \mathrm{E}$ *and* $\varepsilon_i=\pm 1,\quad i=1,\ldots,r$ *such that*

$$
mes\{(x,y)\in I^2 : |H(\sum_{i=1}^{r}\varepsilon_i f(E_i x, E_i^{'} y)|>\lambda\}\geq \frac{1}{8}.
$$

Lemma 3 *[18]Let* $\{H_m\}_{m=1}^{\infty}$ *be a sequence of linear continues operators, acting from Orlicz space* $L_\Phi(I^2)$ *in to the space* $L^0(I^2)$*. Suppose that there exists the sequence of functions* $\{\xi_k\}_{k=1}^{\infty}$ *from unit bull* $S_\Phi(0,1)$ *of space* $L_\Phi(I^2)$*, sequences of integers* $\{m_k\}_{k=1}^{\infty}$ *and* $\{\nu_k\}_{k=1}^{\infty}$ *increasing to infinity such that*

$$
\varepsilon_0=\inf_k mes\{(x,y)\in I^2 : |H_{m_k}\xi_k(x,y)|>\nu_k\}>0.
$$

Then B - the set of functions f from space $L_\Phi(I^2)$, for which the sequence $\{H_m f\}$ converges in measure to an a. e. finite function is of first Baire category in space $L_\Phi(I^2)$.

Lemma 4 *[18]Let* L_{Φ} *be an Orlicz space and let* $\varphi:[0,\infty)\to[0,\infty)$ *be measurable function with condition* $\varphi(x)=o(\Phi(x))$ *as* $x\to\infty$*. Then there exists Orlicz space* L_{ω}*, such that* $\omega(x)=o(\Phi(x))$ *as* $x\to\infty$*, and* $\omega(x)\geq\varphi(x)$ *for* $x\geq c\geq 0$*.*

5. Proof of the Theorem

Proof of Theorem 1. By Lemma 3 the proof of Theorem 1 will be complete if we show that there exists sequences of integers $\{m_k : k\geq 1\}$ and $\{\nu_k : k\geq 1\}$ increasing to infinity, and a sequence of functions $\{\xi_k : k\geq 1\}$ from the unit bull $S_{\Phi}(0,1)$ of Orlicz space $L_{\Phi}(I^2)$, such that for all k

$$mes\{(x,y)\in I^2 : \left|\tilde{\sigma}^{(2/3,2/3)}_{2^{m_k},2^{m_k}}(\xi_k;x,y)\right|>\nu_k\}\geq\frac{1}{8}. \tag{4}$$

First we prove that there exists $c>0$ such that

$$mes\{(x,y)\in I^2 : \left|\tilde{\sigma}^{(2/3,2/3)}_{2m,2m}\left(D_{2^{2m}}\otimes D_{2^{2m}};x,y\right)\right|>2^m\}>c\frac{m}{2^m}. \tag{5}$$

Denote

$$G_m=\bigcup_{l=1}^{m}\left(\frac{1}{2^l},\frac{1}{2^{l-1}}\right).$$

Since

$$\tilde{\sigma}^{(2/3,2/3)}_{2m,2m}(D_{2^{2m}}\otimes D_{2^{2m}},x,y)=\tilde{F}^{(2/3)}_{2m}(x)\tilde{F}^{(2/3)}_{2m}(y),$$

then for $(x,y)\in G_m\times G_m$ we have from Lemma 1 the following estimation

$$\left|\tilde{\sigma}^{(2/3,2/3)}_{2m,2m}(D_{2^{2m}}\otimes D_{2^{2m}},x,y)\right|=\left|\tilde{F}^{(2/3)}_{2m}(x)\tilde{F}^{(2/3)}_{2m}(y)\right|\geq\frac{c}{xy}.$$

It is easy to show that

$$mes\{(x,y)\in I^2 : \left|\tilde{\sigma}^{(2/3,2/3)}_{2m,2m}(D_{2^{2m}}\otimes D_{2^{2m}},x,y)\right| > c2^m\}$$
$$\geq mes\{(x,y)\in G_m\times G_m : \left|\tilde{\sigma}^{(2/3,2/3)}_{2m,2m}(D_{2^{2m}}\otimes D_{2^{2m}},x,y)\right| > c2^m\}$$
$$\geq mes\{(x,y)\in : \frac{c}{xy} > c2^m\}$$
$$\geq c\sum_{l=1}^{m}\sum_{s=m-l}^{m}\frac{1}{2^{s+l}} \geq \frac{cm}{2^m}.$$

Hence (5) is proved.

From the condition of the theorem we write [11]

$$\liminf_{u\to\infty}\frac{\Phi(u)}{u\log u}=0.$$

Consequently, there exists a sequence of integers $\{m_k\}_{k=1}^{\infty}$ increasing to infinity, such that

$$\lim_{k\to\infty}\Phi(2^{2m_k})2^{-2m_k}m_k^{-1}=0,\quad \Phi(2^{2m_k})\geq 2^{2m_k},\quad \forall k.$$

From (5) we write

$$mes\{(x,y)\in I^2 :|\tilde{\sigma}^{(2/3,2/3)}_{2m_k,2m_k}(D_{2^{2m_k}}\otimes D_{2^{2m_k}},x,y)|> c2^{m_k}\} > c\frac{m_k}{2^{m_k}}.$$

Then by the virtue of Lemma 2 there exists $e_1,...,e_r,e_1^{'},...,e_r^{'}\in[0,1]$ and $\varepsilon_1,...,\varepsilon_r=\pm1$ such that

$$mes\left\{(x,y)\in I^2 :|\sum_{i=1}^{r}\varepsilon_i\tilde{\sigma}^{(2/3,2/3)}_{2m_k,2m_k}(D_{2^{2m_k}}\otimes D_{2^{2m_k}},x\oplus e_i,y\oplus e_i^{'})|> 2^{m_k}\right\} > \frac{1}{8},$$

where $r=\left[\frac{2^{m_k}}{cm_k}\right]+1.$

Denote

$$\nu_k=\frac{2^{3m_k-1}}{r\Phi\left(2^{2m_k}\right)}$$

and

$$\xi_k(x,y)=\frac{2^{2m_k-1}}{\Phi\left(2^{2m_k}\right)}M_k(x,y),$$

where

$$M_k(x,y)=\frac{1}{r}\sum_{i=1}^{r}\varepsilon_i D_{2^{2m_k}}(e_i\oplus x)D_{2^{2m_k}}\left(e_i^{'}\oplus y\right).$$

Thus, we obtain (4).

Moreover, since Φ is convex, we have $\xi_k\in S_\Phi(0,1)$. Indeed, the estimation $\|M_k\|_{L^\infty(I^2)}\le 2^{4m_k}$ and $\|M_k\|_{L^1(I^2)}\le 1$ implies

$$\|\xi_k\|_{L_\Phi(I^2)}\le\frac{1}{2}\left[1+\int_{I^2}\Phi\left(\frac{2^{2m_k}|M_k(x,y)|}{\Phi(2^{2m_k})}\right)dxdy\right]\le 1.$$

Theorem 1 is proved.

The validity of Corollary 1 follows immediately from Theorem 1 and Lemma 4.

We note that $\alpha=\beta=\dfrac{2}{3}$ implies that

$$\sum_{i=1}^{\infty}|\alpha_i-\alpha_{i-1}|=+\infty$$

and

$$\sum_{i=1}^{\infty}|\beta_i-\beta_{i-1}|=+\infty$$

If series (6) and (7) are finite then it easy to show that $\tilde{\sigma}_{n,m}^{(\alpha,\beta)}(f)$ converge of the metric of space $L\left(I^2\right)$ for every $f\in L\left(I^2\right)$, in particular converge in measure for every $f\in L\left(I^2\right)$.

We can now formulate the following

Problem 1 *Whether the conditions (6) and (7) guarantees justice of Theorem 1 and Corollary 1?*

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უშანგი გოგინავა

ორგანზომილებიანი შეუღლებული ფურიე-უოლშის მწკრივების ზომით კრებადობის შესახებ

ნაშრომში მტკიცდება, რომ ნებისმიერ ორლიჩის სივრცეში, რომელიც არ არის $L\log L(I^2)$ სივრცის ქვესივრცე, იმ ფუნქციების სიმრავლე, რომელთა ორმაგი ფურიე-უოლშის შეუღლებული მწკრი-ვების ფეიერის საშუალოები ($\alpha = \beta = 2/3$) არის ზომით კრებადი აქვს ბერის პირველი კატეგორია.

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ILIA TAVKHELIDZE

ABOUT SOME GEOMETRIC CHARACTERISTIC OF THE GENERALIZED MOBIUS LISTING'S SURFACES GML_2^n AND ITS CONNECTIONS WITH SET OF RIBBON LINKS

Abstract. We consider the cutting process of a Generalized Möbius-Listing's surfaces GML_2^n along a set of lines "parallel" to its "basic line". We show connection of the resulting mathematical objects with the set of Ribbon knots and links.

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1. Introduction

In previous articles [1-5] a wide class of geometric figures – "Generalized Twisting and Rotated" bodies (sometimes called "surface of Revolution" see [9]) - shortly GTR_m^n - was defined through their analytic representation. In particular cases, this analytic representation gives back many classical objects (torus, helicoid, helix, Möbius strip ... etc.). Aim of this article is to consider some geometric properties of a wide subclass of the already defined surfaces, by using their analytical representation. In previous articles [1-5] a set of the Generalized Möbius Listing's bodies - shortly GML_m^n, which are a particular case of the GTR_m^n bodies, have been defined. In the present paper we show some geometric properties of

Generalized Twisting and Rotated - surfaces and relationships between the set and the sets of Ribbon Knots and Links.

2. Notations and Definitions

In this article we use following notations:

- X, Y, Z, or x, y, z - is the ordinary notation for coordinates;
- τ, ψ, θ - are space values (local coordinates or parameters in parallelogram);

1. $\tau \in [\tau_*, \tau^*]$, *where* $\tau_* \leq \tau^*$ *usually are non – negative constants*;
2. $\psi \in [0, 2\pi]$;
3. $\theta \in [0, 2\pi h]$, *where* $h \in R$ (*Real*);

(1)

But sometimes, as a special case, we suppose that

$$\tau \in [-\tau^*, \tau^*] \qquad (1^*)$$

- $P_m \equiv A_1 A_2 \ldots A_m$ - denotes an "Plane figure with m-symmetry", in particular P_m is a "regular polygon" and m is the number of its angles or vertices. In the general case the edges of "regular polygons" are not always straight lines ($A_i A_{i+1}$ may be, for example: edge of epicycloid, or edge of hypocycloid, or part of lemniscate of Bernoulli, and so on) (see e.g. Fig. 1f);
- $PR_m \equiv A_1 A_2 \ldots A_m A'_1 A'_2 \ldots A'_m$ denotes an orthogonal prism, whose ends $A_1 A_2 \ldots A_m$ and $A'_1 A'_2 \ldots A'_m$ are "Plane m-symmetric figures" P_m (see e.g. Fig. 1b);

For example:

- PR_0 - is a segment and P_0 is a point;
- PR_1 - is an orthogonal cylinder, whose cross section is a P_1 - plane figure without symmetry;
- $PR_2 \equiv A_1 A_2 A'_1 A'_2$ is a rectangle, if $P_2 \equiv A_1 A_2$ is a segment of straight line; but also PR_2 maybe a cylinder with cross section P_2 (ellipse, or lemniscate of Bernoulli and so on);

- PR_∞ - is an orthogonal cylinder, whose cross section is a P_∞-circle.

$$x = p(\tau,\psi), \quad z = q(\tau,\psi) \tag{2}$$

or

$$\begin{aligned} x &= p(\tau,\psi)\cos\psi \\ z &= p(\tau,\psi)\sin\psi \end{aligned} \tag{2*}$$

are the analytic representations of a **"Plane figure with m-symmetry"** P_m, usually $p(0,0) = q(0,0) = 0$ and the point $(0,0)$ is the center of symmetry of this polygon (see [1-5]).

For example, when the function $p(\tau,\psi)$ in formula (2^*) has the form

$$p(\tau,\psi) \equiv \tau \sum_{i=0}^{m-1} \varepsilon(\psi - \psi_i),$$

(2')

where the arguments τ and ψ are defined in (1); $\psi_i \in [0,2\pi)$ are some constants for each $i = \overline{1,m-1}$, with $\psi_i \neq \psi_j$ if $i \neq j$, and

$$\varepsilon(\psi - \psi_i) \equiv \begin{cases} 0 & if \quad \psi = \psi_i \\ 1 & if \quad \psi \neq \psi_i, \end{cases}$$

then the corresponding plane figure P_m (some time in this article P_m^*) is:

1. a **"simple star"** with m "wings" or "vertices" when $\tau_* \equiv 0$ (see e.g. Fig. 1.a. $\overline{i = 0,6}$);

2. a set of m segments of straight lines lying on the radiuses of a circle centered at the origin when $\tau_* > 0$ (see e.g. Fig. 1.e.)).

Conclusion 1. In the case when $\psi_i \equiv \dfrac{2\pi i}{m}, i = \overline{0,m-1}$, then P_m^* is a **"Regular simple star"** (see Figs. 1 b., d., c.);

- if $m = 2$ and $\tau_* \equiv 0$, then (2') is a representation of P_2^* (see e.g. Fig. 1.c.)), which is a segment of straight line $[-\tau^*, \tau^*]$. In this particular case we set

$$p(\tau,\psi) \equiv \tau \tag{2**}$$

where the “argument” τ satisfies (1^*) ;

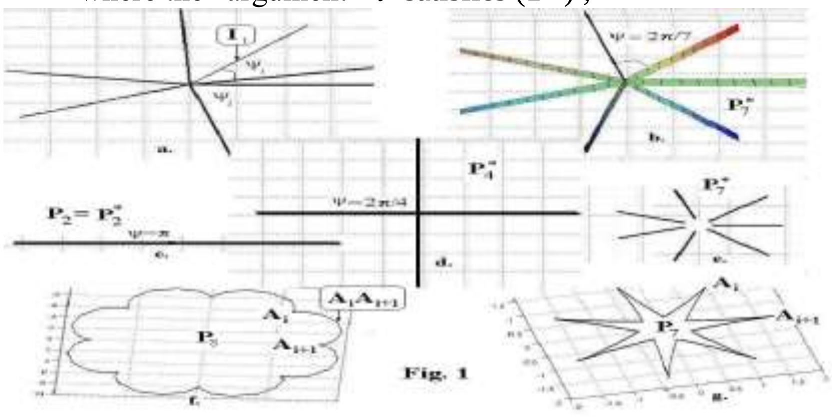


Fig. 1

• $D(p,q)$ or $D(p)$ - diameter of plane figure P_m ;
• OO' - axis of symmetry of the prism PR_m ;
• L_ρ - Family of lines situated on the plane, whose parametric representations are

$$L_\rho = \begin{cases} X = f_1(\rho,\theta) \\ Y = f_2(\rho,\theta) \end{cases} \quad \rho \in [0,\rho^*), \quad \theta \in [0, 2\pi h], \quad h \in Z \tag{3}$$

or

$$L_\rho = \begin{cases} X(\rho,\theta) = \rho_1(\theta)\cos\theta \\ Y(\rho,\theta) = \rho_2(\theta)\sin\theta \end{cases} \tag{3*}$$

We assume the following hypotheses:

i) For any parameters $\rho_1, \rho_2 \in [0, \rho^*], \rho_1 \neq \rho_2$, the lines L_{ρ_1} and L_{ρ_2} have not intersection.

ii) If L_ρ is a closed curve, then for every fixed $\rho \in [0, \rho^*]$ f_i - are $2p$-periodic functions $f_i(\rho, \theta + 2\pi) = f_i(\rho,\theta), (i = 1,2)$.

• $g(\theta)$ - be an arbitrary sufficiently smooth function

$$g(\theta):[0,2h\pi]\to[0,2h\pi] \qquad (4)$$

and if $h=1$, then for every $\Theta\in[0,2\pi]$ there exists $\theta\in[0,2\pi]$, such that $\Theta = g(\theta)$;

- $mod_m(n)$ - natural number $< m$; for every two numbers $m\in N$ (natural) and $n\in Z$ (integer) there exists a unique representation $n = km + j \equiv km + mod_m(n)$, where $k\in Z$ and $j\equiv mod_m(n)\in N\cup\{0\}$;

- $$\mu\equiv\begin{cases} n/m, & when \quad m\in N \quad and \quad n\in Z \\ n & when \quad m=\infty \quad and \quad n\in Z \quad (or\ n\in R\,(\mathrm{Real})) \end{cases} \qquad (5)$$

- **Generalized Twisting and Rotated bodies** - shortly GTR_m^n (sometimes called **"Surfaces of revolution"** see[9]) are defined by the parametric representations:

$$\begin{aligned} X(\tau,\psi,\theta) &= f_1\left(\left[R + p(\tau,\psi)\cos(\mu g(\theta)) - q(\tau,\psi)\sin(\mu g(\theta))\right],\theta\right) \\ Y(\tau,\psi,\theta) &= f_2\left(\left[R + p(\tau,\psi)\cos(\mu g(\theta)) - q(\tau,\psi)\sin(\mu g(\theta))\right],\theta\right) \\ Z(\tau,\psi,\theta) &= Q(\theta) + p(\tau,\psi)\sin(\mu g(\theta)) + q(\tau,\psi)\cos(\mu g(\theta)), \end{aligned} \qquad (6)$$

or

$$\begin{aligned} X(\tau,\psi,\theta) &= \left[\rho_1(\theta) + p(\tau)\cos(\psi + \mu g(\theta))\right]\cos(\theta) \\ Y(\tau,\psi,\theta) &= \left[\rho_2(\theta) + p(\tau)\cos(\psi + \mu g(\theta))\right]\sin(\theta) \\ Z(\tau,\psi,\theta) &= Q(\theta) + p(\tau)\sin(\psi + \mu g(\theta)), \end{aligned} \qquad (6^*)$$

where, respectively:

- the arguments (τ,ψ,θ) are defined in (1);
- the functions f_1 and f_2 or $\rho_1(\theta)$ and $\rho_2(\theta)$ in (3) or (3^*) define the L_R **"Shape of plane basic line"**, more precisely "Shape of orthogonal projection on the plane XOY of the basic line" of corresponding body (see e.g.: circle in – Figs. 2b, 2c, 2g; ellips in – Fig. 2e; spiral in – Figs. 2d, 2f, 2i and square in – Fig. 2h.);
- R is a some fixed real number, which defines the **"Radius"** of the "plane basic line" L_R;

- Functions $p(\tau,\psi)$ and $q(\tau,\psi)$ or $p(\tau)$ in (2) or (2^{*}) define the **"Shape of the radial cross section"** of corresponding figure. In general case this functions may be depends from arguments ψ (see for example (2')) and θ, i.e. "Shape of the radial cross section" depends from the "place" of this cross section (see e.g. Fig 2i.);
- The function $g(\theta)$ from (4) defines the **"Rule of twisting around basic line"**;
- The number μ in (5) defines the **"Characteristic of twisting"**;
- $Q(\theta)$ is a smooth function which defines the **"Law of vertical stretching of figure"**.

Therefore, this parametric representation defines a GTR_m^n body (some examples are shown in Fig. 2) with the following restrictions:

1) The OO'-axis of symmetry (middle line) of the prism PR_m is transformed into a **"Basic line"** (sometimes called **"Profile curve"**) - $(L_R, Q(\theta))$;

2) Rotation at the end of the prism (2) or (2^{*}) is semi-regular along the middle line OO', or the twisting of the shape of radial cross section around the basic line is semi-regular (depending from $g(\theta)$) .

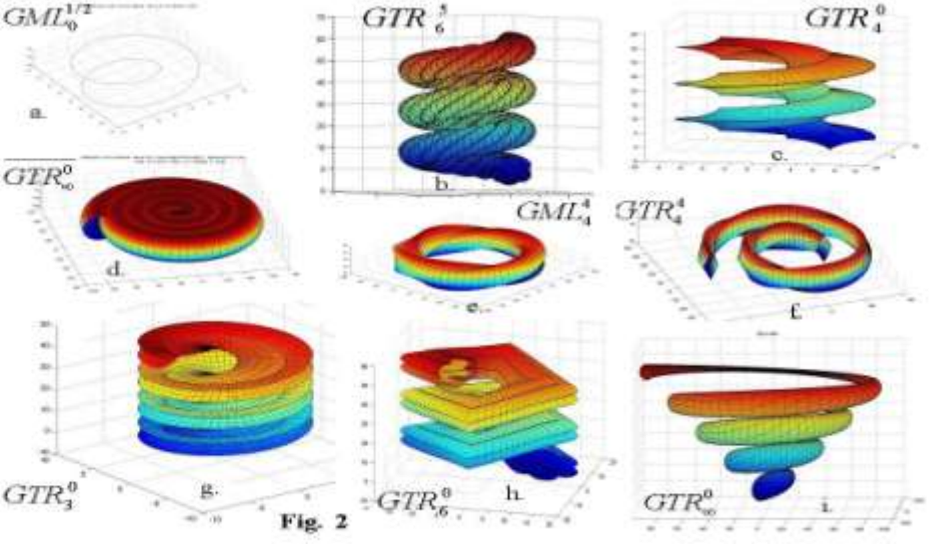


Fig. 2

• **Generalized Möbius Listing's body** - shortly GML_m^n - is obtained by identifying the opposite ends of the prism PR_m in such a way that:

A) For any integer $n \in Z$ and $i = 1, \ldots, m$ each vertex A_i coincides with $A'_{i+n} \equiv A'_{mod_m(i+n)}$, and each edge $A_i A_{i+1}$ coincides with the edge

$$A'_{i+n} A'_{i+n+1} \equiv A'_{mod_m(i+n)} A'_{mod_m(i+n+1)}$$

correspondingly;

B) The integer $n \in Z$ denotes the number of rotations of the end of the prism with respect to the axis OO' before the identification. If $n > 0$, the rotations are counter-clockwise, and if $n < 0$ then rotations are clockwise. Some particular examples of GML_m^n and its graphical realizations can be found in [2-5] (see e.g. Fig 2e.).

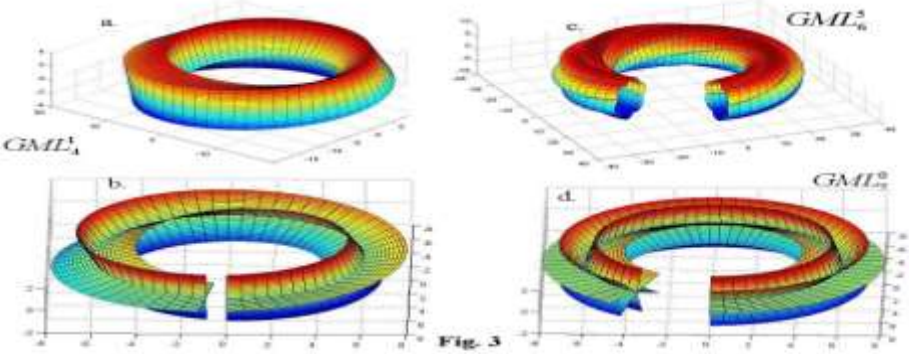


Fig. 3

Conclusion 2. We can assert that:

a.) The GML_m^n body is a particular case of the GTR_m^n body.

b.) The basic line L_R of a GML_m^n body, is always a closed line and the number μ is such that the boundary of this body is a closed surface (see [2]).

c.) The functions f_1, f_2, $\rho_1(\theta), \rho_2(\theta)$, $Q(\theta)$ in parametric representation (6) and (6^*) of a GML_m^n body are always 2π-periodic functions (see e.g. Fig. 3a) or $Q(\theta) \equiv 0$, (some examples are shown in Figs. 3).

- Some additional information about the classification of GRT_m^n bodies are reported in [2,4].

1. **Some Geometric properties of a "Regular" GML_2^n surfaces.**

In this part of our article we study some geometric characteristic of a ``Regular" Generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line. This means that the parametric representations of these surfaces (6) or (6^*) have the following simple form

$$
\begin{aligned}
X(\tau,\theta) &= \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\cos(\theta) \\
Y(\tau,\theta) &= \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin(\theta) \\
Z(\tau,\theta) &= \tau\sin\left(\psi + \frac{n\theta}{2}\right),
\end{aligned}
\tag{7}
$$

where, respectively:

- R - radius of basic circle - is constant (see e.g. Figs. 4a., 4b., 4c.);
- In the general case (6) τ, defined in (1), is variable, but now, according to the notation (2**) (Corollary 1.), the argument τ always belongs to the interval $[-\tau^*, \tau^*]$, see (1*) where $\tau^* < R$ is some non-negative constant. So that, when $n = 1$ formula (7) is the classical (well known) form of analytic representation of the Möbius strip ([2],[9]). Actually, $2\tau^*$ is the width of the surface GML_2^n;
- the variable θ is defined in (1) and in this case $h \equiv 1$, i.e. $\theta \in [0, 2\pi]$;
- The "rule of twisting around basic line" is **"Regular",** i.e. the function, which is defined by eq. (4) is $g(\theta) \equiv \theta$;

- n - the **"Number or twisting"** of GML_2^n - is an arbitrary integer number, i.e. the number defined by eq. (5) is $\mu \equiv n/2$ (see e.g. *n=1*- **"Möbius strip"** Fig. 4a; *n=2* Fig. 4b; *n=14* Fig. 4c; *n=6* Fig. 4d; *n=0* - Figs. 4e, 4f, 4g, 4h, 4i, 4j.);
- in the present case, ψ is a constant defined in (1) (but when $n=0$, the number ψ in eq. (7) defines even the type of the corresponding surface, for example: if $\psi = 0$, then the **"Regular"** Generalized Möbius-Listing's surfaces GML_2^0, with basic line a circle, is a Ring ($R > \tau^*$) (see. e.g. Fig. 4e) or Disk ($R = \tau^*$)(see. e.g. Fig. 4g), and if $\psi = \dfrac{\pi}{2}$, then GML_2^0 is a cylinder (see. e.g. Fig. 4h), in other cases these surfaces are cones or truncated cones (see. e.g. Fig. 4i) (see [3-6])).

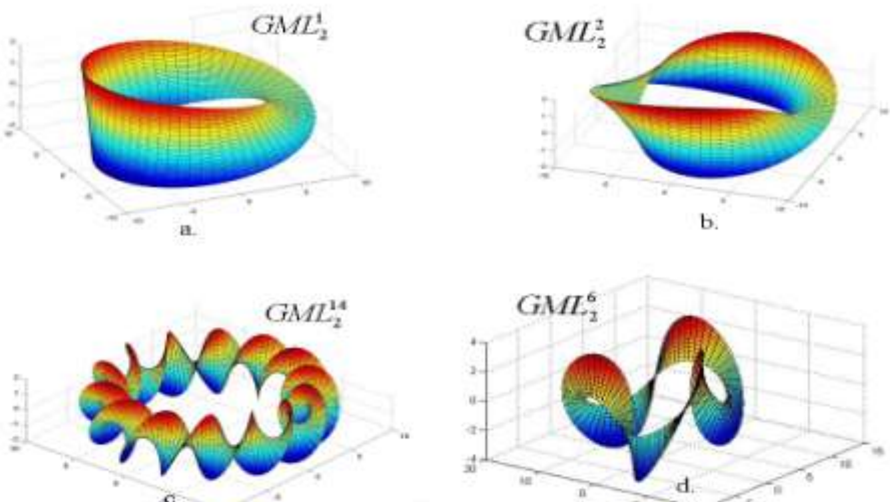
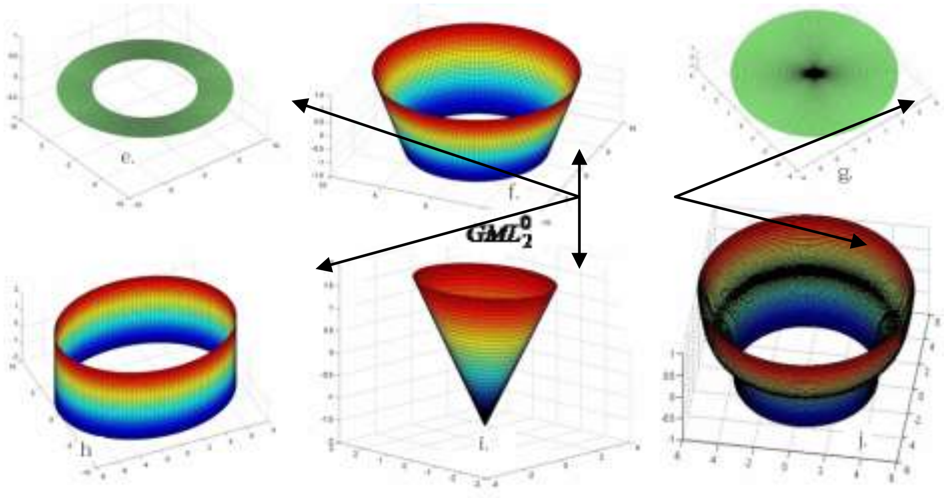


Fig. 4



Remark 1 *Note that:*

a.) For every integer number n eq. (7) defines a one to one correspondence between the strip $[-\tau^*, \tau^*] \times [0, 2\pi)$ and the surface GML_2^n.

b.) If n is an even number, then each function (X, Y, Z) in the representation (7) is a 2π-periodic function of the argument θ.

c.) If n is a odd number, then each function (X, Y, Z) in the representation (7) is a 4π-periodic function satisfying the following properties (Möbius property, see [8])

$$\left(X(\tau,\theta+2\pi);Y(\tau,\theta+2\pi);Z(\tau,\theta+2\pi)\right)=\left(X(-\tau,\theta);Y(-\tau,\theta);Z(-\tau,\theta)\right). \qquad \textbf{(M*)}$$

According to the representation (7), the tangential vectors of the “Regular” Generalized Möbius-Listing's surface GML_2^n, with circle as basic line, are correspondingly

$$\overleftarrow{r}_{\tau} = \left\{\cos\left(\psi+\frac{n\theta}{2}\right)\cos(\theta);\cos\left(\psi+\frac{n\theta}{2}\right)\sin(\theta);\sin\left(\psi+\frac{n\theta}{2}\right)\right\} \qquad (8)$$

and

$$\overleftarrow{r}_\theta = \left\{ \begin{array}{c} -\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\cos(\theta); \\ \left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\sin(\theta); \\ \frac{\tau n}{2}\cos\left(\psi+\frac{n\theta}{2}\right) \end{array} \right\}. \qquad (9)$$

It is easy to check, that the scalar product of these two vectors (8) and (9) is

$$(\overleftarrow{r}_\tau, \overleftarrow{r}_\theta) = 0.$$

Remark 2 *For any integer number* n *two tangential vectors of a regular* GML_2^n*, with circle as basic line, are always orthogonal, i.e. the local system of coordinates* (τ,θ) *in this surface is an orthogonal system.*

Also we may check that

$$\begin{aligned} \frac{\partial(x,y)}{\partial(\tau,\theta)} &= \left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos\left(\psi+\frac{n\theta}{2}\right); \\ \frac{\partial(z,x)}{\partial(\tau,\theta)} &= -\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin\left(\psi+\frac{n\theta}{2}\right)\sin(\theta)-\frac{\tau n}{2}\cos(\theta); \qquad (10) \\ \frac{\partial(y,z)}{\partial(\tau,\theta)} &= -\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin\left(\psi+\frac{n\theta}{2}\right)\cos(\theta)+\frac{\tau n}{2}\sin(\theta); \end{aligned}$$

and the module of the vector product of these two vectors is

$$|\overleftarrow{r}_\tau \times \overleftarrow{r}_\theta| = \sqrt{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4}}. \qquad (11)$$

Therefore, the we may rewrite unit normal vector of a “Regular” Generalized Möbius-Listing's surfaces GML_2^n.

Remark 3 *Note that:*

a.) If n is an even number, then the unit normal vector $\overleftarrow{\nu}(\tau,\theta)$ is a 2π-periodic vector and consequently the GML_2^n body is a two-sided surface; i.e. to each point of the GML_2^n corresponds one (external or internal) normal vector of this surface;

b.) If n is a odd number, then the unit normal vector $\overleftarrow{\nu}(\tau,\theta)$ is a 4π -periodic vector function, with the Möbius property

$$\overleftarrow{\nu}(\tau,\theta+2\pi) = -\overleftarrow{\nu}(\tau,\theta) \qquad (12)$$

so that the GML_2^n is a one-sided surfaces; i.e. to each point of the GML_2^n surface correspond two normal vectors to this surface and, by the geometric point of view, it is impossible to "distinguish" the external from the internal normal vector to this surface.

The first fundamental form of a regular generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line, is given by

$$\begin{aligned} &E(\tau,\theta) = 1; \\ &F(\tau,\theta) = 0; \\ &G(\tau,\theta) = \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]^2 + \frac{(n\tau)^2}{4}, \end{aligned} \qquad (13)$$

so that, it is very easy to see that

Remark 4 *Each point of the corresponding surface (7) is regular, i.e, for each point* (τ,θ) *the corresponding forms satisfy the conditions:* $E(\tau,\theta) > 0, G(\tau,\theta) > 0$ *and* $EG - F^2 > 0$.

The second fundamental form of a regular generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line, is given by

$$L(\tau,\theta)=0;$$
$$M(\tau,\theta)=\frac{nR}{2\sqrt{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4}}};$$
$$N(\tau,\theta)=\frac{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{2}}{\sqrt{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4}}}\sin\left(\psi+\frac{n\theta}{2}\right). \tag{14}$$

So that we may rewrite the mean and Gaussian curvatures of a regular GML_2^n, with circle as basic line, in the form

$$H(\tau,\theta)=\frac{\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{2}}{\left(\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\frac{(n\tau)^2}{4}\right)^{\frac{3}{2}}}\sin\left(\psi+\frac{n\theta}{2}\right) \tag{15}$$

and

$$K(\tau,\theta)=\frac{-n^2R^2}{4\left(\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]^2+\tau^2n^2\right)^2}. \tag{16}$$

Remark 5 *Each point of Regular generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line, is:*

a.) Hyperbolic (saddle) point if the number $n\neq 0$, i.e. always $K(\tau,\theta)<0$;

b.) Parabolic point if the number $n\equiv 0$, i.e. always $K(\tau,\theta)\equiv 0$;

4. **Relations between the set of Generalized Möbius-Listing's Surfaces and the sets of Knots and Links**

We use the following definitions and notations:

Definition 1. *A closed line (similar to the basic or border's line) which is situated on a* GML_2^n *and is "parallel" to the basic (or border's) line of the* GML_2^n *- i.e. the distance between this line and basic or border's lines is constant - is called a **"Slit line"** or shortly an **"s-line"** (see e.g. Fig. 5.d.).*

- If the distance between an s-line and the basic line is zero, then this s-line coincides with the basic line (and sometimes is called **"B-line"**)(see e.g. Fig. 5.c.).

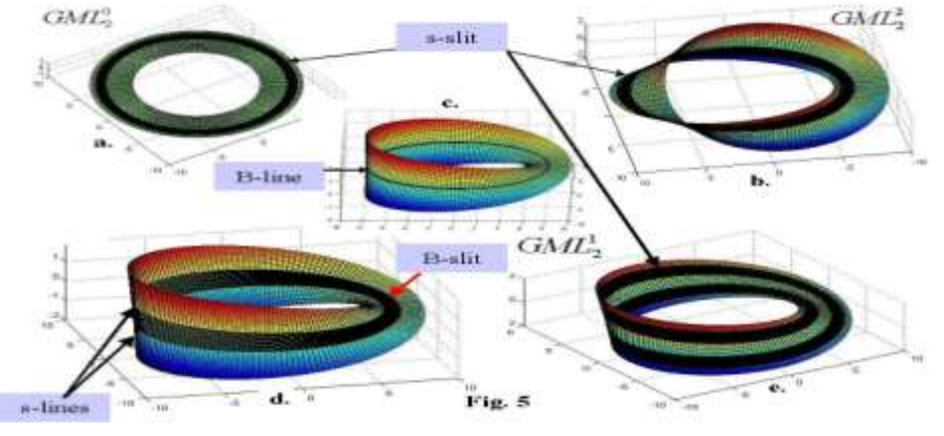


Fig. 5

Definition 2. *A domain situated on the surface* GML_2^n *and such that its border's lines are slit lines, is called a **"Slit zone"** or shortly an **"s-zone"**.*

- The distance between the border's lines of an s-zone is the **``width"** of this s-zone.
- If an s-zone's width equals to zero, then this zone reduces to an s-line.

Definition 3. *If the* ***"B-line"*** *is properly contained inside a* ***"Slit zone"*** *- i.e. his distance to the border's lines is strictly positive - then this* ***"Slit zone"*** *will be called a* ***"B-zone"***.

Definition 4 *The* ***"process of cutting"*** *or shortly the* ***"cutting"*** *is always realized belong some s-lines and produces the vanishing (i.e. elimination) of the corresponding s-zone (which eventually reduces to an s-line)*

. • If a GML_2^n surface is cut along an s-line (sometimes $\xrightarrow{1}$), then the corresponding vanishing zone will be called an **s-slit** (see e.g. Figs. 5.a., 5.b., 5.e.).

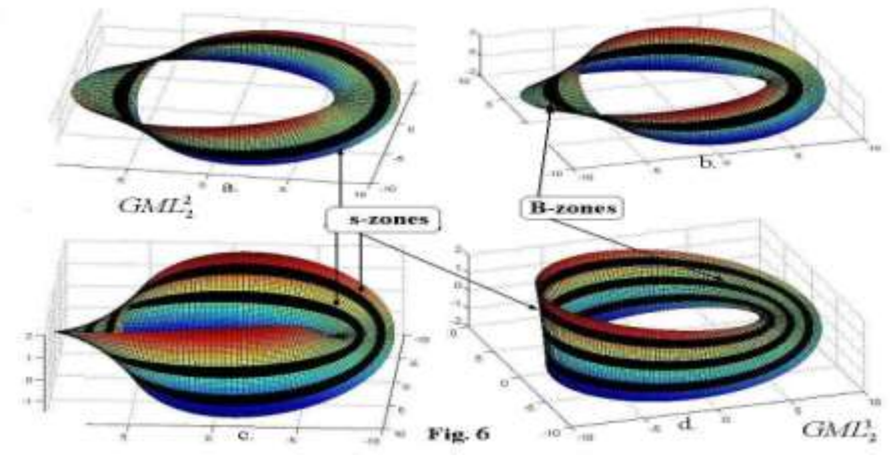


Fig. 6

- If a GML_2^n surface is cut along its B-line (sometimes) $\xrightarrow{B}$, then the corresponding vanishing zone will be called a **B-slit** (see e.g. Fig. 5.d.).
- If the vanishing zone - after an **s-slit** (a **B-slit**) - is given by an **"s-zone"** (a **"B-zone"**), then the cutting process will be called an **s-zone-slit** (a **B-zone-slit**).
- If GML_2^n surface is cut (k+1)-times along (k+1), k=0,1,2,…, different s-lines and none of them coincides with the B-line (for this process we use the symbolic notation $\xrightarrow{k+1}$), then the resulting object is called a **(k+1)-slitting** GML_2^n , and the corres-

ponding vanishing zones are **(k+1)-slits.** In this case the cutting processis called a **(k+1)-zone-slits.**

- If GML_2^n surface is cut (k+1)-times along (k+1), k=0,1,2,…, different s-lines and one of this line coincides with the B-line (for this process we use the symbolic notation $\xrightarrow{B+k}$), then the resulting object is called a **(B+k)-slitting** GML_2^n , and the corresponding vanishing zones are **(B+k)-slits.** In this case the cutting processis called a **(B+k)-zone-slits.**
- For each natural number k the segment [0, τ*] is divided by one of the following rules:

$$\tilde{\tau}_{2j} \equiv j \cdot \left(\frac{\tau^* - \varepsilon}{k+1} + \frac{\varepsilon}{k} \right) \qquad j = 0,1,\ldots,k;$$

$$\tilde{\tau}_{2j+1} \equiv (j+1) \cdot \frac{\tau^* - \varepsilon}{k+1} + j \cdot \frac{\varepsilon}{k} \qquad j = 0,1,\ldots,k; \tag{17}$$

or

$$\hat{\tau}_{2j} \equiv \frac{j\tau^*}{k} \qquad j = 0,1,\ldots,k;$$

$$\hat{\tau}_{2j+1} \equiv \frac{j\tau^* + \varepsilon}{k} \qquad j = 0,1,\ldots,k-1, \tag{18}$$

where ε ∈ [0,τ*) is a real number.

- For each natural number k the domain of definition T ≡{[-τ*,τ*]×[0,2π]} (see (2)) of the reprezentation formula (7) is divided by one of the following rules:

$$T = \tilde{T}_k \cup \tilde{T}_k^{\varepsilon}$$

$$\tilde{T}_k \equiv \left\{ [-\tilde{\tau}_1, \tilde{\tau}_1] \bigcup_{j=1}^{k} [\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1}] \cup [-\tilde{\tau}_{2j+1}, -\tilde{\tau}_{2j}] \right\} \times [0,2\pi)' \tag{19}$$

$$\tilde{T}_k^{\varepsilon} \equiv \left\{ \bigcup_{j=1}^{2k-1} [\tilde{\tau}_{2j-1}, \tilde{\tau}_{2j}] \cup [-\tilde{\tau}_{2j}, -\tilde{\tau}_{2j-1}] \right\} \times [0,2\pi)'$$

or

$$T=\hat{T}_{B,k}\cup\hat{T}^{\varepsilon}_{B,k}$$

$$\hat{T}_{B,k}\equiv\left\{\bigcup_{j=1}^{2k-1}\left[\hat{\tau}_{2j-1},\hat{\tau}_{2j}\right]\cup\left[-\hat{\tau}_{2j},-\hat{\tau}_{2j-1}\right]\right\}\times\left[0,2\pi\right)' \tag{20}$$

$$\hat{T}^{\varepsilon}_{B,k}\equiv\left\{\left[-\hat{\tau}_1,\hat{\tau}_1\right]\bigcup_{j=1}^{k}\left[\hat{\tau}_{2j},\hat{\tau}_{2j+1}\right]\cup\left[-\tilde{\tau}_{2j+1},-\tilde{\tau}_{2j}\right]\right\}\times\left[0,2\pi\right)'$$

or

$$T=\tilde{T}^{*}_{k}\cup\tilde{T}^{\varepsilon,*}_{k}$$

$$\tilde{T}^{*}_{k}\equiv\left\{\left[-\tau^{*},\tilde{\tau}_1\right]\bigcup_{j=1}^{k}\left[\tilde{\tau}_{2j},\tilde{\tau}_{2j+1}\right]\right\}\times\left[0,2\pi\right)' \tag{21}$$

$$\tilde{T}^{\varepsilon,*}_{k}\equiv\left\{\bigcup_{j=1}^{2k-1}\left[\tilde{\tau}_{2j-1},\tilde{\tau}_{2j}\right]\right\}\times\left[0,2\pi\right)'$$

Without loss of generality and for simplifying the process of proofing, in this article we consider the following restrictions:

- The GML_2^n is a regular generalized Möbius-Listing's Surface (see representation (7) and notation (1*));
- B-slit is symmetric (see e.g. Figs. 5.c, 5.d., 6.b., 6.d.), i.e. the "origin" (domain of its parametric representation) (7) of the corresponding B-zone is the domain $\hat{T}^{\varepsilon}_{B,0}$ (20).
- For any fixed number of cutting k, the width of the eliminated slit zones on the GML_2^n surfaces are always identic and equal to 2ε/k (see e.g. Figs. 5.e., 6.c., 6.d.);
- For any fixed number of cutting k, the width of the remaining slit zones on the GML_2^n surfaces are always identic and equal to 2(τ*-ε)/(k+1).

Theorem 1. If the GML_2^n surface is cut (k+1)$-times along (k+1) different (i.e. k=0,1,…) s-lines, and n is an even number, then for each integer numbers n,k,

after (B+k)-zone-slits or (k+1)-zone-slits, an object Link-(k+2) appears, whose each component is a GML_2^n surface (knot with structure $\{0_1\}$); The topologic group of the classic link-(k+2) in this case is at present unknown; only when k=0, the link-2 is of type $\{n^2_1\}$, according the standard classification (see [6,7,8]).}; i.e. if n=2ω is an even number, then for each ω=0,1,2,…, and k:

Case A.

$$GML_2^{2\omega} \xrightarrow{B+k} Link-(k+2)\ \ of\ \ (k+2)\ \ \ objects\ \ GML_2^{2\omega} \tag{22}$$

Case B.

$$GML_2^{2\omega} \xrightarrow{k+1} Link-(k+2)\ \ of\ \ (k+2)\ \ \ objects\ \ GML_2^{2\omega} \tag{23}$$

Proof. The representation (7) is a one to one correspondence between the points of the strip T and the points of the GML_2^n surface, and according to the remak 1, if n is an even number (n≡2ω), then each of the functions X(τ,θ),Y(τ,θ),Z(τ,θ) is a 2π-periodic function of the argument θ.

Let us consider the particular *case A*, when k=0 - This means that there exists only one B-slit-zone.

So that, according to (20), a the B-zone-slit corresponds to the elimination of the

$\hat{T}_{B,0}^{\varepsilon}$ in the domain of definition T. But in this case, the domain of definition $\hat{T}_{B,0}^{\varepsilon}$ in (20) consists of two parts and define by (7) two different objects GML_2^n. The one to one correspondence (7) guarantees that the new objects have not self-cross points.

In this case, according to the above restrictions, both new objects have identic widths.

Let us consider the particular *case B*, when k=0 - This means that there exists only one s-slit-zone.

So that, according to (7) and (21), a s-slit-zone corresponds to the elimination of the $\tilde{T}_1^{\varepsilon,*}$ in the domain of definition T. But in this case, the domain of definition $\tilde{T}_1^{*}$ in (21) consists of two parts and define by (7) two different objects GML_2^n with different widths. The widths of components of B-zone-slits are equal, but the widths of components of s-zone-slits are different.

Let us consider now the general *case A*, when k is an arbitrary natural number. This means that there exist (B+k)-zone-slits.

– Recall that in the particular case A, after a B-zone-slit, an object Link-2 appears, whose both components are GML_2^n surfaces. So that the following cutting or the (B+1)-zone-slits (k=1) is a cutting of one of the new objects GML_2^n, which appears in the particular case A. Consequently, applying the same arguments of the particular cases

A or B to each of the new GML_2^n surfaces (we do not know if the new slit-zone includes the B-line of this surface or not, but this is not important, since the number n is even and both results are the same), we find that an object Link-2 appears, whose both components are GML_2^n surfaces. Therefore, the domain of definition $\hat{T}_{B,1}$ in (20) consists of 3 parts and define by (7) 3 different objects GML_2^n.

So that, in the general case, according to (20), to a (B+k)-zones-slit or to each number k corresponds the elimination of $\hat{T}^{\varepsilon}_{B,k+1}$ sub-domains in the domain T. But in this case, the domain of definition $\hat{T}_{B,k+1}$ in (20) consists of (k+2) parts and defines, by (3), k+2 different objects GML_2^n, and the structure of these new objects is always identic to the original surface. The one to one correspondence (7) guarantees that the new objects have not self-crossing points.

Some examples are given in Figs. 7.

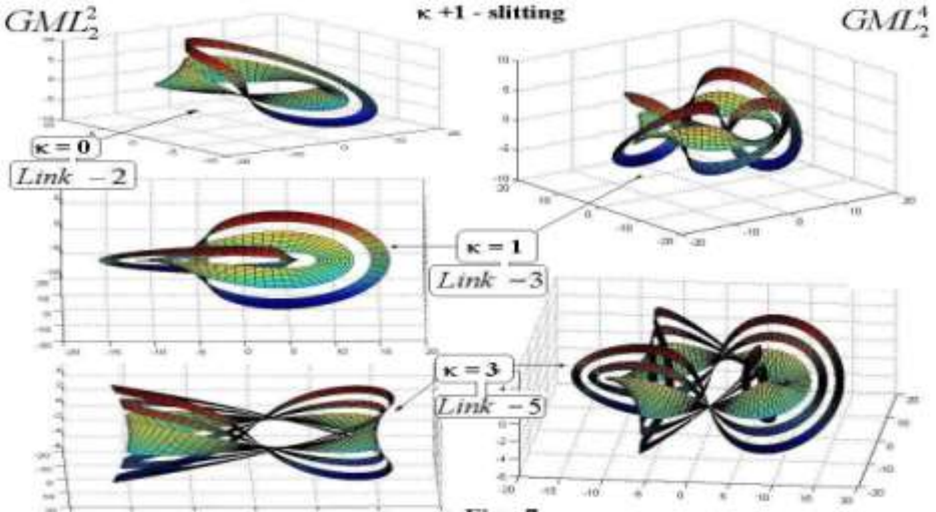


Fig. 7

Let us consider now the general *case B*, when k is an arbitrary natural number.

This means that there exist (k+1)-slit-zones. In this case, by using the same arguments of the previous case A, but considering the domain of definition $\widetilde{T}^*_{k+1}$ in (21), we find identical results. Some examples are given in Figs. 8.

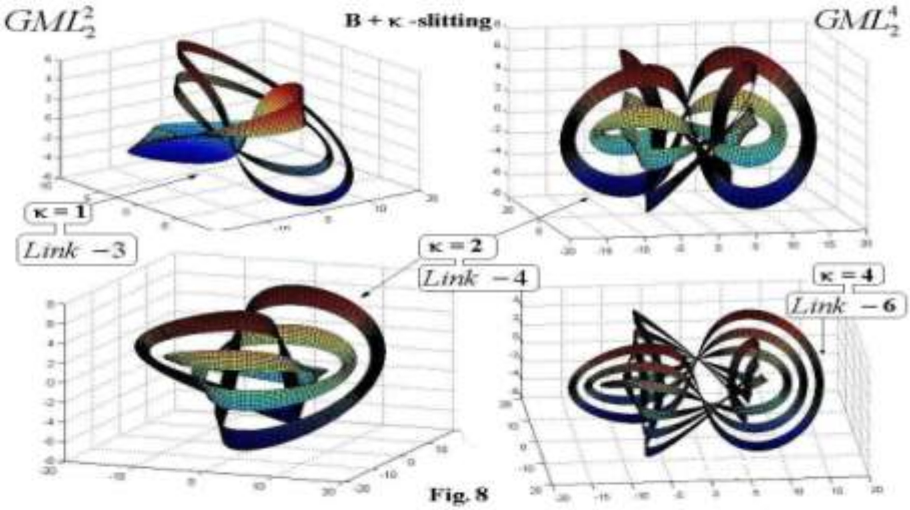


Fig. 8

Theorem 2. If the GML_2^n surface is cut (k+1)-times along (k+1) different (i.e. k=0,1,…,) s-lines and n is an odd number, then for each integer numbers n, k, after:

Case A. - a (B+k)-zone-slits an object Link-(k+1) appears, whose each component is a GML_2^n surface (Ribbon knot with structure $\{n_1\}$); The topologic group of the link- (k+1) in this case is at present unknown; only when k=0, the knot is of type $\{n_1\}$, when n> 1, and of type $\{0_1\}$, when n=1, according the standard classification (see [6-8]); i.e. for every natural numbers ω=0,1,2,…, and k

$$GML_2^{2\omega+1} \xrightarrow{B+k} Link-(k+1) \;\; of \;\; (k+1) \;\;\; objects \;\; GML_2^{4\omega+4} \qquad (24)$$

Case B. - a (k+1)-zones-slit an object Link-(k+2) appears, whose one component is a GML_2^n surface (knot with structure $\{0_1\}$, and each other component is a GML_2^n surface (ribbon knot with structure $\{n_1\}$, except when n=1, since in this case the topological group is $\{0_1\}$); The general topological group, in this case, is at present unknown; i.e. for every natural numbers ω=0,1,2,…, and k,

$$GML_2^{2\omega+1} \xrightarrow{k+1} Link-(k+2) \;\; of \;\; one\, GML_2^{2\omega+1} \; and \; (k+1)\, objects \;\; GML_2^{4\omega+4} \qquad (25)$$

Proof. If n is an odd number (n=2ω+1), then the functions X(τ,θ),Y(τ,θ),Z(τ,θ) is a 4π-periodic function of the argument θ with property (M*); i.e. for each τ∈[-τ*,τ*]

$$X(-\tau,0) = X(\tau,2\pi);\, Y(-\tau,0) = Y(\tau,2\pi);\, Z(-\tau,0) = X(\tau,2\pi); \qquad (26)$$

Let us consider the particular *case A*, when k=0 - this means that there exists only one B-slit-zone. In this case, the one to one correspondence (3) defines a single object, in spite of the fact that the domain of definition $\hat{T}_{B,0}$ in (20) is disconnected. This new object has a new basic line (in particular, according to equations

(7) and restriction (17) which is given by

$$X\left(\frac{\tau^*-\varepsilon}{2},\theta\right)=\left[R+\frac{\tau^*-\varepsilon}{2}\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos\theta$$
$$Y\left(\frac{\tau^*-\varepsilon}{2},\theta\right)=\left[R+\frac{\tau^*-\varepsilon}{2}\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin\theta \qquad (27)$$
$$Z\left(\frac{\tau^*-\varepsilon}{2},\theta\right)=\frac{\tau^*-\varepsilon}{2}\sin\left(\psi+\frac{n\theta}{2}\right)$$

This is a representation of a really closed line, but now $\theta\in[0,4\pi]$ (see Fig. 2.a. or Remark 2, when $\mu\in Q$ in [6]), and therefore the unit normal vector (12) makes 2n+2 rotations around the new basic line (27), since it belongs to a GML_2^n surface.

The following cutting of the GML_2^n body or (B+1)-zones-slit (k=1) is a cutting of the new GML_2^{2n+2} surface, which appears after a B-zones-slit. Since the number of rotations 2n+2 is an even number, and by using the same arguments of the previous theorem, we find that, after a (B+1)-zones-slit, an object Link-(k+2) appears, whose both components GML_2^{2n+2} surfaces.

In general, after a (B+k)-zones-slit an object Link-(k+1) appears, whose each component is a GML_2^{2n+2} surface. Some examples are given in Figs. 9.

Let us consider the particular *case B*, when k=0 - i.e. there exists only one s-slit-zone. Since n is an odd number, then according to (19) after an s-zone-slit the new domain of definition (7) is given by

$$\tilde{T}_1\equiv\{[-\tilde{\tau}_1,\tilde{\tau}_1]\times[0,2\pi)\}\cup\{[\tilde{\tau}_2,\tilde{\tau}_3]\times[0,2\pi)\}\cup\{[-\tilde{\tau}_3,-\tilde{\tau}_2]\times[0,2\pi)\} \qquad (28)$$

This is a disconnected domain, which consist to three parties. The domain defined from the central part of the right hand side of equation (28), according to the representation formulas (7), defines an object GML_2^n . But its width is smaller with respect to the original surface; Each of the remaining parts of the right hand side of eq. (28), similarly to the case A of this theorem, define a geometrical object GML_2^{2n+2}. So that after an s-zone-slit of a GML_2^n , when n is an odd number, appears a link-2, where one of the components is a GML_2^n and the second one is a GML_2^{2n+2} **.**

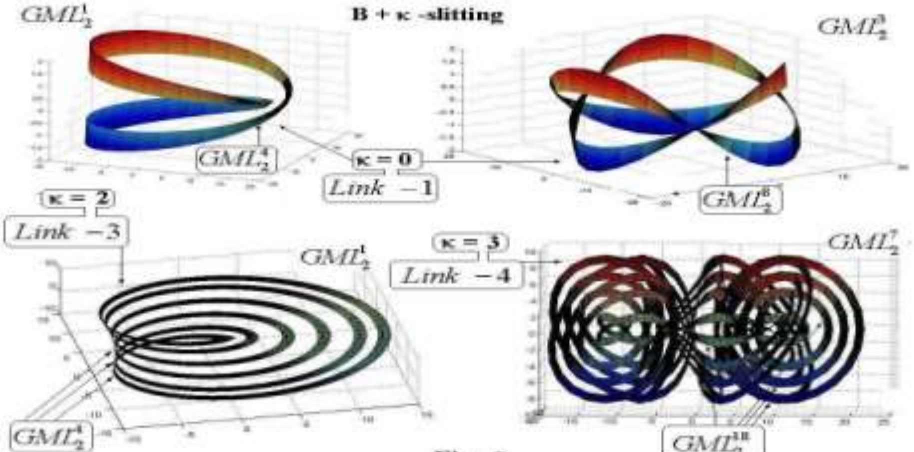


Fig. 9

The following cutting of the GML_2^n body or 2-zones-slit (k=1) is a cut of one of the new GML_2^{2n+2} or GML_2^n surfaces, which appear after a B-zones-slit.

If the second cut is a cutting of the GML_2^{2n+2} surface, then since 2n+2 is a even number, by using the same arguments of previous theorem, we find that after a 2-zones-slit an object Link-3 appears, whose one component is a GML_2^n surface, and each other component is a GML_2^{2n+2} surface.

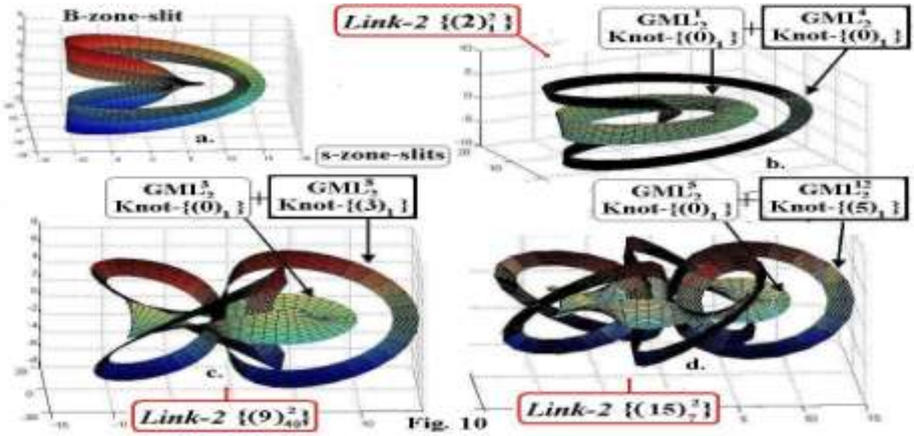


Fig. 10

But if second cut is a cutting of the GML_2^n surface, then, since n is an odd number, by using the same argument of the particular case B of this theorem, we find that after a 2-zones-slit an object Link-3 appears, whose one component is a GML_2^n surface, and all other components are GML_2^{2n+2} surfaces.

Let us consider the general case B, when k is an arbitrary natural number. This means that there exist (k+1)-slit-zones. In this case we can use the previous arguments (when k=1) and remark that the domain of definition $\tilde{T}_{k+1}$ in (19) consists of (2k+3) sub-strips and defines an object Link-(k+2). Therefore, we can conclude that one object is a GML_2^n surface, and each other object is a GML_2^n surface. Some examples are given in Figs. 10.

Lastly, from the above Theorems and Remarks 1,2,3 (also Remarks 1,2,3,4 of the [13]) we can deduce the following facts

Remark 6. Both the previous Theorems still hold when the basic line is a closed sufficiently smooth space line.

After cutting each regular Möbius-Listing surface GML_2^n, whose basic line is a circle, appear objects (object) with the following properties:

- the tangential vectors of the new objects (object) $\vec{r}_\tau$ and $\vec{r}_\theta$ (8), (9) are orthogonal;
- each point of these objects are Hyperbolic (saddle) points if n≠0;
- each point of these objects are Parabolic points if n=0.

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ილია თავხელიძე

GML_2^n – მებუს-ლისტინგის განზოგადებული ზედაპრების ზოგიერთი გეომეტრიული თვისების შესახებ და მათი კავშირი “ლენტისებრი” ხლართების სიმრავლესთან

კვანძების (Knots) და ხლართების (Links) კლასიფიკაცია კარგადაა ცნობილი, მაგრამ ძირითადად მათი სტრუქტურა „ერთგანზომილებიანია“, ანუ მათი განიკვეთი წერტილია. უკანასკნელ წლებში განიხილება ამგვარი სტრუქტურები – „ლენტისებრი“ (Ribbon) ხლართები, რომელთა განიკვეთი ან რაიმე წირია ან მრავალკუთხედი. წარმოდგენილ ნაშრომში განიხილება ისეთი „ლენტისებრი“ კვანძები და ხლართები, რომლებიც მებიუს- ლისტინგის განზოგადებული ზედაპირების საბაზისო წირის პარალელური წირების „გაჭრისას“ წარმოიშვებიან. ნაშრომში შესწავლილია თუ რამდენი გეომეტრიული ობიექტი წარმოიშვება k – „გაჭრის“ შედეგად, რა ტიპის იქნებიან „გაჭრის“ შედეგად წარმოქმნილი ობიექტები და ზოგიერთ შემთხვევაში პასუხი გაეცა შეკითხვას თუ რომელ კლასს, ცნობილი კლასიფიკაციიდან, განეკუთვნება მიღებული სტრუქტურა. მებიუს-ლისტინგის რეგულარული განზოგადებული ზედაპირების შემთხვევაში, როდესაც საბაზისო წირი წრეა შესწავლილია „გაჭრის“ შედეგად მიღებული გეომეტრიული ობიექტების პირველი და მეორე ძირითადი ფორმები და გაუსისა და ნორმალური სიმრუდეები.

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მათემატიკისა და კომპიუტერულ მეცნიერებათა სერია

TEMUR CHILACHAVA, NUGZAR KERESELIDZE

NON-PREVENTIVE CONTINUOUS LINEAR MATHEMATICAL MODEL OF INFORMATION WARFARE

Abstract. In the given work the new direction in the theory of information warfare (mathematical modeling of information warfare) is offered. In particular, the antagonism is meant ”information warfare” mass media (an electronic and printing press, the Internet) two states or two associations of the states, or two powerful economic structures (consortiums) conducting under the relation to each other purposeful misinformation, propagation. As the third side in process association of the international organizations (the United Nations, OSCE, EU, the WTO etc.) which effort are directed on intensity removal between the antagonistic states, the sides and the termination of information warfare acts.

The general continuous linear mathematical model of information warfare between two antagonistic sides which considers an antagonism case as equipotent associations ("yak-bear"), and it is strong different - ("wolf-lamb") is constructed. As required functions quantities of information at present time, made each of the sides promoting achievement of the purposes, to the chosen strategy are taken. In that specific case the models, each side identical rates wages information warfare and reacts to appeals of the international organizations. In turn, the third side in regular intervals reacts to intensity of information attacks of the antagonistic sides.

Exact analytical solutions of a Cauchy's problem for system of the linear differential equations of the first order with constant factors are received.

Parities between constants of model and initial conditions are revealed, at which:

1. The antagonistic sides, despite increasing appeals of the third side, intensify information attacks.
2. One of the antagonistic sides, under the influence of the third side stops, eventually, information warfare (an exit of the corresponding solution on zero) while another strengthens it.

3. Both antagonistic sides, after achievement of a maximum of activity, reduce it under the influence of the third side, and through final time, and at all stop information attacks (an exit of solutions on zero).

In the first case, it is necessary to expect transformation of information warfare in a hot phase, in the second – it is less probable, in the third - it is at all excluded.

The offered model of information warfare, except theoretical interest has as well the important practical meaning. She allows, on the basis of supervision and the analysis, already at an early stage of information attacks, to establish true intentions of each of the sides and character of development of information warfare.

2000Mathematics Subject Classification: 93A30, 00A71

Key words and phrases: information warfare, antagonistic sides, continuous linear mathematical model, ignoring of an opposite side, non-preventive model, hot phase.

1. Introduction

The theory of the information warfare, which formalization has begun all three-four ten years ago, now has already wide applied meaning. It is considered actively by many countries by working out of information safety. For the first time in the USA the presidential commission for protection of a so-called critical infrastructure has been created. Then on the basis of the conclusion of this commission has been developed № 63 instruction of the president which in 1998, became a basis of the governmental policy of maintenance of information safety [1]. The leading countries already have begun purposeful preparation of narrow experts of information war. In the USA, at national university of defence the school of information warfare and strategy operates. At the Californian sea school to group of information warfare read courses of lectures: principles of information operations; psychological operations; information warfare: planning and an estimation; an estimation of information warfare.

Russia, the truth with delay, but too operates in this direction. The Ministry of Defence of Russia has created the information and propaganda centre which along with other problems, will prepare khakers attacks on information resources of the opponent. This decision of the Ministry of Defence of Russia, was the answer to the task of the president of Russia - prepare the offers connected with creation of the centre of preparation of experts which can to wage information warfare by the newest technologies [2, 3].

According to some sources problems of the information and propaganda centre will be: intimidation of the opponent, destruction of its information communications and preservation of the, creation information and misinformation parts and rendering of influence on public opinion both to the conflict and during a confrontation. To necessity of creation of information armies the management of armed forces of Russia was resulted by the analysis of war of 2008 with Georgia.

Originally the term ”information warfare” Thomas Rona has applied in 1976 in the report

”systems weapons and information warfare” which intended for company **Boeing** [4]. T.Rona has noticed that by then, the information infrastructure became a central component of economy of the USA and simultaneously the idle time, less protected purpose both in military and in a peace time.

For the present the uniform definition of the term ”information warfare” is not accepted, but intuitively it is considered that information warfare is purposeful actions on creation of the information superiority, by means of destruction of the information, information systems of an opposite side, thus simultaneously there is a process of protection of own information and information systems.

By information warfare also mean a complex of actions for creation of information influence on public consciousness to change behaviour of people, to impose them the purposes which do not enter into their interests. On the other hand, protection against the same influence is necessary.

As the state information resources often become objects of an attack and protection, the state is compelled to give to an information technology a great attention. Accordingly, in the theory of information war the great number of researches is devoted safety of the information, information systems and processes.

On the other hand, studying of information streams as the information stream which to fall upon mass consciousness, in most cases, allows to manipulate people is essential [5].

The description a mathematical apparatus various a component of information warfare and its studying already is included into sphere of interests of many scientists. In this direction it is necessary to note use of the mathematical theory of an information transfer on communication channels for construction of model of information influence. Attempts of estimations of efficiency of concrete information influences are undertaken [6]. By means of the theory of counts and games models of information networks and information warfare are made, and for the contradictory parties the mixed strategy are found [7, 8]. Strategy, basically, are calculated on deducing out of operation information infrastructures or their protection by means of as physical and program (viruses, Trojans, cyber attacks) influences.

Our approach

In the present work as our purpose studying of quantity of information streams by means of new mathematical models of information warfare was [9]. We mean an antagonism by information warfare by mass media (an electronic and printing press, the Internet) two states or two associations of the states, or the economic structures (consortiums) conducting under the relation to each other purposeful misinformation, propagation.

In a world information field for ideological, political and economic targets the purposeful information and the misinformation, which allocation from the general background in most cases for the unprepared person very difficult is actively used.

The purposes of information warfare can be:

- Drawing of a loss to image of the opposite country – creation from it an image of the enemy.
- Discredit of a management of the opposite country.
- Demoralization of staff of armed forces and the peace population of other country.
- Public opinion creation, both in the country, and behind its limits, for the argument and the justification of possible power actions in the future.
- Counteraction to geopolitical ambitions of an opposite side etc.

The international organizations react to occurring processes in the modern world in this or that form and activity. Therefore in the course of information warfare as the third side we consider association of the international organizations (the United Nations, OSCE, EU, the WTO etc.) which effort are directed on intensity removal between the antagonistic states, the parties and the termination of information warfare.

In the given work we have constructed the general continuous linear mathematical model of information warfare between two antagonistic sides, in the presence of the third, peace-making party. The model considers an antagonism case as equipotent associations ("yak-bear"), and is strong different ("wolf-lamb").

We consider that information warfare is conducted against each other by the first and second sides, and by the third side mean the international organizations. In that specific case the models, each side identical rates wages information warfare and reacts to appeals of the international organizations. In turn, the third side in regular intervals reacts to intensity of information attacks of the antagonistic sides. Exact analytical solutions are found in model of ignoring of an opposite side. By means of the analysis of solutions character of actions of the sides in information warfare, depending on that what starting conditions of the sides, and parities between indexes of aggression, peace-making readiness and peace-making activity is established.

2. The system of the equations and initial conditions

All three sides involved in process of information warfare extend information for object in view achievement. At the moment of time $t \in [0,+\infty)$ quantity of the information extended by each of the sides we will designate accordingly through $N_1(t)$, $N_2(t)$, $N_3(t)$. The quantity of information at the moment of time t, is defined as the sum, all provoking information which are extended by each of the sides all mass media.

The detailed analysis of mathematical models of dynamics of populations, and also of Lanchester's models of military operations [10], has led us to the following general continuous linear mathematical model of information warfare:

$$\begin{cases} \dfrac{dN_1(t)}{dt} = \alpha_1 N_1(t) + \alpha_2 N_2(t) - \alpha_3 N_3(t) \\ \\ \dfrac{dN_2(t)}{dt} = \beta_1 N_1(t) + \beta_2 N_2(t) - \beta_3 N_3(t) \\ \\ \dfrac{dN_3(t)}{dt} = \gamma_1 N_1(t) + \gamma_2 N_2(t) + \gamma_3 N_3(t) \end{cases} \tag{2.1}$$

with initial conditions

$$N_1(0) = N_{10},\ N_2(0) = N_{20},\ N_3(0) = N_{30}, \qquad (2.2)$$

where, $\alpha_1,\ \alpha_3,\ \beta_2,\ \beta_3\ \geq 0$, $\gamma_i\ \geq 0 \quad i = \overline{1,3}$, $\alpha_2,\ \beta_1$ - constant factors.

These constant factors are model constants, thus we name α_1, β_2 factors of growth of provoking statements according to the first and the second the sides in the absence of the third side (relative growth rates of quantity of statements).

These constant factors are model constants, thus we name factors of growth of provoking statements according to the first and the second the sides in the absence of the third side (relative growth rates of quantity of statements).

In the general linear model (2.1) speed of change of quantity of the information spread by the first and second sides linearly depends on quantity of information extended by the sides and the international peace-making organizations.

Speed of change of quantity of pacifying information extended by the third side linearly grows or is directly proportional to quantity of information spread by all three sides.

In initial conditions (2.2), N_{10}, N_{20}, N_{30} non-negative constants, thus:

If $N_{10} > 0, N_{20} > 0$, then both sides are initiators of information warfare.

If $N_{10} > 0,\ N_{20} = 0$, then the first side is the initiator of information warfare.

If $N_{10} = 0,\ N_{20} > 0$, then the second side is the initiator of information warfare.

The third side initially does not spread any information ($N_{30} = 0$) or does preventive character peace-making statements ($N_{30} > 0$) and then starts to react to the provocative information extended by the antagonistic sides.

3. Model of ignoring of an opposite side.

The antagonistic sides which with identical intensity conduct information warfare, opposite side spread information, but thus both sides can ignore should listen equally to appeals of the third – the peace-making side.

In this case, in the general linear model (2.1) some factors can be put equal to zero. In particular, α_2 and β_1 also are equal to zero. We will put also that $\gamma_3 = 0$ or the third side equally reacts only to widespread provoking information antagonistic sides.

Thus, we will put $\alpha_1 = \beta_2 = \alpha, \quad \alpha_3 = \beta_3 = \beta, \quad \gamma_1 = \gamma_2 = \gamma$.

Then the system (2.1) will written as follows (assume the following air):

$$\begin{cases} \dfrac{d}{dt} N_1(t) = \alpha N_1(t) - \beta N_3(t) \\ \\ \dfrac{d}{dt} N_2(t) = \alpha N_2(t) - \beta N_3(t) \\ \\ \dfrac{d}{dt} N_3(t) = \gamma N_1(t) + \gamma N_2(t) \end{cases} \tag{3.1}$$

The solution of system (3.1) in $[0,\infty)$ area will write down as follows ($N_{30} = 0$, international organizations "are not awake" and react only to already launched information warfare):

a) $D = \alpha^2 - 8\beta\gamma > 0$

$$N_3(t) = \frac{\gamma(N_{10} + N_{20})}{\sqrt{D}} \left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) \tag{3.2}$$

$$N_1(t) = \frac{N_{10} - N_{20}}{2} \; e^{\alpha t} + \beta \; \frac{\gamma(N_{10} + N_{20})}{\lambda_2 \sqrt{D}} \; e^{\lambda_1 t} - \beta \; \frac{\gamma(N_{10} + N_{20})}{\lambda_1 \sqrt{D}} \; e^{\lambda_2 t} \tag{3.3}$$

$$N_2(t)=\frac{N_{20}-N_{10}}{2}\,e^{\alpha t}+\ \beta\,\frac{\gamma\left(N_{10}+N_{20}\right)}{\lambda_2\sqrt{D}}\,e^{\lambda_1 t}-\ \beta\,\frac{\gamma\left(N_{10}+N_{20}\right)}{\lambda_1\sqrt{D}}\,e^{\lambda_2 t} \qquad (3.4)$$

$$\lambda_1=\frac{\alpha+\sqrt{\alpha^2-8\beta\gamma}}{2}>0\ \ ,\ \lambda_2=\frac{\alpha-\sqrt{\alpha^2-8\beta\gamma}}{2}>0.$$

b) $D=\alpha^2-8\beta\gamma=0$

$$N_3(t)=\gamma(N_{10}+N_{20})te^{\frac{\alpha}{2}t} \qquad (3.5)$$

$$N_1(t)=\frac{N_{10}-N_{20}}{2}\,e^{\alpha t}\ +\ \frac{N_{10}+N_{20}}{2\alpha}(\alpha+4\beta\gamma t)\ \ e^{\frac{\alpha}{2}t} \qquad (3.6)$$

$$N_2(t)=\frac{N_{20}-N_{10}}{2}\,e^{\alpha t}\ +\ \frac{N_{10}+N_{20}}{2\alpha}(\alpha+4\beta\gamma t)\ \ e^{\frac{\alpha}{2}t} \qquad (3.7)$$

c) $D=\alpha^2-8\beta\gamma<0$

$$N_3(t)=\frac{2\gamma(N_{10}+N_{20})}{\sqrt{-D}}\,e^{\frac{\alpha}{2}t}\,\sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t\right) \qquad (3.8)$$

$$N_1(t)=\frac{N_{10}-N_{20}}{2}\,e^{\alpha t}+$$

$$\sqrt{2\beta\gamma}\,\frac{(N_{10}+N_{20})}{\sqrt{-D}}\,e^{\frac{\alpha}{2}t}\,\sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\varphi\right) \qquad (3.9)$$

$$N_2(t) = \frac{N_{20} - N_{10}}{2}\, e^{\alpha t} +$$

$$\sqrt{2\beta\gamma}\, \frac{(N_{10} + N_{20})}{\sqrt{-D}}\, e^{\frac{\alpha}{2}t} \sin\left(\frac{\sqrt{8\beta\gamma - \alpha^2}}{2} t + \varphi \right) \tag{3.10}$$

4. **The Analysis of the received results.**

In model of ignoring of an opposite side (3.1) it is possible to consider α as an indicator (index) of aggression of the antagonistic sides, β - an indicator of their readiness for the world, to listen to peace-making appeals of the international organizations, γ - an indicator of peace-making activity of the international organizations. As it will be shown more low, depending on that is more – an aggression index, or indexes of readiness for the world and peace-making activity, character and development of information war essentially varies.

Let's consider a case when the international organizations have not accepted preventive a measure and we investigate development of information warfare at various values D.

4.I. $D = \alpha^2 - 8\beta\gamma > 0$.

In this case, a square of an index of aggression more than eightfold product of indexes of readiness for the world and peace-making activity that unequivocally specifies in high aggression of the antagonistic sides in information warfare.

4.I.I. ($N_{10} = N_{20}$). In that case when the international organizations have not undertaken preventive a measure, and the antagonistic sides have begun information warfare under equal starting conditions influence of the international organizations on the first and second side, is ineffectual – they strengthen information attacks.

Really, from (3.3), (3.4) and with the account $N_{10} = N_{20}$ we will receive:

$$N_1(t) = 2\beta\gamma N_{10} \frac{1}{\sqrt{D}} \left(\frac{e^{\lambda_1 t}}{\lambda_2} - \frac{e^{\lambda_2 t}}{\lambda_1} \right) \to +\infty, \tag{4.1}$$

$$N_2(t)=2\beta\gamma N_{10}\frac{1}{\sqrt{D}}\left(\frac{e^{\lambda_1 t}}{\lambda_2}-\frac{e^{\lambda_2 t}}{\lambda_1}\right)\to+\infty, \qquad (4.22)$$

when $t\to+\infty$.

It is enough to show justice (4.1) for $N_1(t)$, as owing to (4.2) $N_2(t)$ it is identically equal $N_1(t)$.

Really on a semi interval $[0,+\infty)$ function $N_1(t)$ positive, increasing and unlimited from above.

Positivity $N_1(t)$ follows from following parities:

$$N_1(0)=N_{10}>0\ ;\ \ 2\beta\gamma N_{10}\frac{1}{\sqrt{D}}>0, \text{ as } \ \beta,\ \gamma>0,$$

and

$$\frac{e^{\lambda_1 t}}{\lambda_2}-\frac{e^{\lambda_2 t}}{\lambda_1}>0$$

Follows from parities: $\lambda_1>\lambda_2>0,\ \frac{1}{\lambda_2}>\frac{1}{\lambda_1}>0,\ e^{\lambda_1 t}>e^{\lambda_2 t}$.

Positivity of a derivative $N_1(t)$ is an indicator of its increase.

$$N'_1(t)=\frac{2\beta\gamma N_{10}}{\sqrt{D}}\frac{\lambda_2 e^{\lambda_2 t}}{\lambda_1}\left(\frac{\lambda_1^{\ 2}}{\lambda_2^{\ 2}}e^{(\lambda_1-\lambda_2)t}-1\right)>0,$$

As in this expression all factors are positive.

$N_1(t)$ it is unlimited from above, since

$$N_1(t)\cong 2\beta\gamma N_{10}\frac{1}{\sqrt{D}}\ \frac{e^{\lambda_1 t}}{\lambda_2}, \text{ when } t\to\infty.$$

As to the international organizations,

$$N_3(t)=\frac{2\gamma N_{10}}{\sqrt{D}}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)\ge 0,$$

At $t \in [0,+\infty)$, $N_3(t) \to +\infty$, when $t \to +\infty$.

$N_3(0)=0$ and then increases together with t, as its derivative positive.

$$N'_3(t)=\frac{2\gamma N_{10}}{\sqrt{D}}\lambda_2 e^{\lambda_2 t}\left(\frac{\lambda_1}{\lambda_2}e^{(\lambda_1-\lambda_2)t}-1\right)>0$$

$N_3(t)$ unlimited from above, as

$$N_3(t) \cong \frac{2\gamma N_{10}}{\sqrt{D}} e^{\lambda_1 t}, \text{ at } t \to +\infty.$$

Thus, in not preventive model of information warfare, at, $D=\alpha^2-8\beta\gamma>0$ and equal starting a condition ($N_{10}= N_{20}$) the antagonistic sides, they strengthen the activity. Functions - $N_1(t)$, $N_2(t)$, $N_3(t)$ monotonously increase – i.e. information war faredoes not stop, and all expands.

4.I.2. ($N_{10}> N_{20}$). If at the antagonistic sides different launching sites and a starting condition of the first side more than the second, function

$$N_1(t)=\frac{N_{10}-N_{20}}{2}e^{\alpha t}+\beta\frac{\gamma(N_{10}+N_{20})}{\sqrt{D}}\left(\frac{1}{\lambda_2}e^{\lambda_1 t}-\frac{1}{\lambda_1}e^{\lambda_2 t}\right), \qquad (4.3)$$

at $t \in [0,+\infty)$ positive, increasing also it is unlimited from above.

The second member of expression (4.3) we already investigated in IV.I.I and it positive, increasing and is unlimited from above. To it function with positive factor $\frac{N_{10}-N_{20}}{2}>0$, $\alpha>0$, which also is positive is added exhibitor, increasing and is unlimited from above.

Accordingly and their sum, i.e. $N_1(t)$ is positive, increasing and unlimited from above.

As to $N_2(t)$, it in a point $t=0$ is positive $N_2(0)= N_{20}>0$, and its derivative, proceeding from system (3.1) and entry conditions (2.2) positive - $N_2'(0)=\alpha N_{20}>0$, therefore in a certain vicinity on the right, $N_2(t)$ is positive and increasing.

.

$$N_2(t)=\frac{N_{20}-N_{10}}{2}\ e^{\alpha t}+\beta\ \frac{\gamma(N_{10}+N_{20})}{\sqrt{D}}\left(\frac{1}{\lambda_2}e^{\lambda_1 t}-\frac{1}{\lambda_1}e^{\lambda_2 t}\right) \qquad (4.4)$$

But with increase t increases and $N_2(t)$ which reaches the maximum, and then it starts to decrease monotonously and aspires to $-\infty$.

Really, an expression sign

$$N_2(t)=e^{\alpha t}\ \left(\ \frac{N_{20}-N_{10}}{2}+\beta\ \frac{\gamma(N_{10}+N_{20})}{\lambda_2\sqrt{D}}e^{-\lambda_2 t}-\beta\ \frac{\gamma(N_{10}+N_{20})}{\lambda_1\sqrt{D}}e^{-\lambda_1 t}\ \right) \qquad (4.5)$$

Defines a sign on a factor bracketed

$$F(t)\equiv\frac{N_{20}-N_{10}}{2}+\beta\ \frac{\gamma(N_{10}+N_{20})}{\lambda_2\sqrt{D}}e^{-\lambda_2 t}-\beta\ \frac{\gamma(N_{10}+N_{20})}{\lambda_1\sqrt{D}}e^{-\lambda_1 t} \qquad (4.6)$$

Absolute value of the second and third member (4.6) becomes as much as small for enough big t, thus, when $t\to+\infty$ the sign (4.6) is defined by a sign on the first composed $\frac{N_{20}-N_{10}}{2}$. I.e. it will be negative since $\frac{N_{20}-N_{10}}{2}$<0. Proceeding from it, $N_2(t)$ will be monotonously aspires to $-\infty$. $N_2(t)$ it is continuous, for represents the sum of continuous functions and as at $t=0$ it is positive, and at big t - it is negative, it will necessarily cross an absciss, i.e. it has a zero. It will occur in a point t^* which is the decision of a following transcendental equation

$$\frac{N_{20}-N_{10}}{2}+\beta\ \frac{\gamma(N_{10}+N_{20})}{\lambda_2\sqrt{D}}e^{-\lambda_2 t}-\beta\ \frac{\gamma(N_{10}+N_{20})}{\lambda_1\sqrt{D}}e^{-\lambda_1 t}=0, \qquad (4.7)$$

.i.e $N_2(t^*)=0$. At $0\le t<t^*$, $N_2(t)$ positive also reaches a maximum in a point t^{**} which is the equation solution $N'_2(t)=0$,

$$\alpha\ \left(\frac{N_{20}-N_{10}}{2} + \beta\ \frac{\gamma\left(N_{10}+N_{20}\right)}{\lambda_2\sqrt{D}} e^{-\lambda_2 t} - \beta\ \frac{\gamma\left(N_{10}+N_{20}\right)}{\lambda_1\sqrt{D}} e^{-\lambda_1 t} \right) +$$

$$+\beta\ \frac{\gamma\left(N_{10}+N_{20}\right)}{\sqrt{D}} e^{-\lambda_1 t} - \beta\ \frac{\gamma\left(N_{10}+N_{20}\right)}{\sqrt{D}} e^{-\lambda_2 t} = 0 \qquad (4.8)$$

As to the international organizations, the third side, it strengthens the activity

$$N_3(t) = \frac{\gamma(N_{10}+N_{20})}{\sqrt{D}}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) \to +\infty, \qquad (4.9)$$

when $t \to +\infty$.

At $t=0$ $N_3(0)=0$, equals to zero, then $N_3(t)$ increases. Really

$$N'_3(t) = \frac{\gamma(N_{10}+N_{20})}{\sqrt{D}}\left(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}\right) > 0 \qquad (4.10)$$

$N_3(t)$ it is unlimited from above, it is valid -

$$N_3(t) \cong \frac{\gamma\left(N_{10}+N_{20}\right)}{\sqrt{D}} e^{\lambda_1 t} \to +\infty, \text{ at } t \to +\infty.$$

4.I.3. ($N_{10} < N_{20}$). If at the antagonistic sides different launching sites and starting conditions of the second side more the first, i.e. $N_{10} < N_{20}$, then the sides change roles and is received symmetric results, under the relation of the previous point – already second the side strengthens information attacks

$$N_2(t) = \frac{N_{20}-N_{10}}{2} e^{\alpha t} + \beta\ \frac{\gamma\left(N_{10}+N_{20}\right)}{\lambda_2\sqrt{D}} e^{\lambda_1 t} - \beta\ \frac{\gamma\left(N_{10}+N_{20}\right)}{\lambda_1\sqrt{D}} e^{\lambda_2 t} \to +\infty, \qquad (4.11)$$

when $t \to +\infty$. As to the first side, it makes active in the beginning information attacks, leaves on a maximum, further reduces, and then and at all stops information warfare (in t^* leaves on zero)

$$N_1(t)=\frac{N_{10}-N_{20}}{2}\,e^{\alpha t}+\beta\,\frac{\gamma(N_{10}+N_{20})}{\lambda_2\sqrt{D}}\,e^{\lambda_1 t}-\beta\,\frac{\gamma(N_{10}+N_{20})}{\lambda_1\sqrt{D}}\to-\infty, \qquad (4.12)$$

when $t\to+\infty$. t^* represents the solution of the transcendental equation

$$\frac{N_{10}-N_{20}}{2}+\beta\,\frac{\gamma(N_{10}+N_{20})}{\lambda_2\sqrt{D}}e^{-\lambda_2 t}-\beta\,\frac{\gamma(N_{10}+N_{20})}{\lambda_1\sqrt{D}}e^{-\lambda_1 t}=0 \qquad (4.13)$$

The third side strengthens the activity and for it it is fair

$$N_3(t)=\frac{\gamma(N_{10}+N_{20})}{\sqrt{D}}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)\to+\infty, \qquad (4.14)$$

at $t\to+\infty$.

Thus, it is necessary to notice that the third side at not the preventive approach ($N_{30}=0$), can have partial influence on a course of information warfare. In particular, influences one of the sides if the antagonistic sides have begun information warfare under unequal starting conditions $(N_{10}\neq N_{20})$. Thus, the third side influences that antagonistic side which launching site is “weaker". i.e. it is less. And this side, through certain time stops information warfare though initially actively joins in it, leaves on a maximum of actions, however then, reduces information attacks, and in the end and at all them stops.

4.II. $D=\alpha^2-8\beta\gamma=0$.

In this case the aggression index is still high and there are analogies to a case

$D=\alpha^2-8\beta\gamma>0$. Really.

4.II.1. ($N_{10}=N_{20}$ **)**.In that case when the international organizations have not taken preventive measures ($N_{30}=0$ **),** and the antagonistic sides have begun information warfare at equal starting a condition ($N_{10}=N_{20}$), influence international the organizations on the first and second sides without results – the last strengthen information attacks and starting with

(3.5) - (3.7)

$$N_1(t)\to+\infty,\; N_2(t)\to+\infty,\; N_3(t)\to+\infty, \quad \text{при } t\to+\infty.$$

Let's really copy (3.5) - (3.7) as follows

$$N_1(t) = \frac{N_{10}}{2}(\alpha t + 2)\ e^{\frac{\alpha}{2}t} \tag{4.15}$$

$$N_2(t) = \frac{N_{10}}{2}(\alpha t + 2)\ e^{\frac{\alpha}{2}t} \tag{4.16}$$

$$N_3(t) = 2\gamma N_{10} t e^{\frac{\alpha}{2}t} \tag{4.17}$$

On a semi interval $[0,+\infty)$ functions $N_1(t)$, $N_2(t)$ are positive, increasing and are unlimited from above, since, are product defined on a semi interval $[0,+\infty)$, positive, and unlimited from above functions $\frac{N_{10}}{2}(\alpha t + 2)$ and $e^{\frac{\alpha}{2}t}$. The Same it is possible to tell and about function $N_3(t)$ on an interval $(0, +\infty)$.

4.II.2. ($N_{10} > N_{20}$). If at the antagonistic sides different starting conditions, and starting conditions of the first side surpass the second, on a semi interval $[0,+\infty)$ function $N_1(t)$, which has the following appearance

$$N_1(t) = \frac{N_{10} - N_{20}}{2}\ e^{\alpha t} + \frac{N_{10} + N_{20}}{4}(\alpha t + 2)\ e^{\frac{\alpha}{2}t} \tag{4.18}$$

Positive, increasing also it is unlimited from above since represents the sum and positive, increasing and unlimited from above functions $\frac{N_{10} - N_{20}}{2}\ e^{\alpha t}$ and $\frac{N_{10} + N_{20}}{2}(\alpha t + 2)\ e^{\frac{\alpha}{2}t}$ defined on a semi interval $[0, +\infty)$.

As to $N_2(t)$, it in a point $t = 0$ is positive - $N_2(0) = N_{20} > 0$. In this point its derivative owing to system (3.1), and entry conditions (2.2), is positive - $N'_2(0) = \alpha\ N_{20} > 0$. Therefore, on some right vicinity of a point 0, $N_2(t)$ it is positive and increasing.

$$N_2(t) = \frac{N_{20} - N_{10}}{2}\ e^{\alpha t} + \frac{N_{10} + N_{20}}{4}(\alpha t + 2)\ e^{\frac{\alpha}{2}t} \tag{4.19}$$

But already with increase t, $N_2(t)$ reaches the maximum value, and then monotonously decreases and aspires to $-\infty$.

Really, an expression sign

$$N_2(t) = e^{\alpha t}\Big(\frac{N_{20} - N_{10}}{2} + \frac{N_{10} + N_{20}}{4}(\alpha t + 2)\; e^{-\frac{\alpha}{2}t} \Big) \qquad (4.20)$$

defines a sign on the factor bracketed

$$F(t) \equiv \frac{N_{20} - N_{10}}{2} + \frac{N_{10} + N_{20}}{4}(\alpha t + 2)\; e^{-\frac{\alpha}{2}t} \qquad (4.21)$$

Value of the second member (4.21) becomes as much as small at big t, therefore when $t \to +\infty$ the sign (4.21) defines a sign the on the first composed $\frac{N_{20} - N_{10}}{2}$, i.e. will be negative, for $\frac{N_{20} - N_{10}}{2}$ <0. Proceeding from told, $N_2(t)$ aspires to $-\infty$ monotonously. $N_2(t)$ represents the sum of continuous functions and consequently itself it is continuous. Therefore, time it

in $t=0$ is positive, and at big t- is negative, owing to the theorem of Bolzano-Koshi, $N_2(t)$ should will cross an absciss, i.e. it has a zero. It will occur in some point t_2^*, which represents the solution of a following transcendental equation:

$$\frac{N_{20} - N_{10}}{2} + \frac{N_{10} + N_{20}}{4}(\alpha t + 2)\; e^{-\frac{\alpha}{2}t} = 0 \qquad (4.22)$$

i.e. $N_2(t_2^*) = 0$. At $0 \le t < t_2^*$ $N_2(t)$ it is positive and reaches the maximum in a point t_2^{**}, which is the equation solution $N_2^{/}(t) = 0$,

$$\frac{N_{20} - N_{10}}{2} + \frac{N_{10} + N_{20}}{4}(\alpha t + 2)\; e^{-\frac{\alpha}{2}t} + \frac{N_{10} + N_{20}}{4}\; e^{-\frac{\alpha}{2}t} -$$

$$- \frac{N_{10} + N_{20}}{8}(\alpha t + 2)\; e^{-\frac{\alpha}{2}t} = 0 \qquad (4.23)$$

As to the international organizations, they strengthen the activity (see (3.5)).

4.II.3.($N_{10} = N_{20}$) . If at the antagonistic sides different starting conditions, and starting position of the second side more the first the first and second sides change roles and is had symmetric results - already the second side strengthens information attacks.

$$N_2(t) \to +\infty, \tag{4.24}$$

at $t \to +\infty$. And the first side at first makes active information attacks, leaves on a maximum, then reduces, further and at all stops information warfare.

4.III. $D = \alpha^2 - 8\beta\gamma < 0$ **.**

In this case, a square of an index of aggression less than eightfold product of indexes of readiness for the world and peace-making activity, i.e. it is expected that peace-making activity "will pacify" (will block) aggression.

4.**III.1. (** N_{10} **=** N_{20} **).** Really, if the international organizations have not taken preventive measures, and the antagonistic sides have begun information war at equal launching sites, then influence the international organizations on the first and the second the sides the productive.

Really, functions, $N_1(t)$ and $N_2(t)$ owing to (3.9), (3.10)

$$N_1(t) = \sqrt{2\beta\gamma}\,\frac{2N_{10}}{\sqrt{-D}}\, e^{\frac{\alpha}{2}t} \sin\left(\frac{\sqrt{8\beta\gamma - \alpha^2}}{2} t + \varphi\right) \tag{4.25}$$

$$N_2(t) = \sqrt{2\beta\gamma}\,\frac{2N_{10}}{\sqrt{-D}}\, e^{\frac{\alpha}{2}t} \sin\left(\frac{\sqrt{8\beta\gamma - \alpha^2}}{2} t + \varphi\right) \tag{4.26}$$

Leave on zero, when

$$t^* = \frac{2(\pi - \varphi)}{\sqrt{-D}}, \tag{4.27}$$

where, for φ fair

$$\varphi = arctg\frac{\sqrt{-D}}{\alpha} \tag{4.28}$$

As to the international organizations,

$$N_3(t) = \frac{4\gamma N_{10}}{\sqrt{-D}} e^{\frac{\alpha}{2}t} \sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t\right) \tag{4.29}$$

That they leave on zero in $t^{**} = \dfrac{2\pi}{\sqrt{-D}}$, after time moment t^*, i.e. the third side finishes distributions of peace-making appeals after, the antagonistic sides will finish information warfare.

4.III.2. ($N_{10} > N_{20}$). If at the antagonistic sides different launching sites and thus, starting conditions of the first side more than the second from (3.9), having equated to zero function $N_1(t)$, we will find time when the first side stops information warfare

$$\frac{N_{10}-N_{20}}{2} e^{\frac{\alpha}{2}t} + \sqrt{2\beta\gamma}\,\frac{(N_{10}+N_{20})}{\sqrt{-D}} \sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\varphi\right) = 0, \tag{4.30}$$

From (4.30) we will receive

$$\sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\varphi\right) = -\frac{N_{10}-N_{20}}{N_{10}+N_{20}}\,\frac{\sqrt{-D}}{2\sqrt{2\beta\gamma}} e^{\frac{\alpha}{2}t} \tag{4.31}$$

he solution (4.31) exists only in that case, when

$$1 \geq -\frac{N_{10}-N_{20}}{N_{10}+N_{20}}\,\frac{\sqrt{-D}}{2\sqrt{2\beta\gamma}} e^{\frac{\alpha}{2}t} \geq -1 \tag{4.32}$$

Or, something, in our case

$$0 < \frac{N_{10}-N_{20}}{N_{10}+N_{20}}\,\frac{\sqrt{-D}}{2\sqrt{2\beta\gamma}} e^{\frac{\alpha}{2}t} \leq 1$$

$$0 \le t \le \frac{2}{\alpha}\ln\left(\frac{N_{20}+N_{10}}{N_{10}-N_{20}}\sqrt{\frac{8\gamma\beta}{8\beta\gamma-\alpha^2}}\right) \tag{4.33}$$

The solution (5.1.3.7) t_1^* - should satisfy to a condition

$$0 \le t_1^* \le \frac{2}{\alpha}\ln\left(\frac{N_{20}+N_{10}}{N_{10}-N_{20}}\sqrt{\frac{8\gamma\beta}{8\beta\gamma-\alpha^2}}\right), \tag{4.34}$$

and

$$t_1^* = \frac{2}{\sqrt{-D}}\arcsin\left(\frac{N_{10}-N_{20}}{N_{10}+N_{20}}\cdot\frac{\sqrt{-D}}{\sqrt{8\beta\gamma}}e^{\frac{\alpha}{2}t_1^*}\right)+\frac{2}{\sqrt{-D}}(\pi-\varphi) > \frac{2}{\sqrt{-D}}(\pi-\varphi) \tag{4.35}$$

Performance of conditions (4.34) and (4.35) is possible at selection γ, product $\beta\ \gamma$ should be enough great number.

As to the second side, at big t - the sign $N_2(t)$ in (3.10) coincides with a sign $\frac{N_{20}-N_{10}}{2}$, i.e. negative, therefore at $N_2(t)$ is available zero

$$\frac{N_{20}-N_{10}}{2}\ e^{\alpha t} +$$

$$\sqrt{2\beta\gamma}\ \frac{(N_{10}+N_{20})}{\sqrt{-D}}\ e^{\frac{\alpha}{2}t}\ \sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\varphi\right)=0 \tag{4.36}$$

The solution of the equation (4.35) - t_2^*, should satisfy to conditions

$$0 \le t_2^* \le \frac{2}{\alpha}\ln\left(\frac{N_{20}+N_{10}}{N_{10}-N_{20}}\sqrt{\frac{8\gamma\beta}{8\beta\gamma-\alpha^2}}\right) \tag{4.37}$$

and

$$t_2^* = \frac{2}{\sqrt{-D}}\left(\arcsin\left(\frac{N_{10}-N_{20}}{N_{10}+N_{20}}\frac{\sqrt{-D}}{\sqrt{8\beta\gamma}}e^{\frac{\alpha}{2}t_2^*}\right)-\varphi\right) \tag{4.38}$$

Thus, the international organizations stop the peace-making efforts after end of information warfare by the antagonistic sides.

$$N_3(t) = \frac{2\gamma(N_{10}+N_{20})}{\sqrt{-D}}\ e^{\frac{\alpha}{2}t}\ \sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t\right) = 0 \tag{4.39}$$

The solution of this equation will be

$$t^* = \frac{2\pi}{\sqrt{8\beta\gamma-\alpha^2}} \tag{4.40}$$

I.e., the third side leaves on zero in a point t^*, and is had a bilateral inequality.

4.**III.3.** ($N_{10} < N_{20}$). If at the antagonistic sides different starting conditions, and starting position of the second side more the first, the first and second sides change roles and is had symmetric results for $N_1(t)$ and $N_2(t)$.

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თემურ ჩილაჩავა, ნუგზარ კერესელიძე

საინფორმაციო ომის არაპრევენციული უწყვეტი წრფივი მათემატიკური მოდელი

ნაშრომში შემოთავაზებულია საინფორმაციო ომის თეორიის ახალი მიმართულება (საინფორმაციო ომის მათემატიკური მოდელირება). „საინფორმაციო ომის“ ქვეშ იგულისხმება ორი სახელმწიფოს ან სახელმწიფოთა ორი გაერთიანების, ანაც ორი მძლავრი ეკონომიკური სტრუქტურის (კონსორციუმების) მიერ მასობრივი საინფორმაციო საშუალებების (ბეჭვდითი და ელექტრონული პრესის, ინტერნეტის) გამოყენებით ერთმანეთის წინააღმდეგ მიზანმიმართული დეზინფორმაციისა თუ პროპაგანდის წარმოება. პროცესში მესამე მხარედ გვევლინება საერთაშორისო ორგანიზაციათა გაერთიანება (გაერო, ეუთო, ევროკავშირი, მსო და სხვა), რომლის ძალისხმევა მიმართულია ანტაგონისტურ სახელმწიფოთა, მხარეთა შორის დაძაბულობის ნეიტრალიზაციისა და საინფორმაციო ომის შეწყვეტისაკენ. აგებულია საინფორმაციო ომის ზოგადი, წრფივი უწყვეტი მათემატიკური მოდელი, რომელიც ითვალისწინებს როგორც თანაბარი („იაკი – დათვი“), ასევე მკვეთრად განს-

ხვავებული („კრავი – მგელი“) სიძლიერის მქონე გაერთიანებების დაპირისპირებას.

საძებნ ფუნქციებად აღებულია დროის მოცემულ მომენტში ინფორმაციის ის რაოდენობა, რომელსაც ავრცელებს თვითოეული მხარე, დასახული მიზნის, არჩეული სტრატეგიის მისაღწევად. მოწინააღმდეგე მხარის იგნორირების მოდელში, ორივე მხარე ერთი და იგივე ტემპით აწარმოებს საინფორმაციო ომს და რეაგირებს საერთაშორისო ორგანიზაციების მოწოდებებზე. თავის მხრივ, მესამე მხარე, რომელიც მოქმედებს არაპრევენციულად, თანაბრად რეაგირებს ანტაგონისტურ მხარეთა საინფორმაციო შეტევების ინტენსივობაზე.

პირველი რიგის წრფივი მუდმივკოეფიციენტიან დიფერენციალურ განტოლებათა სისტემისათვის კოშის ამოცანის ზუსტი ამონახსნის შესწავლამ, გამოავლინა მოდელის კონსტანტებსა და საწყის პირობებს შორის ის თანაფარდობები, რომლის დროსაც:

- ანტაგონისტური მხარეები, მიუხედავად მესამე მხარის მოწოდებისა, აძლიერებენ საინფორმაციო შეტევებს;
- ერთ-ერთი ანტაგონისტური მხარე, მესამე მხარის ზემოქმედებით, გარკვეული დროის შემდეგ, ასრულებს საინფორმაციო ომს (შესაბამისი ამონახსნის ნულზე გასვლა), მაშინ როცა მეორე, მას აძლიერებს;
- ორივე ანტაგონისტური მხარე, მას მერე რაც მიაღწევს აქტიურობის მაქსიმუმს, მესამე მხარის ზემოქმედებით ამცირებს საინფორმაციო შეტევებს და სასრულო დროის შემდეგ შეწყვეტენ საინფორმაციო ომს (ამონახსნთა ნულზე გასვლა).

პირველ შემთხვევაში, უნდა ველოდოთ საინფორმაციო ომის ცხელ ფაზაში ტრანსფორმაციას, მეორეში ეს ტრანსფორმაცია ნაკლებად სავარაუდოა, მესამეში კი – გამორიცხული.

შემოთავაზებულ მოდელს თეორიული ინტერესის გარდა გააჩნია პრაქტიკული მნიშვნელობაც. იგი გვაძლევს საშუალებას, საინფორმაციო შეტევების უკვე საწყის ეტაპზე, დაკვირვებებისა და ანალიზის საშუალებით, დავადგინოთ მხარეთა ჭეშმარიტი განზრახვები და საინფორმაციო ომის განვითარების ხასიათი.

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TEMUR CHILACHAVA, NUGZAR KERESELIDZE

CONTINUOUS LINEAR MATHEMATICAL MODEL OF PREVENTIVE INFORMATION WARFARE

Abstract. Recently the working out of mathematical models of information warfare has become under which the struggle between the states by using information arsenal solely, i.e. an information technology which is based on industrial production, propagation and information imposing is mainly meant.

In the given work the new direction in the theory of information warfare (mathematical modeling of information warfare) is offered. In particular, ”information warfare” means the struggle between the two states or two associations of the states, or two powerful economic structures (consortiums) conducting purposeful disinformation, propagation to each other by means of mass media (an electronic and printing press, the Internet). The association of the international organizations (the United Nations, OSCE, EU, the WTO etc.) act as the side the efforts of which are directed towards the removal of tension between the antagonistic states, the sides and the cessation of information warfare acts.

There is constructed the general continuous linear mathematical model of information warfare between two antagonistic sides which considers the case of confrontation as equipotent associations ("yak-bear"), as well as strong differences - ("wolf - lamb").

The number of information at present time, made by each of the sides is taken as an unknown quantity promoting the achievement of purposes, the chosen strategy . In that specific case the model, each side conducts information warfare and reacts to the appeals of international organizations at the same pace. In its turn, the third side equally reacts to the intensity of information attacks of the antagonistic sides.

Exact analytical solutions to a Cauchy's problem for the system of the linear differential equations of the first order with constant factors are received.

Parities between the constants of the model and initial conditions are revealed, at which:

1. The antagonistic sides, despite increasing appeals of the third side, intensify information attacks.
2. One of the antagonistic sides, under the influence of the third side stops information warfare (an exit of the corresponding solution on zero) while another strengthens it.
3. Both antagonistic sides, after achieving maximum activity, reduce it under the influence of the third side, and through finite time, stop information attacks at all (an exit of solutions on zero).

In the first case, the transformation of information warfare into a hot phase is expected, in the second – it is less probable, in the third - it is excluded at all.

The offered model of information warfare, except theoretical interest has as well an important practical meaning. At allows, on the basis of observation and the analysis, at an early stage of information attacks, to establish true intentions of each side and the character of the development of information warfare.

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Key words and phrases: information warfare, antagonistic sides, continuous linear mathematical model, ignoring of an opposite side, preventive model, hot phase.

1. **Introduction**

The theory of the information warfare, which formalization of which began thirty-forty years ago, now has a wide applied meaning. It is actively considered by many countries at elaboration of information safety. The presidential commission for protection of a so-called critical infrastructure was created in the USA at first. Then on the basis of the conclusion of this commission directive № 63 of the president was elaborated which in 1998, became the basis of the governmental policy of maintenance of information safety [1]. The leading countries have already begun purposeful preparation of experts in a narrow field of information warfare. In the USA, at national university of defence operates the school of information warfare and strategy. At Californian marine school courses of lectures on: principles of information operations; psychological operations; information warfare: planning and an estimation; an estimation of information warfare are delivered to the group of information warfare.

Russia, though with delay, operates in this direction to. The Ministry of Defense of Russia has created the information and propaganda centre which along with tasks will prepare hacker attacks on information resources of the opponent. This decision of the Ministry of Defense of Russia was the answer to the task of the president of Russia - to prepare the offers concerning the creation of the centre of preparation of experts which will be able to conduct information warfare using the newest technologies [2, 3].

According to some sources tasks of the information and propaganda centre will be: intimidation of the opponent, destruction of its information communications and preservation of its own, creation of information and disinformation parts and rendering of influence on public opinion before the conflict as well as during the confrontation. For the necessity of creation of information forces the management of armed forces of Russia cited the analysis of war with Georgia in 2008.

Originally the term ”information warfare” was used Thomas Rona in his report in 1976 “Systems of weapons and information warfare” which meant for company **Boeing** [4]. T. Rona noted that by then, the information infrastructure was becoming the central and at same time simple component of the economy of the USA, less protected target both in state and in peaceful time.

The common definition of the term ”information warfare” has not yet been accepted, but intuitively it is considered that information warfare is a purposeful action for creation of information superiority, by means of destruction of information, information systems of an opposite side, while the protection of its own information and information systems is in process.

Information warfare also implies complex of actions for creation of information influence on public consciousness to change behaviour of people, to impose purposes on them which are not their interests. On the other hand, protection against the same influence is necessary.

As the state information resources often become objects of an attack and protection, the state is compelled to pay a great attention to information technologies. Accordingly, in the theory of information warfare the great number of researches is devoted to the safety of information, information systems and processes.

On the other hand, studying of information streams is essential, as the information stream which falls upon mass consciousness, in most cases, allows manipulating people [5].

The description of various components of information warfare and its study by a mathematical apparatus is already included into the sphere of interests of many scientists. In this regard the usage of the mathematical theory of an information transfer by communication channels for the construction of the model of information influence should be noted. Attempts of estimations of efficiency of specific information influences are undertaken [6]. By means of the theory of graphs and games models of information networks and information warfare are drawn up, as for the contradictory sides mixed strategies are found [7, 8]. Strategies, basically, are calculated for extermination of information infrastructures or their protection by means of physical and as well as program (viruses, trojans, cyber attacks) influences.

Our approach

In the present work the studying of quantity of information streams by means of new mathematical models of information warfare was our purpose [9] . By information warfare we mean an antagonism by means of mass media (an electronic and printing press, the Internet) between the two states or the two associations of states, or the economic structures (consortiums) conducting purposeful misinformation, propagation against each other.

In a world information field for ideological, political and economic targets the purposeful information and the misinformation is actively used, from which in most cases the separation of the general background for the unprepared person is very difficult.

The aims of information warfare can be:

– Infliction of losses to the image of the antagonist country – creating the image of the enemy.
– Discredit of the management of the antagonist country.
– Demoralization of the personnel of the armed forces and the civilians of the antagonist country.
– Creation of public opinion, inside and outside of the country, for justification of argumentation of possible military operations.
– Opposition to the geopolitical ambitions of the antagonist country etc.

International organizations react to occurring processes in the modern world in this or that form and activity. Therefore as the third side in the course of information warfare we consider association of international organizations (the United Nations, OSCE, EU, the WTO etc.) efforts of which are directed on remonal of tension between the rival states, the sides and the cessation of information warfare.

In the given work we have constructed the general continuous linear mathematical model of information warfare between two antagonistic sides, in the presence of the third, peace-making side. The model considers the case of confrontation between the unions of equal ("yak-bear") as well as strong different ("wolf-lamb") strength.

We consider that information warfare is conducted against each other by the first and second sides, and by the third side we mean the international organizations. In that specific case the model, each side conducts information warfare and reacts to the appeals of the international organizations equally. The third side, in its turn reacts equally to the intensity of information attacks of the antagonistic sides. Exact analytical solutions are found in the model of ignoring of an opposite side. By means of the analysis of solutions the character of actions of the sides in information warfare is establish, depending on starting conditions of the sides, and parities between indexes of aggression, peace-making readiness and peace-making activity.

2. **The system of the equations and initial conditions**

All the three sides involved in the process of information warfare spread information for the achievement of the set goal. At the moment of time $t \in [0,+\infty)$ quantity of the information spread by each of the sides we will accordingly designate by $N_1(t)$, $N_2(t)$, $N_3(t)$. The quantity of information at the moment of time t, is defined as the sum of all provoking information which is spread by each of the sides through every kind of mass media.

The detailed analysis of mathematical models of dynamics of population as well as Lanchester's models of military operations [10 - 12], has led us to the following general continuous linear mathematical model of information warfare:

$$\begin{cases} \dfrac{dN_1(t)}{dt} = \alpha_1 N_1(t) + \alpha_2 N_2(t) - \alpha_3 N_3(t) \\ \\ \dfrac{dN_2(t)}{dt} = \beta_1 N_1(t) + \beta_2 N_2(t) - \beta_3 N_3(t) \\ \\ \dfrac{dN_3(t)}{dt} = \gamma_1 N_1(t) + \gamma_2 N_2(t) + \gamma_3 N_3(t) \end{cases} \tag{2.1}$$

with initial conditions

$$N_1(0) = N_{10},\ N_2(0) = N_{20},\ N_3(0) = N_{30}, \tag{2.2}$$

where, $\alpha_1, \alpha_3, \beta_2, \beta_3, \gamma_1, \gamma_2 > 0$, $\gamma_3 \geq 0$, α_2, β_1 - constant factors.

These constant factors are constants of model, thus we name α_1, β_2 factors of growth of provoking statements by the first and second sides according in the absence of the third side (relative growth rates of quantity of statements).

In the general linear model (2.1) speed of quantity change of the information spread by the first and second sides linearly depends on quantity of information spread by the sides and the international peace-making organizations.

The speed of quantity change of pacifying information spread by the third side linearly grows or is directly proportional to the quantity of information spread by all three sides.

In initial conditions (2.2), N_{10}, N_{20}, N_{30} non-negative constants, thus:

If $N_{10} > 0, N_{20} > 0$, then both sides are initiators of information warfare.

If $N_{10} > 0$, $N_{20} = 0$, then the first side is the initiator of information warfare.

If $N_{10} = 0$, $N_{20} > 0$, then the second side is the initiator of information warfare.

The third side initially does not spread any information ($N_{30} = 0$) or does preventive peace-making statements ($N_{30} > 0$) and then starts to reacting to the provocative information spread by the antagonistic sides.

3. **The model of ignoring of an opposite side**

The antagonistic sides which wage information warfare with the same intensity may ignore information spread by the opposite side, but meanwhile both sides should equally listen to the appeals of the third – the peace-making side.

In this case, in the general linear model (2.1) some factors can be considered as equal to zero. In particular, α_2 and β_1 are equal to zero. Let's consider that $\gamma_3 = 0$ or the third side equally reacts only to provoking information spread by antagonistic sides.

Thus, let's consider $\alpha_1 = \beta_2 = \alpha,\ \alpha_3 = \beta_3 = \beta,\ \gamma_1 = \gamma_2 = \gamma$.

Then the system (2.1) will be the following :

$$\begin{cases} \frac{d}{dt} N_1(t) = \alpha N_1(t) - \beta N_3(t) \\ \frac{d}{dt} N_2(t) = \alpha N_2(t) - \beta N_3(t) \\ \frac{d}{dt} N_3(t) = \gamma N_1(t) + \gamma N_2(t) \end{cases} \tag{3.1}$$

The solution of the system (3.1) with initial conditions (2.2) in $[0,\infty)$ area will be written down as follows:

1. $D = \alpha^2 - 8\beta\gamma > 0$

$$N_3(t)=\frac{\gamma\left(N_{10}+N_{20}\right)-\lambda_2 N_{30}}{\sqrt{D}}\ e^{\lambda_1 t}-\frac{\gamma\left(N_{10}+N_{20}\right)-\lambda_1 N_{30}}{\sqrt{D}}\ e^{\lambda_2 t} \qquad (3.2)$$

$$N_1(t)=\frac{N_{10}-N_{20}}{2}\ e^{\alpha t}+\beta\ \frac{\gamma\left(N_{10}+N_{20}\right)-\lambda_2 N_{30}}{\lambda_2\sqrt{D}}\ e^{\lambda_1 t}-\ \beta\ \frac{\gamma\left(N_{10}+N_{20}\right)-\lambda_1 N_{30}}{\lambda_1\sqrt{D}}\ e^{\lambda_2 t} \qquad (3.3)$$

$$N_2(t)=\frac{N_{20}-N_{10}}{2}\ e^{\alpha t}+\beta\ \frac{\gamma\left(N_{10}+N_{20}\right)-\lambda_2 N_{30}}{\lambda_2\sqrt{D}}\ e^{\lambda_1 t}-\ \beta\ \frac{\gamma\left(N_{10}+N_{20}\right)-\lambda_1 N_{30}}{\lambda_1\sqrt{D}}\ e^{\lambda_2 t} \qquad (3.4)$$

$$\lambda_1=\frac{\alpha+\sqrt{\alpha^2-8\beta\gamma}}{2}>0\ ,\ \lambda_2=\frac{\alpha-\sqrt{\alpha^2-8\beta\gamma}}{2}>0.$$

2. $D=\alpha^2-8\beta\gamma=0$

$$N_3(t)=\left[N_{30}+(\gamma N_{10}+\gamma N_{20}-\frac{\alpha}{2}N_{30})t\right]e^{\frac{\alpha}{2}t} \qquad (3.5)$$

$$N_1(t)=\frac{N_{10}-N_{20}}{2}\ e^{\alpha t}+\left[\frac{N_{10}+N_{20}}{2}+\frac{2\beta}{\alpha}(\gamma N_{10}+\gamma N_{20}-\frac{\alpha}{2}N_{30})t\right]\ e^{\frac{\alpha}{2}t} \qquad (3.6)$$

$$N_2(t)=\frac{N_{20}-N_{10}}{2}\ e^{\alpha t}+$$
$$+\left[\frac{N_{10}+N_{20}}{2}+\frac{2\beta}{\alpha}(\gamma N_{10}+\gamma N_{20}-\frac{\alpha}{2}N_{30})t\right]\ e^{\frac{\alpha}{2}t} \qquad (3.7)$$

3. $D=\alpha^2-8\beta\gamma<0$

$$N_3(t)=$$
$$\sqrt{N_{30}{}^2+\frac{\left(2\gamma\left(N_{10}+N_{20}\right)-\alpha N_{30}\right)^2}{8\beta\gamma-\alpha^2}}\; e^{\frac{\alpha}{2}t}\sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\theta\right), \qquad (3.8)$$

$$\theta=arctg\,\frac{N_{30}\sqrt{8\beta\gamma-\alpha^2}}{2\gamma\left(N_{10}+N_{20}\right)-\alpha N_{30}}.$$

$$N_1(t)=\frac{N_{10}-N_{20}}{2}\,e^{\alpha t}+\sqrt{2\beta\gamma}\,\frac{(N_{10}+N_{20})}{\sqrt{-D}}\,e^{\frac{\alpha}{2}t}\sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\varphi\right) \qquad (3.9)$$

$$N_2(t)=\frac{N_{20}-N_{10}}{2}\,e^{\alpha t}+\sqrt{2\beta\gamma}\,\frac{(N_{10}+N_{20})}{\sqrt{-D}}\,e^{\frac{\alpha}{2}t}\sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\varphi\right) \qquad (3.10)$$

4. **The analysis of the obtaining results.**

In model of ignoring of an opposite side (3.1) α can be considered as an indicator of aggression of the antagonistic sides, β - an indicator of their readiness for peace, listening to peace-making appeals of international organizations, γ - an indicator of peace-making activity of the international organizations. As it will further be shown, depending on which is more – an aggression index, or indexes of readiness for peace and peace-making activity, character and development of information warfare essentially varies.

Influence on a course of information warfare from international organizations is more effective, even at a big index of aggression from the antagonistic sides if their action has a preventive character ($N_{30}>0$**)**. Let's study a course of information warfare for various D.

IV.I. $D=\alpha^2-8\beta\gamma>0$**.**

Let's consider different cases of starting position of the antagonistic sides.

IV.I.1. $(N_{10}=N_{20})$. In case when international organizations have taken preventive measures, and the antagonistic sides have begun information warfare under equal starting conditions, the influence of international organizations on the first and second side is expressed equally as - $N_1(t)=N_2(t)$.

From (3.2) - (3.4) we will obtain:

$$N_3(t)=\frac{2\gamma N_{10}-\lambda_2 N_{30}}{\sqrt{D}}\; e^{\lambda_1 t}-\frac{2\gamma N_{10}-\lambda_1 N_{30}}{\sqrt{D}}\; e^{\lambda_2 t} \tag{4.1.1}$$

$$N_1(t)=\beta\,\frac{2\gamma N_{10}-\lambda_2 N_{30}}{\lambda_2\sqrt{D}}\; e^{\lambda_1 t}-\beta\,\frac{2\gamma N_{10}-\lambda_1 N_{30}}{\lambda_1\sqrt{D}}\; e^{\lambda_2 t} \tag{4.1.2}$$

$$N_2(t)=\beta\,\frac{2\gamma N_{10}-\lambda_2 N_{30}}{\lambda_2\sqrt{D}}\; e^{\lambda_1 t}-\beta\,\frac{2\gamma N_{10}-\lambda_1 N_{30}}{\lambda_1\sqrt{D}}\; e^{\lambda_2 t} \tag{4.1.3}$$

Functions $N_1(t)$, $N_2(t)$ in a point $t=0$ are equal and positive $N_{10}>0$, and at big t become negative, if

$$N_{30}>\frac{2\gamma N_{10}}{\lambda_2} \tag{4.1.4}$$

Indeed, according to (4.1.2),

$$N_1(t)=\frac{\beta}{\sqrt{D}}\,e^{\lambda_2 t}\left(\frac{2\gamma N_{10}-\lambda_2 N_{30}}{\lambda_2}e^{(\lambda_1-\lambda_2)t}-\frac{2\gamma N_{10}-\lambda_1 N_{30}}{\lambda_1}\right) \tag{4.1.5}$$

The sign $N_1(t)$ at big t defines the factor $\frac{2\gamma N_{10} - \lambda_2 N_{30}}{\lambda_2}$ before $e^{(\lambda_1 - \lambda_2)t}$, it is negative when $2\gamma N_{10} - \lambda_2 N_{30} < 0$ i.e. it is fair (4.1.4). In this case, the continuous function of $N_1(t)$ changes the sign on a semi-interval $[0, +\infty)$, i.e. in some point t^* of this semi-interval is has zero. t^* is from the equation which it is obtained, having equated (4.1.5) to zero and has the following appearance:

$$t^* = \frac{1}{\sqrt{D}} \ln\left(\frac{-2\gamma N_{10} + \lambda_1 N_{30}}{-2\gamma N_{10} + \lambda_2 N_{30}} \cdot \frac{\lambda_2}{\lambda_1} \right) \tag{4.1.6}$$

(4.1.6) it makes sense, since when it is fair (4.1.4), thus

$$\frac{-2\gamma N_{10} + \lambda_1 N_{30}}{-2\gamma N_{10} + \lambda_2 N_{30}} \frac{\lambda_2}{\lambda_1} > 1 \tag{4.1.7}$$

i.e.

$$t^* = \frac{1}{\sqrt{D}} \left[\ln \frac{-2\gamma N_{10} + \lambda_1 N_{30}}{-2\gamma N_{10} + \lambda_2 N_{30}} - \ln\left(\frac{\lambda_1}{\lambda_2} \right) \right] \tag{4.1.8}$$

Analogical research we will establish that under condition of (4.1.4), the third side finishes actions ($N_3(t)$ comes to zero), however a bit later

$$t^{**} = \frac{1}{\sqrt{D}} \ln \frac{-2\gamma N_{10} + \lambda_1 N_{30}}{-2\gamma N_{10} + \lambda_2 N_{30}} \tag{4.1.9}$$

Thus, if international organizations match a measure preventive N_{30}, for the accomplishment of the condition (4.1.4) all the three functions $N_1(t)$, $N_2(t)$, $N_3(t)$ come to zero – i.e. information warfare comes to an end. If the condition (4.1.4) is not carried out, information warfare proceeds, and moreover, amplifies, as from (4.1.1) - (4.1.3) follows:

$$N_1(t) \to +\infty,\; N_2(t) \to +\infty,\; N_3(t) \to +\infty, \text{ when } t \to +\infty .$$

IV.I.2. ($N_{10} > N_{20}$). If the antagonistic sides have different starting position and at the same time starting conditions of the first side are more than of the second one, the function $N_3(t)$ comes to zero. Really, let's rewrite (3.2) as follows

$$N_3(t) = \frac{e^{\lambda_2 t}}{\sqrt{D}} \left[\left(\gamma\left(N_{10} + N_{20}\right) - \lambda_2 N_{30} \right) e^{(\lambda_1 - \lambda_2)t} - \left(\gamma\left(N_{10} + N_{20}\right) - \lambda_1 N_{30} \right) \right] \quad (4.1.10)$$

It is clear that at big t, the sign $N_3(t)$ defines factor before $e^{(\lambda_1 - \lambda_2)t}$, in particular $N_3(t)$ is negative, if $\gamma\left(N_{10} + N_{20}\right) - \lambda_2 N_{30} < 0$ i.e.

$$N_{30} > \frac{\gamma\left(N_{10} + N_{20}\right)}{\lambda_2} \quad (4.1.11)$$

For N_{30}, obeying (4.1.11) $N_3(t)$ changes the sign at a semi-interval $[0, +\infty)$ from positive $N_3(0) = N_{30} > 0$ to negative. Thus, continuous function of $N_3(t)$ has zero in some point t^{**} of this semi-interval which is the solution of the equation $N_3(t) = 0$

$$\left(\gamma\left(N_{10} + N_{20}\right) - \lambda_2 N_{30} \right) e^{(\lambda_1 - \lambda_2)t} = \gamma\left(N_{10} + N_{20}\right) - \lambda_1 N_{30} \quad (4.1.12)$$

From (4.1.12) we have

$$t^{**} = \frac{1}{\sqrt{D}} \ln\left(\frac{-\gamma(N_{10}+N_{20})+\lambda_1 N_{30}}{-\gamma(N_{10}+N_{20})+\lambda_2 N_{30}} \right) \qquad (4.1.13)$$

Let's notice that the expression under the logarithm is more than unit when, N_{30} obeys the condition (4.2.1.11) since $\lambda_1 > \lambda_2$

$$\frac{-\gamma(N_{10}+N_{20})+\lambda_1 N_{30}}{-\gamma(N_{10}+N_{20})+\lambda_2 N_{30}} > 1$$

Thus, $N_2(t)$ equals zero at point t_1^* of the semi-interval $[0, +\infty)$.

Really, continuous function $N_2(t)$ (3.4) changes the sign at semi-interval $[0, +\infty)$: $N_2(0) = N_{20} > 0$, and for accordingly big t it is negative

$$N_2(t) = e^{\alpha t}\left(\frac{N_{20}-N_{10}}{2} + \beta\frac{\gamma(N_{10}+N_{20})-\lambda_2 N_{30}}{\lambda_2\sqrt{D}}\, e^{-\lambda_2 t} - \beta\frac{\gamma(N_{10}+N_{20})-\lambda_1 N_{30}}{\lambda_1\sqrt{D}}\, e^{-\lambda_1 t}\right), \qquad (4.1.14)$$

as, in this case, the sign $N_2(t)$ defines the first composed $\frac{N_{20}-N_{10}}{2}$ of the second factor of expression (4.1.14), which is negative, and the other members of this factor become any smaller in absolute value, for accordingly big t. Thus, t_1^* is the solution of the transcendental equation

$$\frac{N_{20}-N_{10}}{2} + \beta\frac{\gamma(N_{10}+N_{20})-\lambda_2 N_{30}}{\lambda_2\sqrt{D}}\, e^{-\lambda_2 t} -$$
$$\beta\frac{\gamma(N_{10}+N_{20})-\lambda_1 N_{30}}{\lambda_1\sqrt{D}}\, e^{-\lambda_1 t} = 0 \qquad (4.1.15)$$

If $N_{30} > \frac{\alpha N_{20}}{\beta}$, i.e. the derivative $N_2(t)$ is negative at the initial moment, then owing to (3.1), the second side at first reduces information attacks and then function $N_2(t)$ comes to zero.

If $N_{30} < \frac{\alpha N_{20}}{\beta}$ the second side at first increases information attacks, reaches its maximum, and then starts reducing information attacks and laer stops information warfare at all (function $N_2(t)$ to go out on zero).

In case of $N_{30} = \frac{\alpha N_{20}}{\beta}$, the second side with conducts information attacks with constant intensity and then stops them.

As for the first side $N_1(t)$, for enough big t, it is positive and aspires to $+\infty$, when $t \to +\infty$. Really, behavior of $N_1(t)$, when $t \to +\infty$ is defined by function $e^{\alpha t}$ and the sign of quantity $\frac{N_{10} - N_{20}}{2}$, which in this case is positive. It becomes obvious at the following record of $N_1(t)$

$$N_1(t) = e^{\alpha t}\,\Big(\frac{N_{10} - N_{20}}{2}$$
$$+\beta\frac{\gamma(N_{10}+N_{20}) - \lambda_2 N_{30}}{\lambda_2\sqrt{D}}\,e^{-\lambda_2 t} - \beta\frac{\gamma(N_{10}+N_{20}) - \lambda_1 N_{30}}{\lambda_1\sqrt{D}}\,e^{-\lambda_1 t}\Big) \qquad (4.1.16)$$

The first composed $\frac{N_{10} - N_{20}}{2}$ of the second factor (4.1.16) –is positive, and the other members of this factor become any smaller in absolute value for enough big t. But if we match N_{30}, $N_1(t)$ will accordingly cross an abscissa. Really, we will present $N_1(t)$ in the form of product of two functions

$$N_1(t) = e^{\lambda_2 t}\,F(t), \qquad (4.1.17)$$

where $F(t)$ is set by the formula

$$F(t)=\frac{N_{10}-N_{20}}{2}\,e^{\lambda_1 t}-\frac{\beta}{\sqrt{D}}[(N_{30}-\frac{\gamma}{\lambda_2}(N_{10}+N_{20}))e^{\sqrt{D}t}-(N_{30}-\frac{\gamma}{\lambda_1}(N_{10}+N_{20}))] \quad (4.1.18)$$

If the function $F(t)$ crosses an abscissa and $N_1(t)$ will also cross it at the same point,

i.e. function $F(t)$ has zero, and owing to (4.1.17) there will be zeros of the function $N_1(t)$ at the same points. So, we can investigate $F(t)$ and then use these results for $N_1(t)$.

$N_1(0)= F(0)= N_{10}>0$.

Let's enter designations:

$$\text{A}\equiv N_{30}-\frac{\gamma}{\lambda_2}(N_{10}\ \ N_{20})>0 \qquad (4.1.19)$$

$$\text{B}\equiv N_{30}-\frac{\gamma}{\lambda_1}(N_{10}+\ N_{20})>0 \qquad (4.1.20)$$

Let's notice that B > A, then with the account of (4.1.18), (4.1.19), (4.1.20), $F(t)$ will be copied in the following way:

$$F(t)=\frac{N_{10}-N_{20}}{2}\,e^{\lambda_1 t}-\frac{\beta}{\sqrt{D}}[\,\text{A}\,e^{\sqrt{D}t}-\text{B}] \qquad (4.1.21)$$

Let's find stationary points $F(t)$ from the equation $F^{/}(t)=0$

$$F^{/}(t)=\frac{N_{10}-N_{20}}{2}\,\lambda_1\,e^{\lambda_1 t}-\beta\,\text{A}\,e^{\sqrt{D}t}=0 \qquad (4.1.22)$$

Let's divide (4.1.22) into $e^{\sqrt{D}t}$, we will obtain

$$\frac{N_{10}-N_{20}}{2}\ \lambda_1\, e^{\lambda_2 t} - \beta\, \mathrm{A} = 0,$$

whence

$$e^{\lambda_2 t} = \frac{2\beta A}{(N_{10}-N_{20})\lambda_1} \tag{4.1.23}$$

The equation (4.1.23) at $t > 0$ has the solution when the right side is more than unit, and it will occur when

$$2\beta\,(N_{30} - \frac{\gamma}{\lambda_2}(N_{10}+N_{20})) > (N_{10}-N_{20})\,\lambda_1,$$

i.e.

$$N_{30} > \frac{\gamma(N_{10}+N_{20})}{\lambda_2} + \frac{N_{10}-N_{20}}{2\beta}\ \lambda_1 = \bar{N}_{30} \tag{4.1.24}$$

At (4.1.24), the solution (4.1.23) will be written in the following way

$$t_0 = \frac{1}{\lambda_2}\ln\ \frac{2\beta A}{(N_{10}-N_{20})\lambda_1} \tag{4.1.25}$$

Now let's investigate $F^{/}(t)$ in the neighborhood of the point t_0. We will present $F^{/}(t)$ in the following way

$$F^{/}(t) = e^{\sqrt{D}t}\ \frac{N_{10}-N_{20}}{2}\ \lambda_1 \left(e^{\lambda_2 t} - \frac{2\beta A}{\lambda_1 (N_{10}-N_{20})} \right) \tag{4.1.26}$$

All factors (4.1.26), are positive, except $e^{\lambda_2 t} - \dfrac{2\beta A}{\lambda_1 (N_{10}-N_{20})}$. And this last expression sign-variable in the neighborhood of t_0: to the left of t_0 it is negative, and to the right of t_0 - positive, in t_0 - it is equal to zero, i.e. t_0 is a point of local minimum.

Lemma 1. There are such values of N_{30}, for which function $F(t)$ in a point t_0 of minimum is not positive: $F(t_0) \le 0$.

The proof. Let's introduce the designation

$$\mathrm{K} \equiv \frac{2\beta}{(N_{10} - N_{20})\lambda_1}.$$

Then at big N_{30} - from (4.1.21) we can obtain

$$F(t_0) = \frac{N_{10} - N_{20}}{2} (KA)^{\lambda_1/\lambda_2} - \frac{\beta}{\sqrt{D}} [A(KA)^{\sqrt{D}/\lambda_2} - B] \le 0 \qquad (4.1.27)$$

Really, when $N_{30} >> \frac{\gamma(N_{10} + N_{20})}{\lambda_2}$, from (4.1.19), (4.1.20) we will obtain

$$\mathrm{A} \approx N_{30}, \ \mathrm{B} \approx N_{30}.$$

Owing to (4.1.21) $F(t_0)$ looks like:

$$F(t_0) = \frac{N_{10} - N_{20}}{2} (K)^{\lambda_1/\lambda_2} (N_{30})^{\lambda_1/\lambda_2} - \frac{\beta}{\sqrt{D}} N_{30} [(K)^{\sqrt{D}/\lambda_2} (N_{30})^{\sqrt{D}/\lambda_2} - 1],$$

$$F(t_0) = (N_{30})^{\lambda_1/\lambda_2} (K)^{\sqrt{D}/\lambda_2} [\frac{\beta}{\lambda_1} - \frac{\beta}{\sqrt{D}} + \frac{\beta}{\sqrt{D}} K^{-\frac{\sqrt{D}}{\lambda_2}} (N_{30})^{-\frac{\sqrt{D}}{\lambda_2}}],$$

$$F(t_0) = (N_{30})^{\lambda_1/\lambda_2} (K)^{\sqrt{D}/\lambda_2} \beta [- \frac{\lambda_2}{\lambda_1 \sqrt{D}} + \frac{1}{\sqrt{D}} (KN_{30})^{-\frac{\sqrt{D}}{\lambda_2}}] \le 0,$$

As in the right part of the last parity all factors, except the last are positive, and last – negative for big N_{30}. W.D.P.

Thus (4.1.27) it is fair, when $N_{30} \geq N_{30}^{**}$, thus there is equality $F(t_0, N_{30} = N_{30}^{**}) = 0$. We will also notice that $N_{30}^{**} > \bar{N}_{30}$.

As $F(t)$ in a point t_0 is not positive owing to **lemma 1** it means that it is either equal to zero in this point , or is negative. On the other hand it means that $N_1(t)$ equals to zero either at a point t_0, or at some point t_1 ($t_1 < t_0$).

Thus, the first side, as well as the second and the third ones, finish information warfare.

IV.I.3. ($N_{10} < N_{20}$) .If at the antagonistic sides have different starting conditions, and the starting position of the second one are more, the first and the second sides change roles and we get symmetric results for $N_1(t)$ and $N_2(t)$. In this case the analogue N_{30}^{**} will be designated through N_{30}^{***}.

IV.II. $D = \alpha^2 - 8\beta\gamma = 0$.

In this case the aggression index is still high.

IV.II.1. ($N_{10} = N_{20}$). In case when international organizations have taken preventive measures, and the antagonistic sides have begun information warfare under equal starting conditions influence of international organizations on the first and the second sides is the same and taking into account (3.5) - (3.7) involvement of each side into information warfare will be described as following:

$$N_1(t) = N_2(t) = [N_{10} + (\frac{\alpha}{2} N_{10} - \beta N_{30})t]e^{\frac{\alpha}{2}t} \qquad (4.2.1)$$

$$N_3(t) = [N_{30} + (2\gamma N_{10} - \frac{\alpha}{2} N_{30})t]e^{\frac{\alpha}{2}t} \qquad (4.2.2)$$

All the three functions (4.2.1), (4.2.2) come to zero if the factor before t is negative, which is reached at (4.2.1), when

$$N_{30} > \frac{\alpha}{2\beta} N_{10} \tag{4.2.3}$$

and at (4.2.2), when

$$N_{30} > \frac{4\gamma}{\alpha} N_{10} \tag{4.2.4}$$

Let's notice that the right sides (4.2.3) and (4.2.4) - are equal when $D = 0$. Functions $N_1(t)$ and $N_2(t)$ come to zero at some point t^*, where

$$t^* = \frac{N_{10}}{\beta N_{30} - \frac{\alpha}{2} N_{10}} = \frac{N_{10}}{\beta (N_{30} - \frac{\alpha}{2\beta} N_{10})}, \tag{4.2.5}$$

and $N_3(t)$ comes to zero in t^{**} where

$$t^{**} = \frac{N_{10}}{\frac{\alpha}{2} N_{30} - 2\gamma N_{10}} = \frac{N_{30}}{\frac{\alpha}{2} (N_{30} - \frac{\alpha}{2\beta} N_{10})} \tag{4.2.6}$$

It is obvious that in case of (4.2.3) or (4.2.4) $t^{**} > t^*$. Thus, when $D = 0$, $N_{10} = N_{20}$ и $N_{30} > \frac{4\gamma}{\alpha} N_{10}$, all the three sides finish information warfare.

Thus, if the bilateral inequality is carried out

$$\frac{\alpha}{\beta} N_{10} > N_{30} > \frac{\alpha}{2\beta} N_{10}, \tag{4.2.7}$$

then the antagonistic sides make their attacks active at first, but then, after definite time, under the pressure of international organizations, reduce, and then and stop information warfare at all.

If the inequality (4.2.7) is not carried out in the left part, i.e. there is

$$N_{30} > \frac{\alpha}{\beta} N_{10},$$

then, the antagonistic sides under the pressure of international organizations reduce information influence from the very beginning and cease information warfare once and for all. If the inequality (4.2.7) is not carried out in the right part, i.e.

$$0 < N_{30} < \frac{\alpha}{2\beta} N_{10},$$

then according to (4.2.1), (4.2.2), information warfare develops

$$N_1(t) \to +\infty,\ N_2(t) \to +\infty,\ N_3(t) \to +\infty,\ \text{when } t \to +\infty.$$

IV.II.2. ($N_{10} > N_{20}$).If at the antagonistic sides have different starting position and at the same time the starting conditions of the first side are more than of the second one, under certain conditions all the three required functions come to zero.

For $N_3(t)$ coming to zero is reached proceeding from (3.5) when the factor before t will be negative, i.e. in case of

$$N_{30} > \frac{\alpha(N_{10} + N_{20})}{4\beta}, \qquad (4.2.8)$$

thus $N_3(t)$ comes to zero at a point t^{**}

$$t^{**} = \frac{N_{30}}{\frac{\alpha}{2} N_{30} - \gamma(N_{10} + N_{20})} \qquad (4.2.9)$$

As for the function $N_1(t)$ it becomes any bigger for big t . With the account of $N_{10} > N_{20}$, it is well visible from the following record of $N_1(t)$

$$N_1(t) = \{\frac{N_{10} - N_{20}}{2} + \left[\frac{N_{10} + N_{20}}{2} + \frac{2\beta}{\alpha}(\gamma N_{10} + \gamma N_{20} - \frac{\alpha}{2} N_{30})t\right] e^{-\frac{\alpha}{2}t} \} e^{\alpha t} \quad (4.2.10)$$

Let's write down the equation $N_1(t) = 0$, taking into account (4.2.10) and division into $e^{\frac{\alpha}{2}t}$, then

$$(N_{10} - N_{20})\, e^{\frac{\alpha}{2}t} = -\left(N_{10} + N_{20}\right) + \left(2\beta N_{30} - \frac{\alpha}{2}\left(N_{10} + N_{20}\right)\right)t \qquad (4.2.11)$$

Let's enter a designation

$$F(t) \equiv (N_{10} - N_{20})\, e^{\frac{\alpha}{2}t} + (N_{10} + N_{20}) - \left(2\beta N_{30} - \frac{\alpha}{2}\left(N_{10} + N_{20}\right)\right)t \qquad (4.2.12)$$

as

$$N_1(t) = \frac{1}{2}\, e^{\frac{\alpha}{2}t}\, F(t)\ ,$$

therefore zero $F(t)$ will be the zero of $N_1(t)$, and there is also equality

$$F(0) = 2\, N_{10} > 0\,.$$

Let’s find stationary points $F(t)$:

$$F^{/}(t) = \frac{\alpha}{2}\,(N_{10} - N_{20})\, e^{\frac{\alpha}{2}t} - \left(2\beta N_{30} - \frac{\alpha}{2}\left(N_{10} + N_{20}\right)\right), \qquad (4.2.13\ )$$

whence we will obtain that

$$F^{/}(0) = -\,2\beta\left(N_{30} - \frac{\alpha}{2\beta}N_{10}\right) < 0,$$

when

$$N_{30} > \frac{\alpha}{2\beta}\, N_{10} \qquad (4.2.14)$$

At performance (4.2.14), the decrease $F(t)$ begins from zero. The decrease interval $F(t)$ is calculated from the following inequality

$$F^{/}(t) < 0$$

$$1 < e^{\frac{\alpha}{2}t} < \frac{2\left[2\beta N_{30} - \frac{\alpha}{2}\left(N_{10} + N_{20}\right)\right]}{\alpha\left(N_{10} - N_{20}\right)} \qquad (4.2.15)$$

Thus, when it is fairly (4.2.14) has the following inequality

$$\frac{2\left[2\beta N_{30} - \frac{\alpha}{2}\left(N_{10} + N_{20}\right)\right]}{\alpha\left(N_{10} - N_{20}\right)} > 1$$

In case of (4.2.14), $F(t)$ decreases at an interval $\left(0, t^{+}\right)$, thus t^{*} is the solution

the equation $F^{/}(t) = 0$, i.e. t^{*} is a point of a local minimum $F(t)$ and looks like:

$$t^{*} = \frac{2}{\alpha} \ln \frac{2\left(2\beta N_{30} - \frac{\alpha}{2}\left(N_{10} + N_{20}\right)\right)}{\alpha\left(N_{10} - N_{20}\right)} \qquad (4.2.16)$$

Let's define conditions for N_{30}, at which $F(t)$ becomes negative at t^{*} .

$$F\left(t^{*}\right) = \frac{4\beta N_{30}}{\alpha} - \left(\frac{4\beta}{\alpha} N_{30} - \left(N_{10} + N_{20}\right)\right) \ln \frac{2\left(2\beta N_{30} - \frac{\alpha}{2}\left(N_{10} + N_{20}\right)\right)}{\alpha\left(N_{10} - N_{20}\right)} \leq 0 \qquad (4.2.17)$$

Lemma 2. The inequality (4.2.17) is just, when $N_{30} \geq N_{30}^{*}$,

$n^{*} \frac{\alpha}{4\beta}\left(N_{10} + N_{20}\right) = N_{30}^{*}$, $n^{*} \cong 4{,}5911$.

The proof.

The solution to inequality (4.2.17) concerning N_{30}, we will search as follows

$$n\frac{\alpha}{4\beta}\left(N_{10}+N_{20}\right)= N_{30}$$

and match such n that $F\left(t^*\right)\le 0$. Then

$$F\left(t^*\right)= n\,(N_{10}+N_{20})-(N_{10}+N_{20})(n-1)\ln\frac{(n-1)(N_{10}+N_{20})}{N_{10}-N_{20}}= (N_{10}+N_{20})$$

$$[n-(n\text{-}1)\ln\frac{(n-1)(N_{10}+N_{20})}{N_{10}-N_{20}}]\le(N_{10}+N_{20})[n-(n-1)\ln(n-1)]\le 0 \qquad (4.2.18)$$

The inequality (4.2.18) for $n>1$ is just, if

$$n-(n-1)\ln(n-1)\le 0$$

Let's designate

$$G(n)\equiv n-(n-1)\ln(n-1).$$

Then

$G(n)\to 1+$, при $n\to 1+$;

$G^{/}(n)<0$, при $n>2$;

$G^{/}(n)>0$, когда $1<n<2$;

$G^{/}(n)=0$, когда $n=2$,

i.e. $n=2$ is the point of a local maximum for function $G(n)$.

It's clear that there is such point as n^*, for which

$$G(n^*)=0,$$

as $G(4)>0$, $G(5)<0$, that $4<n^*<5$, more precisely $n^*\cong 4{,}5911$.

Thus, when, $N_{30} \geq n^* \frac{\alpha}{4\beta}\left(N_{10}+N_{20}\right)$ i.e. $n \geq n^*$, $G(n) \leq 0$ and $F\left(t^*\right) \leq 0$.

The lemma is proved.

It is obvious that as there is (4.2.17), the function $N_1(t)$ has zero. As for $N_2(t)$, it is for any N_{30} and big t aspires to $-\infty$, and consequently its zero t^{**}, is the solution of the following transcendental equation

$$N_2(t) = \frac{N_{20}-N_{10}}{2} e^{\alpha t} + \left[\frac{N_{10}+N_{20}}{2} + \frac{2\beta}{\alpha}(\gamma N_{10} + \gamma N_{20} - \frac{\alpha}{2} N_{30})t\right] e^{\frac{\alpha}{2}t} = 0 .$$

IV.II.3. ($N_{10} < N_{20}$). If at the antagonistic sides have different starting conditions and the starting position of the second side is more, the first and the second sides change roles and there are symmetric results for $N_1(t)$ and $N_2(t)$. N_{30}^1 will be the analogue for N_{30}^*.

IV.III. $D = \alpha^2 - 8\beta\gamma < 0$.

IV.III.1. ($N_{10} = N_{20}$). In case when the antagonistic sides have begun information warfare under equal starting conditions the influence of international organizations on the first and second sides is productive. In this case, taking into account (3.8) - (3.10), functions $N_3(t)$, $N_1(t)$, $N_2(t)$, will become

$$N_3(t) = \sqrt{N_{30}{}^2 + \frac{\left(4\gamma N_{10} - \alpha N_{30}\right)^2}{8\beta\gamma - \alpha^2}}\; e^{\frac{\alpha}{2}t} \sin\left(\frac{\sqrt{8\beta\gamma - \alpha^2}}{2} t + \theta_1\right), \qquad (4.3.1)$$

$$\theta_1 = arctg\, \frac{N_{30}\sqrt{8\beta\gamma - \alpha^2}}{4\gamma N_{10} - \alpha N_{30}} .$$

$$N_1(t)=N_2(t)=\sqrt{\frac{\beta}{2\gamma}}\sqrt{N_{30}{}^2+\frac{(4\gamma N_{10}-\alpha N_{30})^2}{8\beta\gamma-\alpha^2}}\;e^{\frac{\alpha}{2}t}\sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\theta_1+\varphi\right), \quad (4.3.2)$$

$$\varphi=arctg\,\frac{\sqrt{8\beta\gamma-\alpha^2}}{\alpha}.$$

According to (4.3.2), functions $N_1(t)$ and $N_2(t)$ come to zero at point t^*

$$t^* = \frac{2(\pi-\varphi-\theta_1)}{\sqrt{-D}}, \qquad (4.3.3)$$

and $N_3(t)$ in t^{**} , which is more than t^*

$$t^{**} = \frac{2(\pi-\theta_1)}{\sqrt{-D}} \qquad (4.3.4)$$

and information warfare stops.

IV.III.2. ($N_{10} > N_{20}$). If at the antagonistic sides have different starting positions and thus, the starting conditions of the first side are more than of the second one the function $N_3(t)$ comes to zero at point t_1^{**} according to (3.8)

$$t_1^{**} = \frac{2(\pi-\theta)}{\sqrt{-D}} ,$$

and $N_2(t)\to-\infty$, at $t\to+\infty$, therefore the function $N_2(t)$ has zero at point t_1^* , which is from the equation

$$N_2(t_1^*)=0,$$

where - $N_2(t)$, looks like (3.10).

As for $N_1(t)$, $N_1(t) \to +\infty$, at $t \to +\infty$, but at selection N_{30}, the function $N_1(t)$ can come to zero.

Indeed, the parity

$$N_1(t) = 0$$

owing to (3.9), gives the following equation

$$\sin\left(\frac{\sqrt{8\beta\gamma-\alpha^2}}{2}t+\theta+\varphi\right)=\left(\frac{N_{20}-N_{10}}{2}e^{\frac{\alpha t}{2}}\right)/\sqrt{\frac{\beta}{2\gamma}}\sqrt{N_{30}{}^2+\frac{\left(2\gamma\left(N_{10}+N_{20}\right)-\alpha N_{30}\right)^2}{8\beta\gamma-\alpha^2}} \qquad (4.3.5)$$

According to (4.3.5), the condition of existence of zero for $N_1(t)$, leads to the following inequality

$$0 \le \frac{\left(N_{10}-N_{20}\right)e^{\frac{\alpha t}{2}}}{\sqrt{\frac{\beta}{2\gamma}}\sqrt{N_{30}^2+\frac{\left(2\gamma\left(N_{10}+N_{20}\right)-\alpha N_{30}\right)^2}{8\beta\gamma-\alpha^2}}} \le 2,$$

$$e^{\frac{\alpha}{2}t} \le \frac{2\cdot\sqrt{\frac{\beta}{2\gamma}}\sqrt{N_{30}^2+\frac{(2\gamma(N_{10}+N_{20})-\alpha N_{30})^2}{8\beta\gamma-\alpha^2}}}{N_{10}-N_{20}}$$

$$t \le \frac{2}{\alpha}\ln\frac{2\cdot\sqrt{\frac{\beta}{2\gamma}}\sqrt{N_{30}^2+\frac{(2\gamma(N_{10}+N_{20})-\alpha N_{30})^2}{8\beta\gamma-\alpha^2}}}{N_{10}-N_{20}} \qquad (4.3.6)$$

The solution (4.3.5) concerning t, which satisfies (4.3.6), always exists for big N_{30}.

Thus the function $N_1(t)$ has a zero, hence the first side finishes information warfare too.

IV.III.3.($N_{10} < N_{20}$). If at the antagonistic sides have different starting conditions and the starting position of the second side is more the first and second sides change roles and there are symmetric results for $N_1(t)$ and $N_2(t)$.

Thus, the analysis of the received results shows that functions $N_1(t)$, $N_2(t)$ and $N_3(t)$ come to zero at selection of corresponding N_{30} and γ (with the help of prevention and increase in peace-making activity). If there are no preventive activities from the international organizations, then coming to zero of all the three required functions is possible only in case, when $D < 0$ (aggression of the antagonistic sides is weaker than peace-making activity). Thus, in the last case, coming to zero is provided with increase of γ (peace-making activity), even in case when information warfare begins under different starting conditions.

International organizations capable of extinguishing information warfare, i.e. functions $N_1(t)$ and $N_2(t)$, do not come to zero, when $D \geq 0$ (high aggression of the antagonistic sides) and there are no preventive actions from their side.

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თემურ ჩილაჩავა, ნუგზარ კერესელიძე

პრევენციული საინფორმაციო ომის უწყვეტი წრფივი მათემატიკური მოდელი

ნაშრომში აგებულია ორ ანტაგონისტურ სახელმწიფოთა შორის საინფორმაციო ომის წრფივი უწყვეტი მათემატიკური მოდელი, იმის გათვალისწინებით, რომ არსებობს მესამე, მშვიდობისმყოფელი მხარე, რომელიც მოქმედებს პრევენციულად. მოდელი ითვალისწინებს როგორც თანაბარი („იაკი – დათვი“), ასევე მკვეთრად განსხვავებული („კრავი – მგელი“) სიძლიერის მქონე გაერთიანებების დაპირისპირებას.

საძებნ ფუნქციებად აღებულია დროის მოცემულ მომენტში ინფორმაციის ის რაოდენობა, რომელსაც ავრცელებს თვითოეული მხარე, დასახული მიზნის, არჩეული სტრატეგიის მისაღწევად. მოდელში, ორივე მხარე ერთი და იგივე ტემპით აწარმოებს საინფორმაციო ომს და რეაგირებს საერთაშორისო ორგანიზაციების მოწოდებებზე. თავის მხრივ, მესამე მხარე თანაბრად რეაგირებს ანტაგონისტურ მხარეთა საინფორმაციო შეტევების ინტენსივობაზე.

მიღებულია პირველი რიგის წრფივი მუდმივკოეფიციენტიან დიფერენციალურ განტოლებათა სისტემისათვის კოშის ამოცანის

ზუსტი ამონახსნი. ანალიზმა გამოავლინა მოდელის კონსტანტებსა და საწყის პირობებს შორის ის თანაფარდობები, რომლის დროსაც:

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HAMLET MELADZE

ON TWO-LAYER FACTORIZED DIFFERENCE SCHEMES FOR SYSTEM OF DIFFERENTIAL EQUATIONS WITH PARTIAL DERIVATIVES OF PARABOLIC TYPE

Abstract. The mixed problem with first sort boundary conditions for systems of equations of parabolic type is considered

$$\frac{\partial u}{\partial t} = Lu + f,$$

where L - strong elliptic operator with variable coefficients, containing the mixed derivatives, $u = \left(u^{(1)}, u^{(2)}, \dots, u^{(n)}\right)$, $f = \left(f^{(1)}, f^{(2)}, \dots, f^{(n)}\right)$ - n-dimensional vectors. The two-layer factorized scheme is constructed. The received algorithms can be effectively realized for multiprocessing computing systems. For solution of difference scheme the aprioristic estimation on layer in norm of mesh space $\overset{0}{W}{}_2^{(1)}$ is received, on which basis convergence of solution of difference scheme to the solution of an initial problem is proved.

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Key words and phrases: difference scheme, two-layer factorized scheme, parabolis type.

1. **Introduction**

As it is known, at the mathematical formulation of many scientific and technical problems there are arising the differential equations with partial derivatives of parabolic type [1,2]. Construction and investigation of the difference schemes for linear system of equations of parabolic type with the mixed derivatives is a subject of the present article.

The solution of such problems is one of challenges in numerical mathematics and demands, as a rule, a lot of computing resources. One of the ways of reduction the time of the solution of such problems is to use the parallel computations on multiprocessing computing systems.

Usage of parallel computing systems demands construction of algorithms in the form accessible to parallel processing of the data. In computational mathematics enough considerable quantity of such methods is developed. One of such methods is the method of decomposition of challenging tasks on more simple tasks moving implementations on parallel processors.

At construction of the difference schemes with given properties the method of the regularization offered by A.A.Samarskim [3,4], possesses the big efficiency. In the present paper this method is used for construction of the difference schemes, which can be easily implemented on parallel computing systems.

According to the regularization method:

- At first, the initial difference scheme, meeting requirements of approximation of the given order is constructed.
- the regularizator, that is operator, providing absolute stability of the difference scheme, is selected.
- By means of the factorization method of the operator on the upper layer passage to the economic stable difference scheme is made.

Thus, the basic attention is given to following problems:

1. Reception economic difference scheme at minimal requirements on the spatial operator. It is required only that the spatial operator will be strong elliptic.
2. Absolute stability at any $\tau < \infty$ (τ - a step on time) and $h_\alpha < \infty$ (h_α - a step in a direction on x_α , $\alpha = 1,2,\ldots,p$).
3. Application of one-dimensional double-sweep algorithm for solution of received difference equations. These algorithms can be used for parallel computing systems.
4. The convergence proof of difference schemes at smaller smoothness of the solution of initial system of the differential equations that is reached by refusal of an estimation of local approximation.

To the problem of construction of difference splitting schemes the extensive literature is devoted. We will note some monograph [3,5,6]. It is possible to find the extensive list of works in this direction in these monographs. Let's note also some works on parallel algorithms of the solution of parabolic equations [8-12]. It is natural, that this list is incomplete.

2. **The statement of the problem**

1°. Let, $D_p = \{x = (x_1, x_2, \dots, x_p),\quad 0 < x_\alpha < \ell_\alpha,\quad \alpha = 1,2,\dots,p\}$ is open p–dimensional parallelepiped in p–dimensional Euclid space E_p. Γ - surface of parallelepiped D_p and $\overline{D}_p = D_p \cup \Gamma$ -- is the D_p area's closure.

$G_T = D_p \times (0, T]$ -- cylinder in (p +1)-dimensional Euclid space with basis E_p. $\overline{G}_T = \overline{D}_p \times [0, T]$ -- is the closure of the area G_T and (x, t) – any point in $\overline{G}_T$. $C^{\ell,k}(\overline{G}_T)$ is the set of continuous in $\overline{G}_T$ functions, which have in $\overline{G}_T$ continuous derivatives to an order ℓ (inclusive) on x and to an order k (inclusive) on t.

In cylinder $\overline{G}_T$ we considered the following problem for system of equations of parabolic type with mixed derivatives:

Let's find continuous in $\overline{G}_T$ solution of system of the equations

$$\frac{\partial u^{(i)}}{\partial t} = \sum_{j=1}^{n} \sum_{\alpha,\beta=1}^{p} \frac{\partial}{\partial x_\alpha}\left(K_{\alpha,\beta}^{i,j}(x,t) \frac{du^{(j)}}{dx_\beta} \right) + f^i(x,t), \quad i = 1,2,\dots,n, \tag{1.1}$$

satisfying to boundary conditions

$$u^{(i)}(x', t) = g^{(i)}(x', t), \text{ when } x' \in \Gamma \times [0, T],\ i = 1,2,\dots,n, \tag{1.2}$$

and to initial conditions

$$u^{(i)}(x,0) = u_0^{(i)}(x), \text{ when } x \in \overline{D}_p,\ \ i = 1,2,\dots,n. \tag{1.3}$$

Let's assume that following conditions are satisfied:

I. Coefficients $K_{\alpha,\beta}^{i,j}(x,t)$ are Lipschitz-continuous with respect to t. Let's assume also that matrixes from coefficients of the spatial operator $K_{\alpha\beta} = \left(K_{\alpha,\beta}^{i,j}\right)_{i,j=1}^{n}$ $(\alpha, \beta = 1,2,\dots,p)$ are symmetric and that the problem (1.1)-(1.3) has the unique solution $u \equiv u(x,t)$, continuous in $\overline{G}_T$ and differentiated necessary number of times.

II. Spatial operator of system of equations (1.1) is strong-elliptic, so that for any $t \in [0,T]$ the following inequality is true

$$\nu_1 \sum_{\alpha=1}^{p} \sum_{i=1}^{n} \left(\xi_\alpha^i\right)^2 \le \sum_{\alpha,\beta}^{1\div p} \sum_{i,j}^{1\div n} K_{\alpha\beta}^{ij}\, \xi_\beta^j\, \xi_\alpha^i \le \nu_2 \sum_{\alpha=1}^{p} \sum_{i=1}^{n} \left(\xi_\alpha^i\right)^2, \tag{1.4}$$

where $\left\{\xi_\alpha^i\right\}$ -- any real numbers, ν_1 and ν_2 -- positive constants.

3. **The two layer difference scheme for problem (1.1)-(1.3)**

1°. In cylinder $\overline{G}_T$ let's introduce the difference mesh. Construction of spatial-time mesh in $\overline{G}_T$ is carried out by means of one-dimensional meshes on intervals $[0,\ell_\alpha]$, $\alpha = 1,2,\ldots,p$ and $[0,T]$:

$$\overline{\omega}_\alpha = \left\{x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha, \quad i_\alpha = 0,1,\ldots,N_\alpha, \quad N_\alpha h_\alpha = \ell_\alpha\right\}, \alpha = 1,2,\ldots,p.$$

$$\omega_\alpha = \left\{x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha, \quad i_\alpha = 1,\ldots,N_\alpha - 1, \quad N_\alpha h_\alpha = \ell_\alpha\right\}, \alpha = 1,2,\ldots,p.$$

The difference mesh in parallelepiped $\overline{D}_p$ is constructing in following way:

$$\overline{\omega}_h = \prod_{\alpha=1}^{p} \overline{\omega}_\alpha = \left\{x = \left(i_1 h_1,\ldots,i_p h_p\right) \in \overline{D}_p, \quad i_\alpha = 0,1,\ldots,N_\alpha, \quad N_\alpha h_\alpha = \ell_\alpha\right\}.$$

$\omega_h = \prod_{\alpha=1}^{p} \omega_\alpha$ -- set of internal points of difference grid in D_p.

Denote by $\gamma_h = \overline{\omega}_h \setminus \omega_h \equiv \{x \in \Gamma\}$ the set of knots, belonging to the boundary Γ, which are named the boundary knots. On interval $[0,T]$ let's introduce the difference mesh

$$\overline{\omega}_\tau = \left\{t_j = j\tau, \quad j = 0,1,\ldots,k, \quad k\tau = T\right\}.$$

Let $\Omega_{h\tau} = \omega_h \times \omega_\tau$ be the partial-time mesh of internal knots of cylinder G_T and $\left(x, t_j\right)$ is any knot of $\Omega_{h\tau}$. $\Gamma_{h\tau} = \gamma_h \times \overline{\omega}_\tau$ be the set of knots of difference grid on lateral surface of the cylinder $\overline{G}_T$ and $\left(x', t_j\right)$ is knot, belonging to $\Gamma_{h\tau}$. Then

$$\overline{\Omega}_{h\tau} = \Omega_{h\tau} \cup \Gamma_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$$

is the partial-time mesh in $\overline{G}_T$.

2°. We will deal with functions of discrete argument, defined in knots of difference mesh and named mesh functions.

We will consider the set of net vector-functions $y = \left(y^{(1)}, y^{(2)}, \ldots, y^{(n)}\right)$, defined on $\overline{\Omega}_{h\tau}$. For them the symbol $y^{(j)}$ means value of a mesh vector-function in knots $\left(x, t_j\right)$.

We will use designations:

$$x^{(\pm 1_\alpha)} = \left(x_1, \ldots, x_{\alpha-1}, x_\alpha \pm h_\alpha, x_{\alpha+1}, \ldots, x_p\right),$$

$$y^{(\pm 1_\alpha)} = y\left(x^{(\pm 1_\alpha)}\right) = \left(y_1\left(x^{(\pm 1_\alpha)}\right), \ldots, y_n\left(x^{(\pm 1_\alpha)}\right)\right),$$

$$|h|^2 = \sum_{\beta=1}^{p} h_\alpha^2 .$$

Let's enter the difference relations:

$$y_{\bar{x}_\alpha} = \frac{y - y^{(-1_\alpha)}}{h_\alpha}, \quad y_{x_\alpha} = \frac{y^{(+1_\alpha)} - y}{h_\alpha},$$

$$\left(y_{\bar{x}_\alpha}\right)_{x_\beta} = y_{\bar{x}_\alpha x_\beta} = \frac{y^{(-1_\alpha)} - 2y + y^{(+1_\alpha)}}{h_\alpha^2} = \Lambda_\alpha^0 y ,$$

$$-\Lambda_\alpha^0 y = A_\alpha^0 y, \quad A^0 = \sum_{\alpha=1}^{p} A_\alpha^0 .$$

For the mesh vector-functions defined on $\overline{\Omega}_{h\tau}$, besides the specified designations, we will use still the following:

$$\hat{y} = y\left(x, t_{j+1}\right), \quad y = y\left(x, t_j\right), \quad \breve{y} = y\left(x, t_{j-1}\right), ,$$

$$y_{\bar{t}} = \frac{y - \breve{y}}{\tau}, \quad y_t = \frac{\hat{y} - y}{\tau}, \quad y_{\bar{t}t} = \frac{\hat{y} - 2y + \breve{y}}{\tau^2}, \quad y_{\overset{\circ}{t}} = \frac{\hat{y} - \breve{y}}{2\tau}.$$

3°. Set of the mesh functions, defined on the mesh $\overline{\omega}_p$, we will designate by *H*, and its subset consisting of mesh functions, becoming zero on γ_p - through $\overset{\circ}{H}$.

On set $\overset{\circ}{H}$ we will enter the scalar product

$$(y,v)=\sum_{i=1}^{n}\left(y^i,v^i\right),$$

Where

$$\left(y^i,v^i\right)=\sum_{x\in\omega_p} y^i(x)v^i(x)H, \quad H=\prod_{\alpha=1}^{p} h_\alpha .$$

It is obvious, that this scalar product induces the norm $\|y\|_0^2=(y,y)$.

Denote by $\omega_p^{+\alpha}=\omega_p\cup\gamma_\alpha^+$, where γ_α^+ -- set of knots of border γ_p, at which $x_\alpha=l_\alpha$. $\gamma_{\alpha\beta}^+(\alpha\neq\beta)$ -- set of knots of border γ_p, at which $x_\alpha=l_\alpha$, $x_\beta=l_\beta$ and etc. Similarly, we will designate through γ_α^- -- set of knots of border γ_p, at which $x_\alpha=0$ and etc.

Let's enter the designation $\omega_p^+=\omega_p\cup\gamma_{1,2,\dots,p}^+$. Scalar product also is required to us:

$$(y,v]_\alpha=\sum_{i=1}^{n}\left(y^i,v^i\right], \quad \left(y^i,v^i\right]=\sum_{x\in\omega_p^+} y^i(x)v^i(x)H.$$

Let's enter the mesh space of Sobolev $\overset{\circ}{W}{}_2^{(1)}\left(\omega_p\right)$, consisting the functions of set $\overset{\circ}{H}$. The norm in this space is set by means of following equality

$$\|u\|_1^2=\|u\|_{\overset{\circ}{W}{}_2^{(1)}}^2=\sum_{\alpha=1}^{p}\left\|u_{\bar{x}_\alpha}\right\|_0^2, \quad \left\|u_{\bar{x}_\alpha}\right\|_0^2=\left(1,\left(u_{\bar{x}_\alpha}\right)^2\right].$$

Let's notice that $\overset{\circ}{W}{}_2^{(1)}$ -- is full Hilbert space concerning this norm.

4°. Let A -- the linear self-conjugate operator and A>0, operating in Hilbert space $\overset{\circ}{H}$. We will define new scalar product

$$(y,v)_A = (Ay,v).$$

Owing to such definition $\overset{\circ}{H}$ turns in new Hilbert space, which we will denote through H_A. Energetic space H_A consists of the same elements, as the space $\overset{\circ}{H}$. Norm in H_A we will denote by the symbol $\|\cdot\|_A$:

$$\|u\|_A^2 = (Au,u).$$

5°. For problem (1.1)-(1.4) let's consider two-layer difference scheme, which order of approximation in a class $C^{4,2}\left(\overline{G_T}\right)$ of solutions (1.1) is a value of the order $O\left(\tau + |h|^2\right)$:

$$(E+\tau R)y_t + Ay = f(x,t), \quad (x,t) \in \Omega_{h\tau}, \qquad (2.1)$$

where $y = \left(y^{(1)}, y^{(2)}, \ldots, y^{(n)}\right)$, $f = \left(f^{(1)}, f^{(2)}, \ldots, f^{(n)}\right)$ – n-dimensional vectors, and operator A is defined by equality

$$Ay = -\frac{1}{2}\sum_{\alpha,\beta}^{1\div p}\left[\left(K_{\alpha\beta}(x,t)y_{\bar{x}_\beta}\right)_{x_\alpha} + \left(K_{\alpha\beta}(x,t)y_{x_\beta}\right)_{\bar{x}_\alpha}\right], \qquad (2.2)$$

R – operator-regularizator, which choice provides absolute stability of difference scheme (2.1).

The vector function $y(x,t)$ satisfies the following boundary and initial conditions:

$$y(x',t) = g(x',t), \quad if \quad x' \in \Gamma_{h\tau}, \quad y(x,0) = u_0(x). \qquad (2.3)$$

In work [4] is proved the fairness of following relations

$$\nu_1\left(A^0 y, y\right) \le (Ay,y) \le \nu_2\left(A^0 y, y\right), \text{ or } \nu_1 A^0 \le A \le \nu_2 A^0, \qquad (2.4)$$

for any vector $y \in H_0$.

Hence, operators A and A^0 in $\overset{\circ}{H}$ are energetically equivalent with constants ν_1 and ν_2.

On the basis of the results §2 from [7] and inequalities (2.4) it is possible to conclude, that the difference scheme (2.1) with regularizator

$R = \sigma A^0$, at $\sigma \geq 0.5\nu_2$, is absolutely stabile in space $\overset{\circ}{H}$ (stabile under the initial data and the right part).

6°. Let's pass to construction economic factorized two-layer difference scheme, considering the scheme (2.1) as initial. As

$$R = \sigma A^0 = \sum_{\alpha=1}^{p} R_\alpha, \qquad R_\alpha = \sigma A_\alpha^0, \qquad \sigma \geq 0.5\nu_2 , \tag{2.5}$$

then, replacing the operator $E + \tau R = E + \tau \sum_{\alpha=1}^{p} R_\alpha$ with the factorized operator $\prod_{\alpha=1}^{p}(E + \tau R_\alpha)$, we receive the two-layer factorized difference scheme

$$\prod_{\alpha=1}^{p}(E + \tau R_\alpha) y_t + Ay = f(x,t), \tag{2.6}$$

which in a canonical form can be written down as follows

$$(E + \tau\tilde{R}) y_t + Ay = f(x,t), \quad (x,t) \in \Omega_{h\tau} , \tag{2.7}$$

where $\tilde{R} = R + \tau Q_p$, and

$$Q_p = \sum_{\alpha<\beta} R_\alpha R_\beta + \tau \sum_{\alpha<\beta<\gamma} R_\alpha R_\beta R_\gamma + \cdots + \tau^{p-2} \prod_{\alpha=1}^{p} R_\alpha . \tag{2.8}$$

It's evident, that in $\overset{\circ}{H}$ the inequality $\tilde{R} > R$ is valid, therefore the factorized difference scheme (2.6) or (2.7) will be absolutely stabile under the initial data and the right part.

7°. We will notice, that for difference schemes (2.6) it's possible to construct the effective computing algorithm by means of decomposition of the operator $\prod_{\alpha=1}^{p}(E + \tau R_\alpha)$, which is easily implemented on the parallel computing system.

The difference scheme (2.6) we will rewrite as follows

$$\prod_{\alpha=1}^{p} B_\alpha y_t = F,$$

where $B_\alpha = E + \tau R_\alpha = E - \tau\sigma\Lambda_\alpha^0$, $\Lambda_\alpha^0 y = y_{\bar{x}_\alpha x_\alpha}$, $F = f - Ay$.

Such record of difference schemes gives the chance to construct the following computing algorithm

$$B_1 v_{(1)} = F, \quad B_\alpha v_{(\alpha)} = v_{(\alpha-1)} \quad (\alpha = 2,3,\dots,p), \qquad v_{(p)} = y_t,$$
$$y^{j+1} = y^j + \tau v_{(p)}, \quad j = 1,2,3\dots \tag{2.9}$$

For definition of functions $v_{(\alpha)}$ at $x \in \Gamma_{h\tau}$ it is necessary to use the following formulas:

$$v_{(\alpha)} = B_{\alpha+1} \cdots B_p g_{tt}(x,t) \text{ при } x \in \gamma_\alpha^+ \cup \gamma_\alpha^- \cup [0,T]\ . \tag{2.10}$$

This algorithm is especially convenient for solution of considered problem in case, when the function $g(x,t)$ does not depend from *t*. In this case we receive homogeneous boundary conditions for functions $v_{(\alpha)}$ $(\alpha = 1,2,\dots,p)$.

So, the initial problem (2.6) breaks into a number of subtasks, each of which is possible to solve, using parallel computing algorithms [12, 13].

4. **Convergence of the difference scheme**

1°. The initial difference scheme (2.1) has an error of approximation of the order $O\left(\tau + |h|^2\right)$, if the solution of system of the differential equations (1.1) belongs to the class $C^{4,2}\left(\overline{G}_T\right)$. An approximation error of factorized scheme (2.6) we will present in such kind $\psi = \psi_0 + \psi_1$, where ψ_0 -- an error of approximation of initial difference scheme (2.1), which order in the considered class is equal to $O\left(\tau + |h|^2\right)$, and $\psi_1 = \tau^2 Q_p u_t$. From here it is visible that at *p*>2 requirement $\psi_1 = O\left(\tau + |h|^2\right)$ imposes additional restrictions on smoothness of the solution of the system of equations (1.1). However, the aprioristic estimation and the theorem of convergence in norm of space $\overset{\circ}{W}{}_2^{(1)}$ can be received at weaker restrictions, than the condition of local approximation of order $O\left(\tau+|h|^2\right)$.

2°. To proof the convergence of factorized scheme (2.6) we will consider the vector function of error $z=y-u,\quad (x,t)\in\overline{\Omega}_{h\tau}$, where u -- the solution of the initial problem (1.1)-(1.3), and y - the solution of the problem (2.6), (2.3).

For the net function z we will receive the following problem

$$\begin{aligned}&\left(E+\tau\tilde{R}\right)z_t+Az=\psi,\\&z(x,0)=0,\quad z(x',t)=0,\quad x'\in\Gamma_{h\tau}\end{aligned}\qquad(3.1)$$

where $\psi=\psi_0+\psi_1$ -- the error of approximation of the difference scheme (2.6).

For the further statement it is useful to enter following denotations:

$A_\alpha^0=T_\alpha T_\alpha^*$ or $A_\alpha^0 u=T_\alpha T_\alpha^* u=u_{\bar{x}_\alpha x_\alpha}$, $\quad T_\alpha u=u_{\bar{x}_\alpha}$, $\quad T_\alpha^* u=u_{x_\alpha}$,

$$\|y\|_{q_{2,p}}^2=\sum_{\alpha<\beta}^{1\div p}\left\|T_\alpha T_\beta y\right\|_0^2,\qquad \|y\|_{q_{3,p}}^2=\sum_{\alpha<\beta<\gamma}^{1\div p}\left\|T_\alpha T_\beta T_\gamma y\right\|_0^2,\ldots,\quad \|y\|_{q_{p,p}}^2=\sum_{\alpha<\beta}^{1\div p}\left\|T_1T_2\cdots T_p y\right\|_0^2.$$

<u>Theorem 3.1.</u> Let, conditions I-II are satisfied, regularizator R is defined by equality $R=\sigma A^0$ and $\sigma\geq 0.5\nu_2$. Then for the solution of the problem (3.1) on any sequence of meshes $\overline{\Omega}_{h\tau}$ the aprioristic estimation is fair

$$\left\|z^{j+1}\right\|_A\leq M_1\max_{0<t'\leq t_j}\left\|\psi_0(t')\right\|_0+M_2\,\tau\max_{0<t'\leq t_j}\left\{\sum_{s=2}^{p}\tau^{s-2}\left\|u_{\bar{t}}(t')\right\|_{q_{s,p}}^2\right\}^{1/2},$$

where M_1, M_2 -- the constant number which is not dependent on the mesh.

Considering the estimation $\left\|z^{j+1}\right\|_A\geq\nu\left\|z^{j+1}\right\|_1$, where ν - the positive constant, which is not dependent on the mesh, on the basis of the theorem 3.1 it is easily possible to receive the aprioristic estimation for the solution of the problem (3.1) in space $\overset{\circ}{W}{}_2^{(1)}$.

Theorem 3.2. Let, conditions of the theorem 3.1 are satisfied. Then, for solution of the problem (3.1) on any sequence of meshes $\overline{\Omega}_{h\tau}$ the aprioristic estimation is fair:

$$\left\|z^{j+1}\right\|_1 \le M_1' \max_{0<t'\le t_j} \left\|\psi_0(t')\right\|_0 + M_2' \,\tau \max_{0<t'\le t_j} \left\{\sum_{s=2}^{p} \tau^{s-2} \left\|u_{\bar t}(t')\right\|_{q_{s,p}}^2\right\}^{1/2},$$

where M_1', M_2' -- positive constants, which are not depend on mesh.

Theorem 3.3. Let, conditions of the theorem 3.1 are satisfied. Besides, let solution of system of the equations (1.1) $u \in C^{4,2}\left(\overline{G}_T\right)$, when $p \le 4$, and $u \in C^{p,2}\left(\overline{G}_T\right)$ when $p \ge 2$. Then the solutions of difference problem (2.6), (2.3) converge in norm $\overset{\circ}{W}{}_2^{(1)}$ with speed $O\left(\tau + |h|^2\right)$ to the solution of problem (1.1)-(1.3) on any sequence of grids $\overline{\Omega}_{h\tau}$.

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ჰამლეტ მელაძე

ორშრიანი ფაქტორიზებული სხვაობიანი სქემები პარაბოლური ტიპის კერძოწარმოებულიანი განტოლებათა მრავალგანზომილებიანი სისტემისათვის

ნაშრომში განხილულია შერეული სასაზღვრო ამოცანა პირველი გვარის სასაზღვრო პირობებით პარაბოლური ტიპის კერძოწარმოებულიანი განტოლებათა მრავალგანზომილებიანი სისტემისათვის:

$$\frac{\partial u}{\partial t} = Lu + f,$$

სადაც L - ცვალებადკოეფიციენტებიანი, ძლიერად ელიფსური ოპერატორია, ხოლო $u = \left(u^{(1)}, u^{(2)}, \ldots, u^{(n)}\right)$, $f = \left(f^{(1)}, f^{(2)}, \ldots, f^{(n)}\right)$ - წარმოადგენს n-განზომილებიან ვექტორებს.

ნაშრომში აგებულია აბსოლუტურად მდგრადი სხვაობიანი ფაქტორიზებული სქემები. ამ სხვაობიანი სქემების აგების დროს გამოყენებულია რეგულარიზაციის მეთოდი, რომელიც დამუშავებული იყო ა. სამარსკის მიერ. მიღებული ალგორითმები შეიძლება ეფექტურად იქნეს რეალიზებული მრავალპროცესორებიან გამოთვლით სისტემებზე. ენერგეტიკულ უტოლობათა მეთოდის გამოყენებით ბადურ $\overset{0}{W}{}_2^{(1)}$ სივრცეში მიღებულია აპრიორული შეფასებები, რომელთა საფუძველზეც დამტკიცებულია თეორემა სხვაობიანი სქემის კრებადობის შესახებ.

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TINA GOROZIA, TINATIN DAVITASHVILI

BOUNDARY VALUE PROBLEM FOR ONE CLASS OF ORDINARY DIFFERENTIAL EQUATIONS, GIVEN ON GRAPHS

Abstract. In the present paper the boundary value problem for one class of the ordinary differential equations, given on graphs, and the difference method for solution of this problem is investigated. The explicit solutions of the differential problem in the form of quadratures, and also exact solution of the difference scheme are constructed.

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Key words and phrases: boundary value problem**,** differential equations on graphs, analytical solutions, difference schemes.

1. **Introduction**

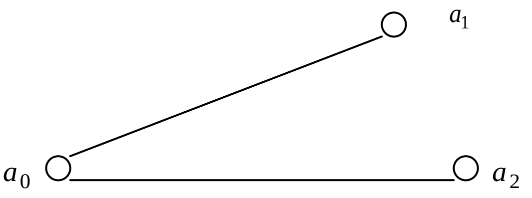
In papers [1,2] the mathematical model of an electropower system which represents to boundary value problem for the ordinary differential equations set on graphs is considered. The problem in view correctness is investigated. The correspondent finite-difference scheme is constructed and investigated. The double-sweep method type formulas for finding the solution of the finite-difference scheme are offered.

Let's mark, that the boundary value problems on graphs are not investigated theoreticaly to the full in the scientific literature. See, for example [3 −7] and the literature mentioned there.

In the present paper the boundary value problem for one class of the ordinary differential equations set on graphs is considered. The explicit solution of this problem in the form of quadratures is constructed. Further the difference method for solution of this problem is considered. The exact solution of the difference scheme, which is the difference analogue of solution of the differential problem is constructed. We will mark that it’s important to have exact solution of this problem for testing, and also for an estimation of accuracy and efficiency of the difference methods used for numerical solution of the ordinary differential equations, given on graphs.

3. **Analytical solutions of the differential equations given on graphs**

For simplicity of presentation we will consider the graph that consists of two ribs. Tops of the graph are the points a_1, a_2, and the ribs of the graph are $a_0 a_1 (\Gamma_1)$, $a_0 a_2 (\Gamma_2)$.



Let us state the following problem: find twice continuously differentiated functions $u(x_1)$, $v(x_2)$, where $x_\alpha -$ the local coordinate along Γ_α, α=1,2, which satisfy:

1) the differential equations

$$\frac{d^2u}{dx_1^2} = -f_1(x_1)\,,\ x_1 \in (0, b_1), \quad \frac{d^2v}{dx_2^2} = -f_2(x_2)\,,\ x_2 \in (0, b_2), \qquad (1)$$

where $\ell_\alpha -$ is length of the rib Γ_α, and $f_\alpha(x_\alpha)$, α=1,2, − are given, continuous on $(0, b_\alpha)$ functions, $\alpha = 1{,}2$;

2) the boundary conditions

$$u(b_1) = \mu_1,\ v(b_2) = \mu_2\,, \qquad (2)$$

3) and conditions of conjunction

$$u(0) = v(0)\,,\ \left.\frac{du}{dx_1}\right|_{x_1=0} + \left.\frac{dv}{dx_2}\right|_{x_2=0} = 0\,. \qquad (3)$$

It’s easy to construct the solution of the problem (1)-(3) in the form of quadratures. We will present functions $u(x_1)$, $v(x_2)$ in the form of sum of two functions:

$$\begin{aligned} u(x_1) &= u^{(1)}(x_1) + u^{(2)}(x_1) \\ v(x_2) &= v^{(1)}(x_2) + v^{(2)}(x_2) \end{aligned}, \qquad (4)$$

where

$$\frac{d^2u^{(1)}(x_1)}{dx_1^2} = 0, \quad x_1 \in (0, b_1), \qquad \frac{d^2v^{(1)}(x_2)}{dx_2^2} = 0, \quad x_2 \in (0, b_2)$$

$$u^{(1)}(b_1) = \mu_1, \quad v^{(1)}(b_2) = \mu_2, \qquad (5)$$

$$u^{(1)}(0) = v^{(1)}(0), \qquad \left.\frac{du^{(1)}}{dx_1}\right|_{x_1=0} + \left.\frac{dv^{(2)}}{dx_2}\right|_{x_2=0} = 0$$

and

$$\frac{d^2u^{(2)}(x_1)}{dx_1^2} = -f_1(x_1), \quad x_1 \in (0, b_1), \qquad \frac{d^2v^{(2)}(x_2)}{dx_2^2} = -f_2(x_2), \quad x_2 \in (0, b_2).$$

$$u^{(2)}(b_1) = 0, \quad v^{(2)}(b_2) = 0, \qquad (6)$$

$$u^{(2)}(0) = v^{(2)}(0), \qquad \left.\frac{du^{(2)}}{dx_1}\right|_{x_1=0} + \left.\frac{dv^{(2)}}{dx_2}\right|_{x_2=0} = 0.$$

The solutions of the problem (5) are the linear functions:

$$u^{(1)}(x_1) = \frac{x_1 + b_2}{b_1 + b_2}\mu_1 + \frac{b_1 - x_1}{b_1 + b_2}\mu_2 \ , \qquad (7)$$

$$v^{(1)}(x_2) = \frac{b_2 - x_2}{b_1 + b_2}\mu_1 + \frac{x_2 + b_1}{b_1 + b_2}\mu_2 \ . \qquad (8)$$

For construction the solution of the problem (6) let's integrate the first equation:

$$u'^{(2)}(t) = u'^{(2)}(0) - \int_0^t f_1(s)ds \, .$$

Integrating once again the previous relation, we will receive

$$u^{(2)}(x_1) = u^{(2)}(0) + x_1 u'^{(2)}(0) - \int_0^{x_1}\left(\int_0^t f_1(s)ds\right)dt \, .$$

Let's denote by $u^{(2)}(0) = v^{(2)}(0) = \alpha$. Then we will have

$$u^{(2)}(b_1) = \alpha + b_1 u'^{(2)}(0) - \int_0^{b_1}\left(\int_0^t f_1(s)ds\right)dt \, .$$

Considering the condition $u^{(2)}(b_1) = 0$, we will receive, that

$$u'^{(2)}(0) = -\frac{\alpha}{b_1} + \frac{1}{b_1}\int_0^{b_1}\left(\int_0^t f_1(s)ds\right)dt \, . \qquad (9)$$

Therefore,

$$u^{(2)}(x_1)=\left(1-\frac{x_1}{b_1}\right)\alpha+\frac{x_1}{b_1}\int\limits_0^{b_1}\left(\int\limits_0^t f_1(s)ds\right)dt-\int\limits_0^{x_1}\left(\int\limits_0^t f_1(s)ds\right)dt\,. \qquad (10)$$

Let's similarly receive

$$v'^{(2)}(0)=-\frac{\alpha}{b_2}+\frac{1}{b_2}\int\limits_0^{b_2}\left(\int\limits_0^t f_2(s)ds\right)dt\,, \qquad (11)$$

$$v^{(2)}(x_2)=\left(1-\frac{x_2}{b_2}\right)\alpha+\frac{x_2}{b_2}\int\limits_0^{b_2}\left(\int\limits_0^t f_2(s)ds\right)dt-\int\limits_0^{x_2}\left(\int\limits_0^t f_2(s)ds\right)dt\,. \qquad (12)$$

Taking into account the condition $\left.\frac{du^{(2)}}{dx_1}\right|_{x_1=0}+\left.\frac{dv^{(2)}}{dx_2}\right|_{x_2=0}=0$ and relations (9), (11), it's possible to define the value of a constant α:

$$\alpha=\frac{1}{b_1+b_2}\left[b_2\int\limits_0^{b_1}\left(\int\limits_0^t f_1(s)ds\right)dt+b_1\int\limits_0^{b_2}\left(\int\limits_0^t f_2(s)ds\right)dt\right]. \qquad (13)$$

Finally, taking into account equalities (7), (8), (10), (12), (13), we obtain the explicit solution of the problem (1)-(3) in the form of quadratures:

$$u(x_1)=\frac{x_1+b_2}{b_1+b_2}\mu_1+\frac{b_1-x_1}{b_1+b_2}\mu_2+\frac{b_1-x_1}{b_1(b_1+b_2)}\left[b_2\int_0^{b_1}\left(\int_0^t f_1(s)ds\right)dt+b_1\int_0^{b_2}\left(\int_0^t f_2(s)ds\right)dt\right]+$$
$$+\frac{x_1}{b_1}\int_0^{b_1}\left(\int_0^t f_1(s)ds\right)dt-\int_0^{x_1}\left(\int_0^t f_1(s)ds\right)dt \qquad (14)$$

$$v(x_2)=\frac{b_2-x_2}{b_1+b_2}\mu_1+\frac{x_2+b_1}{b_1+b_2}\mu_2+\frac{b_2-x_2}{b_2(b_1+b_2)}\left[b_2\int_0^{b_1}\left(\int_0^t f_1(s)ds\right)dt+b_1\int_0^{b_2}\left(\int_0^t f_2(s)ds\right)dt\right]+$$
$$+\frac{x_2}{b_2}\int_0^{b_2}\left(\int_0^t f_2(s)ds\right)dt-\int_0^{x_2}\left(\int_0^t f_2(s)ds\right)dt \qquad (15)$$

3. **The difference scheme for problem (1) - (3)**

For numerical solution of problem (1)-(3) let's enter on segments $[0,b_1]$ and $[0,b_2]$ the uniform mesh with steps h_1 and h_2. Then replace $\frac{d^2u}{dx_1^2}$ and $\frac{d^2v}{dx_2^2}$ by second difference derivative [8]. Then for problem (1)-(3) we will receive the following difference scheme:

$$\frac{y^{(i-1)}-2y^{(i)}+y^{(i+1)}}{h_1^2}=-f_1(x_1^{(i)}),\quad \frac{z^{(j-1)}-2z^{(j)}+z^{(j+1)}}{h_2^2}=-f_2(x_2^{(j)}), \qquad (16)$$

$$i=1,\ldots,N_1-1,\quad N_1h_1=b_1,\qquad j=1,\ldots,N_2-1,\quad N_2h_2=b_2,$$

the boundary conditions

$$y^{(N_1)}=\mu_1,\quad z^{(N_2)}=\mu_2, \qquad (17)$$

and conditions of conjunction

$$y^{(0)} = z^{(0)}\ , \quad \frac{y^{(1)} - y^{(0)}}{h_1} + \frac{z^{(1)} - z^{(0)}}{h_2} = 0\,, \qquad (18)$$

where $y^{(i)} = y(ih_1), \quad z^{(j)} = z(jh_2), \quad x_1^{(i)} = ih_1, \quad x_2^{(j)} = jh_2$.

Let's construct by analogy (14)-(15) exact solution of difference scheme (16)-(18). We will present the functions $y^{(i)}, \quad z^{(j)}$ in the form of the sum of two functions:

$$y^{(i)} = y_1^{(i)} + y_2^{(i)}\ , \qquad z^{(j)} = z_1^{(j)} + z_2^{(j)}\ ,$$

where

$$\frac{y_1^{(i-1)} - 2y_1^{(i)} + y_1^{(i+1)}}{h_1^2} = 0\,, \qquad \frac{z_1^{(j-1)} - 2z_1^{(j)} + z_1^{(j+1)}}{h_2^2} = 0\,,$$

$$i = 1,\dots,N_1 - 1, \quad N_1 h_1 = b_1\,, \qquad j = 1,\dots,N_2 - 1, \quad N_2 h_2 = b_2\,,$$

$$y_1^{(N_1)} = \mu_1, \quad z_1^{(N_2)} = \mu_2\,, \qquad (19)$$

$$y_1^{(0)} = z_1^{(0)}\ , \quad \frac{y_1^{(1)} - y_1^{(0)}}{h_1} + \frac{z_1^{(1)} - z_1^{(0)}}{h_2} = 0\,.$$

and

$$\frac{y_2^{(i-1)} - 2y_2^{(i)} + y_2^{(i+1)}}{h_1^2} = -f_1(x_1^{(i)})\,,$$

$$\frac{z_2^{(j-1)} - 2z_2^{(j)} + z_2^{(j+1)}}{h_2^2} = -f_2(x_2^{(j)})\,,$$

$$i = 1,\dots,N_1 - 1, \quad N_1 h_1 = b_1\,, \qquad j = 1,\dots,N_2 - 1, \quad N_2 h_2 = b_2\,,$$

$$y_2^{(N_1)} = 0\,, \quad z_2^{(N_2)} = 0\,, \tag{20}$$

$$y_2^{(0)} = z_2^{(0)}\ , \quad \frac{y_2^{(1)} - y_2^{(0)}}{h_1} + \frac{z_2^{(1)} - z_2^{(0)}}{h_2} = 0\,.$$

It is easy to show that solution of the problem (19) are following mesh functions:

$$y_1^{(i)} = \frac{b_2 + x_1^{(i)}}{b_1 + b_2}\mu_1 + \frac{b_1 - x_1^{(i)}}{b_1 + b_2}\mu_2\,, \quad i = 1, \ldots, N_1 - 1,$$

$$z_1^{(j)} = \frac{b_2 - x_2^{(j)}}{b_1 + b_2}\mu_1 + \frac{b_1 + x_2^{(j)}}{b_1 + b_2}\mu_2\,, \quad j = 1, \ldots, N_2 - 1. \tag{21}$$

Let's find explicit expression for $y_2^{(i)}, \quad z_2^{(i)}$. For this rewrite the equation

$\left(y_2^{(i)}\right)_{\bar{x}x} = \dfrac{y_2^{(i-1)} - 2y_2^{(i)} + y_2^{(i+1)}}{h_1^2} = -f_1(x_1^{(i)})$ in the form:

$$\left(y_2^{(i+1)}\right)_{\bar{x}} - \left(y_2^{(i)}\right)_{\bar{x}} = -h_1 f_1\left(x_1^{(i)}\right),\ i = 1, \ldots, N_1 - 1,$$

and summarizing on i from 1 to k. Then we will receive

$$\left(y_2^{(k+1)}\right)_{\bar{x}} = \left(y_2^{(1)}\right)_{\bar{x}} - \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right)$$

or

$$y_2^{(k+1)} - y_2^{(k)} = h_1\left(y_2^{(1)}\right)_{\bar{x}} - h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right).$$

Summarizing the last equation on k from 1 to j-1, we will receive:

$$y_2^{(j)} = y_2^{(1)} + (j-1)h_1\left(y_2^{(1)}\right)_{\bar{x}} - \sum_{k=1}^{j-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right)$$

or

$$y_2^{(j)} = y_2^{(0)} + jh_1\left(y_2^{(1)}\right)_{\bar{x}} - \sum_{k=1}^{j-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right). \qquad (22)$$

Taking into account the condition $y_2^{(N_1)} = 0$, when $j = N_1$, from the last equality it is possible to write:

$$y_2^{(0)} + N_1 h_1\left(y_2^{(1)}\right)_{\bar{x}} - \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right) = 0\,.$$

Let's denote $y_2^{(0)} = z_2^{(0)} = \alpha$. Then we will have

$$\left(y_2^{(1)}\right)_{\bar{x}_1} = -\frac{\alpha}{b_1} + \frac{1}{b_1} \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right). \qquad (23)$$

Therefore, from equality (22) and (23), we will receive:

$$y_2^{(j)} = \left(1 - \frac{x_1^{(j)}}{b_1}\right)\alpha + \frac{x_1^{(j)}}{b_1} \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right) - \sum_{k=1}^{j-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right),$$

$$j = 2, 3, \ldots, N_1 - 1, \qquad (24)$$

$$y_2^{(1)} = \left(1 - \frac{x_1^{(1)}}{b_1}\right)\alpha + \frac{x_1^{(1)}}{b_1} \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right).$$

Similarly we will receive

$$\left(z_2^{(1)}\right)_{\bar{x}_2} = -\frac{\alpha}{b_2} + \frac{1}{b_2}\sum_{k=1}^{N_2-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right) \qquad (25)$$

and

$$z_2^{(j)} = \left(1 - \frac{x_2^{(j)}}{b_2}\right)\alpha + \frac{x_2^{(j)}}{b_2}\sum_{k=1}^{N_2-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right) - \sum_{k=1}^{j-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right),$$

$$j = 2,3,\ldots,N_2 - 1, \qquad (26)$$

$$z_2^{(1)} = \left(1 - \frac{x_2^{(1)}}{b_2}\right)\alpha + \frac{x_2^{(1)}}{b_2}\sum_{k=1}^{N_2-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right).$$

Using the condition of conjunction $\left(y_2^{(1)}\right)_{\bar{x}_1} + \left(z_2^{(1)}\right)_{\bar{x}_2} = 0$ and equalities (23), (25), it is possible to define value of a constant α:

$$\alpha = \frac{1}{b_1 + b_2}\left[b_2 \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right) + b_1 \sum_{k=1}^{N_2-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right)\right]. \qquad (27)$$

Taking into account equalities (21), (24), (26) and (27), we will receive the explicit solution of the difference problem (16)-(18):

$$y^{(j)} = y_1^{(j)} + y_2^{(j)} = \frac{b_2 + x_1^{(j)}}{b_1 + b_2}\mu_1 + \frac{b_1 - x_1^{(j)}}{b_1 + b_2}\mu_2 +$$

$$+\frac{b_1 - x_1^{(j)}}{b_1(b_1 + b_2)}\left[b_2 \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right) + b_1 \sum_{k=1}^{N_2-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right)\right] +$$

$$+\frac{x_1^{(j)}}{b_1} \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right) - \sum_{k=1}^{j-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right), \quad j = 2,3,\ldots, N_1 - 1.$$

$$y^{(1)} = y_1^{(1)} + y_2^{(1)} = \frac{b_2 + x_1^{(j)}}{b_1 + b_2}\mu_1 + \frac{b_1 - x_1^{(j)}}{b_1 + b_2}\mu_2 +$$

$$+\frac{b_1 - x_1^{(1)}}{b_1(b_1 + b_2)}\left[b_2 \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right) + b_1 \sum_{k=1}^{N_2-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right)\right] +$$

$$+\frac{x_1^{(1)}}{b_1} \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right).$$

$$z^{(j)} = z_1^{(j)} + z_2^{(j)} = \frac{b_2 - x_2^{(j)}}{b_1 + b_2}\mu_1 + \frac{b_1 + x_2^{(j)}}{b_1 + b_2}\mu_2 +$$

$$+\frac{b_2 - x_2^{(j)}}{b_2(b_1 + b_2)}\left[b_2 \sum_{k=1}^{N_1-1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right) + b_1 \sum_{k=1}^{N_2-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right)\right] +$$

$$+\frac{x_2^{(j)}}{b_2} \sum_{k=1}^{N_2-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right) - \sum_{k=1}^{j-1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right), \quad j = 2,3,\ldots, N_2 - 1.$$

$$
\begin{aligned}
z^{(1)} = z_1^{(1)} + z_2^{(1)} &= \frac{b_2 - x_2^{(1)}}{b_1 + b_2}\mu_1 + \frac{b_1 + x_2^{(1)}}{b_1 + b_2}\mu_2 + \\
&+ \frac{b_2 - x_2^{(1)}}{b_2\left(b_1 + b_2\right)}\left[b_2 \sum_{k=1}^{N_1 - 1} h_1 \sum_{i=1}^{k} h_1 f_1\left(x_1^{(i)}\right) + b_1 \sum_{k=1}^{N_2 - 1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right) \right] + \\
&+ \frac{x_2^{(1)}}{b_2} \sum_{k=1}^{N_2 - 1} h_2 \sum_{i=1}^{k} h_2 f_2\left(x_2^{(i)}\right).
\end{aligned}
$$

Let's mark that results of given article can be transferred for the differential equations, given on graphs, which consists from n ribs.

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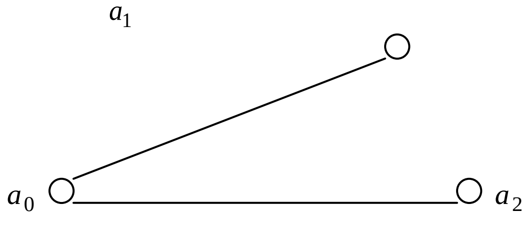
თინა გოროზია, თინათინ დავითაშვილი

სასაზღვრო ამოცანა გრაფებზე განსაზღვრული ჩვეულებრივ დიფერენციალურ განტოლებათა ერთი კლასისთვის

ნაშრომში გამოკვლეულია სასაზღვრო ამოცანა გრაფებ-ზე განსაზღვრული ჩვეულებრივ დიფერენციალურ განტოლებათა ერთი კლასისთვის.

სიმარტივისათვის განვიხილოთ ორი წიბოსაგან შემდგარი გრაფი. გრაფის წვეროებია a_1, a_2 წერტილები, ხოლო წიბოებია $a_0a_1(\Gamma_1)$ და $a_0a_2(\Gamma_2)$.



დავსვათ შემდეგი ამოცანა: ვიპოვოთ ორჯერ უწყვეტად დი-ფერენცირებადი ფუნქციები $u(x_1)$, $v(x_2)$ (სადაც $x_\alpha -$ ლოკალური კოორდინატაა Γ_α-ის გასწვრივ, α=1,2), რომლებიც აკმაყოფილებენ ა) შემდეგ დიფერენციალურ განტოლებებს:

$$\frac{d^2u}{dx_1^2} = -f_1(x_1)\,,\ x_1 \in (0,b_1),\quad \frac{d^2v}{dx_2^2} = -f_2(x_2)\,,\ x_2 \in (0,b_2). \qquad (1)$$

სადაც $\ell_\alpha - \Gamma_\alpha$ წიბოს სიგრძეა, ხოლო $f_\alpha(x_\alpha)$, α=1,2 − მოცემული ფუნქციებია;

ბ) სასაზღვრო პირობებს

$$u(b_1) = \mu_1,\ v(b_2) = \mu_2, \tag{2}$$

გ) შეუღლების პირობებს

$$u(0) = v(0)\,,\ \left.\frac{du}{dx_1}\right|_{x_1=0} + \left.\frac{dv}{dx_2}\right|_{x_2=0} = 0\,. \tag{3}$$

შესწავლილია დასმული ამოცანის ამოხსნის სხვაობიანი მეთოდი. აგებულია დიფერენციალური ამოცანის ცხადი ამონახსნები კვადრატურებში და სხვაობიანი სქემის ზუსტი ამონახსნები. ნაშრომში მიღებული შედეგები შეიძლება მიღებულ იქნეს დიფერენციალური განტოლებებისთვის, რომლებიც განსაზღვრულია n წიბოს შემცველ გრაფზე. შევნიშნოთ, რომ ანალოგიური ამოცანები წარმოიქმნება ელექტროენერგეტიკული სისტემების, გაზსადენების, წყალსადენების და ა.შ. ქსელებში სხვადასხვა პროცესების მოდელირებისას.

სოხუმის სახელმწიფო უნივერსიტეტის შრომები
ტ. VII, 2009
მათემატიკისა და კომპიუტერულ მეცნიერებათა სერია

თემურ ჩილაჩავა, ციალა ძიძიგური, ლეილა სულავა, მაია ჩაკაბერია

ადმინისტრაციული ზეწოლის არაწრფივი მათემატიკური მოდელი

რეზიუმე. ნაშრომში შემოთავაზებულია ახალი არაწრფივი უწყვეტი მათემატიკური მოდელი, რომელსაც შეუძლია აღწეროს მოცემულ სოციუმში (ქვეყანა, სასწავლო დაწესებულება, საწარმოო ობიექტი და სხვა) მმართველი სტრუქტურების მხრიდან ადმინისტრაციული ზეწოლა ადამიანებზე მათი მოქმედებების გაკონტროლების მიზნით. მათემატიკური მოდელი აღიწერება ორუცნობიანი (თავისუფალი (არამართვადი) და მართვადი ადამიანების რაოდენობა დროის t მომენტში) დიფერენციალური განტოლებათა სისტემით. ადმინისტრაციული ზეწოლა ზოგადად განისაზღვრება დროის მოცემული ფუნქციით. მუდმივი ადმინისტრაციული ზეწოლის პირობებში კოშის ამოცანა არაწრფივი დიფერენციალური განტოლებათა სისტემისათვის ამოხსნილია ანალიზურად ზუსტად. მოდელის პარამეტრებისა (თავისუფლების ხარისხის კოეფიციენტი, ადმინისტრაციული ზეწოლის ძალა) და საწყის პირობებს შორის სხვადასხვა დამოკიდებულებების მიხედვით მიღებულია ხუთი განსხვავებული შემთხვევა:

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2000Mathematics Subject Classification: 93A30, 00A71

Key words and phrases: administrative pressure, nonlinear mathematical model, factor of degree of freedom, a complete control case, model of full submission.

1. **შესავალი**

სინერგეტიკა ანუ თვითორგანიზაციის თეორია, დღეისთვის საგანთა შორის კავშირების კვლევის ერთ-ერთ ყველაზე პოპულარული და პერსპექტიული მიდგომაა. ტერმინი “სინერგეტიკა” ბერძნულიდან თარგმანში ნიშნავს “ერთობლივ მოქმედებას”.

ჩვენი მთავარი მიზანია ვაჩვენოთ როგორ აღწევს სინერგეტიკა მნიშვნელოვან შედეგებს განსხვავებულ მეცნიერებებში (მაგალითად, მათემატიკა და სოციოლოგია). რას უნდა დავეყრდნოთ ცვლილებების ეპოქაში? ალბათ, გამოყენებით მათემატიკას, მათემატიკურ და კომპიუტერულ მოდელებს, პროგნოზისა და მონიტორინგის ალგორითმებს, მეცნიერულ მსოფმხედველობას.

სინერგეტიკის განვითარებამ გვიჩვენა, რომ მისი თავისებურება და ორიგინალობა დამოკიდებულია იმაზე, რომ ის სამეცნიო სივრცეში სამი სფეროს თანაკვეთაზე მდებარეობს – საგნობრივი ცოდნის, ფილოსოფიური რეფლექტებისა და მათემატიკური მოდელირების. სწორედ ეს სამთაკავშირი საშუალებას გვაძლევს ერთ შემთხვევაში ეფექტურად ავაშენოთ მეცნიერული სტრატეგია, მეორეში – მოვახდინოთ მსხვილი კვლევითი პროექტების ორგანიზება [1-9].

ამ ნაშრომში ჩვენს ამოცანას წარმოადგენდა ისეთი სოციალური პროცესის აღწერა მათემატიკურ ენაზე (შეგვექმნა მათემატიკური მოდელი), როგორიც არის ადმინისტრაციული ზეწოლა, რომელიც შეიძლება ხორციელდებოდეს მაკროდონეზე (სახელმწიფო ან მისი დიდი რეგიონი, მაკრომოდელი) ან მიკროდონეზე (სასწავლო დაწესებულება, საწარმოო ობიექტი და სხვა, მიკრომოდელი) მრავალფეროვანი იდეოლოგიური და ტექნოლოგიური საშუალებებით. მათემატიკურ მოდელში განიხილება სამი ობიექტი:

1. ადმინისტრაცია (მმართველი სტრუქტურები, ხელისუფლება), რომლის მიზანს მიკრო ან მაკროდონეზე წარმოადგენს მის დაქვემდებარებაში მყოფი ადამიანების მართვა თავისი ინტერესებიდან გამომდინარე. ეს ინტერესები შეიძლება იყოს როგორც სამართლიანი (სწორი მართვა) ისე უსამართლო (არასწორი მართვა).
2. მართვადი ადამიანები, რომელზეც მოქმედებს ადმინისტრაციული ზეწოლა.
3. თავისუფალი ანუ არამართვადი ადამიანები, რომლებიც, მიუხედავად ადმინისტრაციული ზეწოლისა, რჩებიან თავისუფალნი თავის არჩევანში ან მოქმედებაში.

2. განტოლებათა სისტემა და საწყისი პირობები

პოპულაციის ევოლუციის, საომარი მოქმედებების ლანჩესტერის მოდელებისა და სოციალური პროცესების (მათ შორის, დემოგრაფიული, ისტორიული, საინფორმაციო ომი და სხვა) აღმწერი სხვადასხვა მოდელების ანალიზს მივყავართ ადმინისტრაციული ზეწოლის (მართვის) შემდეგ არაწრფივ მათემატიკურ მოდელამდე [1, 10 – 12]

$$\begin{cases} \dfrac{dx(t)}{dt} = \alpha x(t)y(t) - b \\ \\ \dfrac{dy(t)}{dt} = -\alpha x(t)y(t) + b \end{cases} \qquad (2.1)$$

მოცემულ არაწრფივ მათემატიკურ მოდელში ადმინისტრაციულ ზეწოლას აქვს მუდმივი ხასიათი და ის არ არის დამოკიდებული დროზე. მოდელის განხილვის პერიოდში ადამიანთა ორივე ჯგუფის ბუნებრივი ცვლილებები არ არის გათვალისწინებული (ამ მიახლოებაში ჩათვლილია, რომ შობადობისა და სიკვდილიანობის კოეფიციენტები ტოლია).

მოდელი არის უწყვეტი ანუ საძიებელ ფუნქციებს გააჩნიათ სისტემაში შემავალი უწყვეტი წარმოებულები.

$x(t)$ –დროის t მომენტში თავისუფალი (არამართვადი) ადამიანების რაოდენობაა,

$y(t)-$დროის t მომენტში მართვადი ადამიანების რაოდენობაა,

$b-$ ადმინისტრაციული ზეწოლის ძალა,

$\alpha-$ თავისუფლების კოეფიციენტი.

$x(t)$ და $y(t)$ ფუნქციების რაოდენობრივი ცვლილებები დროის t მომენტში აღიწერება (2.1) არაწრფივი განტოლებათა სისტემით. რადგანაც განტოლებათა სისტემა (2.1) არის პირველი რიგის, უნდა დაისვას დამატებითი პირობა (კოშის პირობა)

$$x(0)= x_0 \qquad\qquad y(0)=y_0 \tag{2.2}$$

მივიღეთ კოშის ამოცანა (2.1), (2.2). ამასთან, ბუნებრივია, რომ $b>0,\ \alpha>0.$

ვიპოვოთ ცხადი სახით $x(t)$ და $y(t)$ ფუნქციები.

ადგანაც მოდელის განხილვის პერიოდში ორივე ჯგუფის ბუნებრივი ცვლილებები იძლევიან ერთობლიობაში ნულოვან ეფექტს, გვაქვს:

$$x(t)+y(t)=x_0+y_0=a \tag{2.3}$$

საიდანაც გამოვსახოთ $y(t)$

$$y(t)=a-x(t) \tag{2.4}$$

ჩავსვათ (2.4) სისტემა (2.1)-ის პირველ განტოლებაში. მაშინ მივიღებთ პირველი რიგის განცალებადცვლადებიანი დიფერენციალური განტოლებისათვის კოშის ამოცანას

$$\frac{dx(t)}{dt}=\alpha x(t)\big(a-x(t)\big)-b \tag{2.5}$$

$$x(0)= x_0$$

$x(t)$ ფუნქციის რაოდენობრივი ცვლილება გამოისახა კვადრატული სამწევრით

$$\frac{dx}{dt}=\alpha ax-\alpha x^2-b=-\alpha x^2+\alpha ax-b$$

შემოვიღოთ აღნიშვნა:

$$D = \frac{b}{\alpha} - \frac{a^2}{4} \tag{2.6}$$

განვიხილოთ სამი განსხვავებული შემთხვევა:

I. $D = 0.$

მაშინ (2.5) მიიღებს სახეს

$$\frac{dx}{dt} = -\alpha x^2 + \alpha a x - b = -\alpha\left(x^2 - ax + \frac{b}{\alpha}\right) = -\alpha\left(x^2 - ax + \frac{a^2}{4} - \frac{a^2}{4} + \frac{b}{\alpha}\right) =$$

$$= -\alpha\left[\left(x - \frac{a}{2}\right)^2 + \left(\frac{b}{\alpha} - \frac{a^2}{4}\right)\right]$$

რადგან

$$\frac{b}{\alpha} - \frac{a^2}{4} = 0$$

ვღებულობთ მარტივ დიფერენციალურ განტოლებას

$$\frac{dx}{dt} = -\alpha\left(x - \frac{a}{2}\right)^2$$

რომლის ერთადერთი ამონახსნი, საწყისი პირობის გათვალისწინებით ჩაიწერება

$$x(t) = \frac{a}{2} + \frac{x_0 - \frac{a}{2}}{\alpha t\left(x_0 - \frac{a}{2}\right) + 1} \tag{2.7}$$

ხოლო (2.3) და (2.4) გათვალისწინებით, მივიღებთ

$$y(t) = \frac{a}{2} - \frac{x_0 - \frac{a}{2}}{\alpha t\left(x_0 - \frac{a}{2}\right) + 1} \tag{2.8}$$

საბოლოოდ a -ს მნიშვნელობის გათვალისწინებით, მივიღებთ კოშის ამოცანის (2.1), (2.2) ერთადერთ ზუსტ ანალიზურ ამონახსნს.

$$x(t) = \frac{x_0 + y_0}{2} + \frac{x_0 - y_0}{2 + \alpha t(x_0 - y_0)} \qquad (2.9)$$

$$y(t) = \frac{x_0 + y_0}{2} + \frac{y_0 - x_0}{2 + \alpha t(x_0 - y_0)}$$

ამ შემთხვევაში, (2.9)-ის ანალიზი გვიჩვენებს, რომ სოციუმში მუდმივი ადმინისტრაციული ზეწოლისა და თუნდაც განსხვავებული სასტარტო პირობების მიუხედავად $(x_0 \neq y_0)$ გარკვეული დიდი დროის შემდეგ, დამყარდება მართვადი და არამართვადი ადამიანების თანაბარი რაოდენობა (არასაკმარისი ზეწოლა).

II. განვიხილოთ შემთხვევა, როცა $D > 0$.

$$\frac{b}{\alpha} - \frac{a^2}{4} = p^2 > 0$$

მაშინ (2.5)-დან მივიღებთ

$$\frac{dx}{dt} = -\alpha\left(x^2 - ax + \frac{b}{\alpha}\right) = -\alpha\left(x^2 - ax + \frac{a^2}{4} - \frac{a^2}{4} + \frac{b}{\alpha}\right)$$
$$= -\alpha\left[(x - \frac{a}{2})^2 + (\frac{b}{\alpha} - \frac{a^2}{4})\right]$$

$$\frac{dx}{dt} = -\alpha\left[(x - \frac{a}{2})^2 + p^2\right] \qquad (2.10)$$

მიღებული (2.10) განტოლების ზოგადი ამონახსნია:

$$arc\tan\frac{x - \frac{a}{2}}{p} = -\alpha pt + C \qquad (2.11)$$

მოცემული საწყისი პირობის გათვალისწინებით, ვღებულობთ

$$C = arc\tan\frac{x_0 - \frac{a}{2}}{p} = arc\tan\frac{x_0 - y_0}{2p}$$

ჩავსვათ C-ს მნიშვნელობა ზოგად ამონახსნში (2.11), მივიღებთ

$$arc\tan\frac{x - \frac{x_0 + y_0}{2}}{p} = arc\tan\frac{2x - x_0 - y_0}{2p}$$

$$arc\tan\frac{2x - x_0 - y_0}{2p} = -\alpha pt + arc\tan\frac{x_0 - y_0}{2p}$$

ამრიგად, ამ შემთხვევაში (2.1) სისტემის ამონახსნს ექნება სახე

$$x(t) = \frac{x_0 + y_0}{2} + p\frac{\frac{x_0 - y_0}{2p} - \tan(\alpha pt)}{1 + \frac{x_0 - y_0}{2p}\tan(\alpha pt)} \qquad (2.12)$$

$$y(t) = \frac{x_0 + y_0}{2} - p\frac{\frac{x_0 - y_0}{2p} - \tan(\alpha pt)}{1 + \frac{x_0 - y_0}{2p}\tan(\alpha pt)}$$

(2.12)-ს ანალიზი იძლევა სამ განსხვავებულ შემთხვევას:

- თავისუფალი ადამიანების რაოდენობა, რომელიც თავდაპირველად მართვადი რაოდენობის მეტი იყო $(x_0 > y_0)$, მიისწრაფვის წონასწორობის მდგომარეობისკენ, რომელიც საერთო რაოდენობის ნახევარზე ნაკლებია (ძლიერი, მაგრამ შეძლუდული ზეწოლა თავისუფალ ადამიანებზე);
- თავისუფალი ადამიანების რაოდენობა, რომელიც თავდაპირველად მართვადი რაოდენობის ტოლი ან ნაკლები იყო

$(x_0 \leq y_0)$, მიისწრაფვის ნულისკენ (ძლიერი ზეწოლა, სრული კონტროლის შემთხვევა);

- თავისუფალი ადამიანების რაოდენობა, რომელიც თავდაპირველად მართვადი რაოდენობის მეტი იყო $(x_0 > y_0)$, მიისწრაფვის ნულისკენ (უძლიერესი ზეწოლა თავისუფალ ადამიანებზე , სრული დამორჩილების მოდელი).

III. განვიხილოთ შემთხვევა, როცა $D < 0$.

$$\frac{b}{\alpha} - \frac{a^2}{4} = -p^2 < 0$$

მაშინ (2.5) მიიღებს სახეს

$$\frac{dx}{dt} = -\alpha\left[(x - \frac{a}{2})^2 + (\frac{b}{\alpha} - \frac{a^2}{4})\right] = -\alpha\left[(x - \frac{a}{2})^2 - p^2\right]$$

მიღებული დიფერენციალური განტოლების ზოგადი ამონახსნია:

$$ln\left|\frac{x - \frac{a}{2} - p}{x - \frac{a}{2} + p}\right| = -2\alpha pt + C$$

საწყისი პირობის გათვალისწინებით

$$C = ln\left|\frac{x - \frac{a}{2} - p}{x - \frac{a}{2} + p}\right| = ln\left|\frac{x_0 - y_0 - 2p}{x_0 - y_0 + 2p}\right|$$

ჩავსვათ C-ს მნიშვნელობა ზოგად ამონახსნში, მივიღებთ

$$ln\left|\frac{x-\frac{x_0+y_0}{2}-p}{x-\frac{x_0+y_0}{2}+p}\right| = -2\alpha pt + \; ln\left|\frac{x_0-y_0-2p}{x_0-y_0+2p}\right|$$

საიდანაც

$$\frac{x-\frac{x_0+y_0}{2}-p}{x-\frac{x_0+y_0}{2}+p} = \frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}$$

აღვნიშნოთ $k = x - \frac{x_0+y_0}{2}$

$$\frac{k-p}{k+p} = \frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}$$

$$(k-p)(x_0-y_0+2p) = (k+p)(x_0-y_0-2p)\cdot e^{-2\alpha pt}$$

$$k(x_0-y_0+2p) - k(x_0-y_0-2p)\cdot e^{-2\alpha pt}$$
$$= p(x_0-y_0+2p) - p(x_0-y_0-2p)\cdot e^{-2\alpha pt}$$
$$k(x_0-y_0+2p-(x_0-y_0-2p)\cdot e^{-2\alpha pt})$$
$$= p(x_0-y_0+2p+(x_0-y_0-2p)\cdot e^{-2\alpha pt})$$

ტოლობის ორივე ნაწილი გავყოთ (x_0-y_0+2p)-ზე

$$k\left(1-\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}\right) = p\left(1+\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}\right)$$

საიდანაც

$$k=\frac{p\left(1+\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}\right)}{1-\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}}$$

და საბოლოოდ მივიღებთ ტოლობას

$$x-\frac{x_0+y_0}{2}=\frac{p\left(1+\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}\right)}{1-\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}}$$

ამრიგად, საბოლოოდ ვღებულობთ კოშის ამოცანის (2.1), (2.2)-ის ზუსტ ანალიზურ ამონახსნს

$$x(t)=\frac{x_0+y_0}{2}+\frac{p\left(1+\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}\right)}{1-\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}} \tag{2.13}$$

$$y(t)=\frac{x_0+y_0}{2}-\frac{p\left(1+\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}\right)}{1-\frac{x_0-y_0-2p}{x_0-y_0+2p}\cdot e^{-2\alpha pt}}$$

(2.13) ანალიზი გვიჩვენებს, რომ მიუხედავად სასტარტო პირობებისა თავისუფალი ადამიანების რაოდენობა მიისწრაფვის გარკვეულ წონასწორობის რაოდენობისკენ, რომელიც საერთო რაოდენობის ნახევარზე მეტია (სუსტი ზეწოლა).

ამრიგად, მიღებული ზუსტი ანალიზური ამოხსნა იძლევა საშუალებას მხარეებმა (მმართველი სტრუქტურები, თავისუფალი ადამიანები) არჩეული სტრატეგიის მიხედვით შესაბამისად შეარჩიონ მართვის პარამეტრები (ადმინისტრაციული ზეწოლის ძალა,

თავისუფლების კოეფიციენტი) და ეცადონ მიაღწიონ მათთვის სასურველ შედეგს. ბუნებრივია, რომ მოქმედ მმართველ სტრუქტურებზე არ არის დამოკიდებული მხარეების სასტარტო პირობები, ამიტომ მისთვის მართვის (მხოლოდ მასზე დამოკიდებული) პარამეტრია იდეოლოგიური თუ ტექნოლოგიური (ბეჭვდითი და ელექტრონული პრესა, ინტერნეტი) შესაძლებლობები, რომლის გამოყენების ყველანაირი ეკონომიკური ბერკეტი მას გააჩნია.

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TEMUR CHILACHAVA, TSIALA DZIDZIGURI,
LEILA SULAVA, MAIA CHAKABERIA

NONLINEAR MATHEMATICAL MODEL OF ADMINISTRATIVE PRESSURE

Abstract

In work the new nonlinear continuous mathematical model which can describe in the given society (the country, an educational institution, industrial object etc.) administrative pressure upon people from outside administrative structures for the purpose of the control of their actions is offered. The mathematical model is described by nonlinear system of the differential equations with two unknown (quantity free (non-ruled) and ruled people at the moment of t time). Administrative pressure which can have various forms, is generally defined by the given function of time. In case of constant administrative pressure the problem of Cauchy's for system of the nonlinear differential equations of the first order is solved analytically exactly. Depending on various correlations between model parameters (the factor of degree of freedom, force of administrative pressure) and initial conditions are received five various cases:

- without dependence from starting conditions the quantity of free people aspires to certain equilibrium value which is more than half of their total quantity (weak pressure);
- despite constant administrative pressure and various starting conditions in society the equal quantity of ruled and non-ruled people will be established (insufficient pressure);
- the quantity of free people which was initially more quantities ruled, aspires to equilibrium value which is less than half of their total quantity (strong, but the limited pressure upon free people);
- the quantity of free people which was initially less or equally quantities ruled, aspires to zero (strong pressure, a complete control case);
- the quantity of free people which was initially more quantities ruled, aspires to zero (the strongest pressure upon free people, model of full submission).

The offered mathematical model except theoretical interest has also the important practical meaning as both sides (administration, free people) can use results of mathematical model in conformity of the purposes.

ა ვ ტ ო რ ე ბ ი

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A U T H O R S

- **Elizbar Nadaraya** – Doctor of Physics and Mathematics Sciences, Corresponding Member Georgian National Academy of Sciences, Professor of Ivane Javakhishvili Tbilisi State University
- **Petre Babilua** – Doctor, Assistant-Professor of Ivane Javakhishvili Tbilisi State University
- **Grigol Sokhadze** – Doctor of Physics and Mathematics Sciences, Associate Professor of Ivane Javakhishvili Tbilisi State University
- **Mzia Patsatsia** – Associate Professor of Sokhumi State University
- **Gogi Pantsulaia** – Doctor of Physics and Mathematics Sciences, Professor of Georgian Technical University
- **Ushangi Goginava** – Doctor of Physics and Mathematics Sciences, Professor of Ivane Javakhishvili Tbilisi State University
- **Ilia Tavkhelidze** – Doctor of Physics and Mathematics Sciences, Associate Professor of Ivane Javakhishvili Tbilisi State University
- **Hamlet Meladze** – Doctor of Physics and Mathematics Sciences, Sokhumi State University, Institute of Mathematics, Professor of St. Andrew the First Called Georgian University
- **Temur Chilachava** – Doctor of Physics and Mathematics Sciences, Professor of Sokhumi State University, Director of Institute of Mathematics
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სტატია მოგვაწოდეთ DOC(MS-Word) ფაილის სახით, ფურცლის ზომა – 4, შრიფტი AcadNusx, 12 ზომა, სტრიქონებს შორის მანძილი – 1,5; ინგლისური ტექსტის შრიფტი – Times New Roman, ზომა 12 დაშორება ფურცლის ყველა კიდიდან – 2,5სმ. გრაფიკული მასალა jpg, gif ან bmp ფაილები.

სტატიის შინაარსის სტრუქტურა:

„UDC“ კლასიფიკატორი, სათაური (ცენტრირებული, Bold, ზომა 14); მონაცემები ავტორის (თითოეული თანაავტორის) შესახებ: გვარი და სახელი (ცენტრირებული, ზომა 12), ორგანიზაციის დასახელება და მისამართი (ცენტრირებული, ზომა 10); ანოტაცია (რეზიუმე) სტატიის ენაზე (ზომა 12). ანოტაცია არ უნდა შეიცავდეს ციტატებს გამოყენებული ლიტერატურიდან და რთულ ფორმულებს. საკვანძო სიტყვები სტატიის ენაზე (ზომა 12); სტატიის შინაარსი; გამოყენებული ლიტერატურის ნუსხა. ლიტერატურის ნუსხა შეადგინეთ ტექსტში მისი გამოყენების თანმიმდევრობის მიხედვით; დასასრულ, მოიყვანეთ რეზიუმეები. ოსტატიის მოცულობა არ უნდა აღემატებოდეს 15 გვერდს (რეზიუმეების ჩათვლით). სტატია წარმოდგენილი უნდა იყოს: 1 ეგზემპლიარი ამობეჭდილი + ელექტრონული ვერსია. სტატიის გადმოგზავნა შეიძლება ელ-ფოსტით:

ingagabi@mail.ru; temo_chilachava@yahoo.com; temo.chila@gmail.com

გამომცემლობის რედაქტორები: **რობერტ მესხი,**
ლელა მირცხულავა

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